ISSN: 1017-060X (Print)

ISSN: 1735-8515 (Online)

Bulletin of the

Iranian Mathematical Society

Vol. 43 (2017), No. 5, pp. 1101–1126

Title:

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Published by the Iranian Mathematical Society
http://bims.ims.ir

Bull. Iranian Math. Soc.
Vol. 43 (2017), No. 5, pp. 1101–1126
Online ISSN: 1735-8515

# AFFINIZATION OF SEGRE PRODUCTS OF PARTIAL LINEAR SPACES

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(Communicated by Mohammad Bagher Kashani)

Abstract. Hyperplanes and hyperplane complements in the Segre product of partial linear spaces are investigated. The parallelism of such a complement is characterized in terms of the point-line incidence. Assumptions, under which the automorphisms of the complement are the restrictions of the automorphisms of the ambient space, are given. An affine covering for the Segre product of Veblenian gamma spaces is established. A general construction that produces non-degenerate hyperplanes in the Segre product of partial linear spaces embeddable into projective space is introduced.
**Keywords:** Segre product, hyperplane, hyperplane complement.
**MSC(2010):** Primary: 51A15; Secondary: 51A45, 15A69, 15A75.

## 1. Introduction

The term *affinization* is not widely used. Its idea however, is not only well known but also applied very often in geometry. It has been spotted in [19] and means construction of the complement of a hyperplane in some point-line space, inspired by construction of an affine space as a reduct of a projective space. To be fair we should cite a lot more papers here. Those of a great impact for our work are [5] and [6]. A problem that is closely related to the removal of a point subset or a line subset or both is reconstruction of the ambient space from the remainder. This is addressed in [20] for projective spaces and for Grassmann spaces, while [23] deals with the Segre product of Grassmann spaces.

The main part of the paper starts with the characterization of hyperplanes in the Segre product $\mathfrak{M}$ of partial linear spaces (Theorem 4.1). These are similar to structures investigated in [1] and [2]. Generalized projective geometries introduced in [1] are products of two geometries, such that some distinguished subsets of this product have the structure of an affine space. In the case of

Article electronically published on 31 October, 2017.
Received: 5 April 2016, Accepted: 3 May 2016.
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projective spaces, these subsets are directly affine spaces that emerge as hyperplane complements. Likewise, we are interested in locally affine structures obtained as a hyperplane complement $\mathfrak{M} \setminus \mathcal{H}$, the result of affinization. In this context issues typical to affine geometry with parallelism arise. Our goal is to solve some of them.

The automorphism group of $\mathfrak{M} \setminus \mathcal{H}$ is characterized (Theorem 4.18). We prove that the parallelism $\|_{\mathcal{H}}$ is definable in terms of point-line incidence of the product $\mathfrak{M}$ (Proposition 4.17) like it is in most of geometries that resemble affine spaces. One of the exceptions could be a spine space with affine lines only (cf. [24]).

The next problem concerns the existence of a hyperplane $\mathcal{H}$ in $\mathfrak{M}$ such that the complement $\mathfrak{M} \setminus \mathcal{H}$ is not isomorphic to the Segre product of the related hyperplane complements taken in the components of $\mathfrak{M}$. Those isomorphic to such products are relatively easy to find (cf. Proposition 4.9). In any case, under assumption that $\mathfrak{M}$ is the product of Veblenian gamma spaces with lines thick enough, the complement $\mathfrak{M} \setminus \mathcal{H}$ is covered by affine spaces (Fact 5.1).

In the last section we focus on the Segre product $\mathfrak{M}$ which components are embeddable into projective space. For such $\mathfrak{M}$ we introduce a general construction of a hyperplane, which idea is based on the characterisation of hyperplanes in Grassmann spaces provided in [27] (see also [7, 11, 12]). This makes possible to show that numerous non-degenerate hyperplanes in $\mathfrak{M}$ exist. The complete characterization of hyperplanes in $\mathfrak{M}$ is challenging and worth to be done, but it is not the goal of this paper. This characterization frequently involves computations related to multilinear forms and hyperdeterminants. Many results in this area can be found in the literature or the Internet, but none of them gives an ultimate answer to our problems. So, we can only explicitly characterize hyperplanes in the Segre product of projective spaces, which is, with no doubt, the most significant class.

## 2. Generalities

2.1. **Partial linear spaces, hyperplanes and parallelism.** A structure $\mathfrak{M} = \langle S, \mathcal{L} \rangle$, $\mathcal{L} \subseteq \wp(S)$, where the elements of $S$ are called *points* and the elements of $\mathcal{L}$ are called *lines*, is a *partial linear space* if and only if there are two or more points on every line, there is a line through every point, and any two lines that share two or more points coincide. We say that the points $a, b \in S$ are *collinear* or *adjacent* in $\mathfrak{M}$ and write $a \sim b$ when they are on a line of $\mathfrak{M}$, denoted by $\overline{a, b}$. It might be confusing which line is which when substructures of $\mathfrak{M}$ come into play but this should be clear from the context. The set of all the points adjacent to a given point $a$ is $[a]_\sim := \{b \in S \colon a \sim b\}$. A partial linear space where every two points are collinear is a *linear space*. Two lines $L, K$ are said to be *adjacent*, in symbols $L \sim K$, whenever they share a point.

We say that three pairwise distinct points $a, b, c \in S$ form a *triangle* in $\mathfrak{M}$ if they are pairwise adjacent and not collinear. A *subspace* of $\mathfrak{M}$ is a subset $X$ of $S$ with the property that if a line $L$ shares two or more points with $X$, then $L$ is entirely contained in $X$. A subspace $X$ of $\mathfrak{M}$ is *strong* if any two points in $X$ are collinear. We call a subspace $X$ of $\mathfrak{M}$ a *hyperplane* if it is proper and every line of $\mathfrak{M}$ meets $X$. In other words a hyperplane is a set $X$ of points such that every line meets $X$ in one or all points.

A partial linear space is *connected* if and only if adjacency relation $\sim$ is connected, i.e., when any two points $p, q$ can be joined by a sequence $p = a_0 \sim a_1 \sim \cdots \sim a_n = q$. It is *strongly connected* if and only if given at least 2-element strong subspace $X$ and a point $p$, there is a sequence of strong subspaces $Y_0, \dots, Y_n$ such that $X = Y_0$, $p \in Y_n$, and $2 \leq |Y_{i-1} \cap Y_i|$ for all $i = 1, \dots, n$. It is clear that every strongly connected partial linear space is connected. In what follows we restrict ourselves to connected partial linear spaces.

Our results involve two specific properties of hyperplanes which we define here in general setting. A subset $X$ of $S$ is called

- *spiky* when every point $a \in X$ is adjacent to some point $b \notin X$,
- *flappy* when for every line $L \subseteq X$ there is a point $a \notin X$ such that $L \subseteq [a]_\sim$.

**Lemma 2.1.** *A flappy hyperplane of a partial linear space is spiky.*

*Proof.* Let $\mathfrak{M} = \langle S, \mathcal{L} \rangle$ be a partial linear space, let $\mathcal{H}$ be a hyperplane in $\mathfrak{M}$. Suppose that $\mathcal{H}$ is not spiky. Then there is a point $q \in \mathcal{H}$ such that each line through $q$ is entirely contained in $\mathcal{H}$. Let $q \in L \in \mathcal{L}$. As $\mathcal{H}$ is flappy there is $a \notin \mathcal{H}$ such that $L \subseteq [a]_\sim$. In particular, $a \sim q$, which is impossible. □

However, a spiky hyperplane need not to be flappy.

**Example 2.2.** Let $\mathfrak{P} = \langle S, \mathcal{L} \rangle$ be a projective 3-space, $L$ be a line, and $\mathcal{H}$ a hyperplane in $\mathfrak{P}$ such that $L \subseteq \mathcal{H}$. Take a line $M$, which is skew to $L$. There is a bijection $f\colon L \longrightarrow M$. Let $\mathcal{L}_1 = \big\{\overline{a, f(a)}\colon a \in L\big\}$ and $\mathcal{L}_2 = \{K \in \mathcal{L}\colon K \cap L = \emptyset\}$. Then $\mathcal{H}$ is spiky but non-flappy in $\langle S, \mathcal{L}_1 \cup \mathcal{L}_2 \rangle$.

Obviously, in case of linear spaces all hyperplanes are flappy and thus spiky as well. Moreover, in this case hyperplanes are maximal proper subspaces, so there are no distinct hyperplanes such that one is contained in the other. However, it is possible in partial linear spaces.

**Example 2.3.** Take a projective space $\mathfrak{P} = \langle S, \mathcal{L} \rangle$ that is at least a plane. Let $X_1, X_2$ be two distinct hyperplanes in $\mathfrak{P}$. Set $\mathcal{H}_1 := X_1 \cap X_2$, $\mathcal{H}_2 := X_2$, and

$$\mathfrak{M} := \big\langle X_1 \cup X_2, \{L \in \mathcal{L}\colon L \subseteq X_1 \cup X_2\}\big\rangle.$$

It is clear that $\mathfrak{M}$ is a partial linear space where $\mathcal{H}_1, \mathcal{H}_2$ are two distinct hyperplanes with $\mathcal{H}_1 \subsetneq \mathcal{H}_2$.

One can also say that hyperplanes in linear spaces are minimal sets satisfying their definition. Spiky hyperplanes, which are of our principal concern in this paper, exhibit similar behaviour in partial linear spaces, they are minimal sets.

**Lemma 2.4.** *Let $\mathcal{H}_1, \mathcal{H}_2$ be hyperplanes in a partial linear space with $\mathcal{H}_1 \subseteq \mathcal{H}_2$. If $\mathcal{H}_2$ is spiky, then $\mathcal{H}_1 = \mathcal{H}_2$.*

*Proof.* Suppose that there is a point $p \in \mathcal{H}_2 \setminus \mathcal{H}_1$. As $\mathcal{H}_2$ is spiky, there is a line $L$ such that $L \cap \mathcal{H}_2 = \{p\}$. But then $L \cap \mathcal{H}_1 = \emptyset$, a contradiction. □

A hyperplane restricted to a substructure is a hyperplane in that substructure.

**Lemma 2.5.** *Let $S_0 \subseteq S$, $\mathcal{L}_0 \subseteq \mathcal{L} \cap \wp(S_0)$, and $\mathcal{H}$ be a hyperplane in $\mathfrak{M}$. If $S_0 \nsubseteq \mathcal{H}$, then $\mathcal{H} \cap S_0$ is a hyperplane in $\langle S_0, \mathcal{L}_0 \rangle$.*

*Proof.* Set $\mathfrak{M}_0 := \langle S_0, \mathcal{L}_0 \rangle$ and $\mathcal{H}_0 := \mathcal{H} \cap S_0$. Let $L \in \mathcal{L}_0$ be such that $|L \cap \mathcal{H}_0| \geq 2$. Then $|L \cap \mathcal{H}| \geq 2$. Since $L \subseteq S_0$, we have $L \subseteq \mathcal{H}_0$. Therefore $\mathcal{H}_0$ is a subspace of $\mathfrak{M}_0$. The assumption $S_0 \nsubseteq \mathcal{H}$ implies that $\mathcal{H}_0$ is a proper subspace of $\mathfrak{M}_0$. Next, let $K \in \mathcal{L}_0$. Notice that there is $p \in K \cap \mathcal{H}$. As $K \subseteq S_0$ we get $p \in K \cap \mathcal{H}_0$ which completes the proof. □

A *gamma space* is a partial linear space where $[a]_\sim$ is a subspace for all $a \in S$. Gamma spaces are also known as those partial linear spaces satisfying *none-one-or-all* axiom. A partial linear space is said to be *Veblenian* if and only if for any two distinct lines $L_1, L_2$ through a point $p$ and any two distinct lines $K_1, K_2$ not through the point $p$ whenever $L_1, L_2 \sim K_1, K_2$, then $K_1 \sim K_2$. Note that a projective space is a Veblenian linear space with lines of size at least 3.

A structure

$$\mathfrak{A} = \langle S, \mathcal{L}, \parallel \rangle$$

is a *partially affine partial linear space* if and only if $\langle S, \mathcal{L} \rangle$ is a partial linear space and $\parallel$ is an equivalence relation on $\mathcal{L}$ such that $L \sim K$ and $L \parallel K$ implies that $L = K$ for all $L, K \in \mathcal{L}$. A partially affine partial linear space $\mathfrak{A}$ is an *affine partial linear space* (cf. [25]) when for all $a \in S, L \in \mathcal{L}$ there is $K \in \mathcal{L}$ such that $a \in K \parallel L$. A partially affine partial linear space $\mathfrak{A}$ is said to satisfy *the Tamaschke Bedingung* when

(2.1) *for any two lines $L_1, L_2$ through a point $p$ and any two other lines $K_1, K_2$ not through $p$ if $K_1 \sim L_1, L_2$, $K_2 \sim L_1$, $K_1 \parallel K_2$ then $L_2 \sim K_2$,*

and it is said to satisfy the *parallelogram completion condition* when

(2.2) *for any two pairs of parallel lines $L_1 \parallel L_2$, $K_1 \parallel K_2$ if $L_1, L_2 \sim K_1$ and $L_1 \sim K_2$,* then $L_2 \sim K_2$.

Observe that, in this approach, an *affine space* is an affine linear space which satisfies the Tamaschke Bedingung and the parallelogram completion condition.

2.2. **Segre products.** Let $I$ be a countable set ($2 \leq |I|$) and let $\mathfrak{M}_i = \langle S_i, \mathcal{L}_i \rangle$ be a partial linear space for $i \in I$. Take

$$S := \times_{i \in I} S_i.$$

To make notation easier we apply the following convention: given $a = (a_1, a_2 \ldots)$ in $S$ and $i \in I$, for a point $x \in S_i$ we write

$$a[i/x] := (a_1, \ldots, a_{i-1}, x, a_{i+1}, \ldots);$$

for a set $A \subseteq S_i$ we write

$$a[i/A] := \{(a_1, \ldots, a_{i-1})\} \times A \times \{(a_{i+1}, \ldots)\};$$

for a family $\mathcal{F} = \{A_j \subseteq S_i : j \in J\}$ of subsets of $S_i$, $J$ being some set of indices, we write

$$a[i/\mathcal{F}] := \Big\{\{(a_1, \ldots, a_{i-1})\} \times A_j \times \{(a_{i+1}, \ldots)\} : j \in J\Big\}.$$

Now take

$$\mathcal{L} := \bigcup_{i \in I} \Big\{a[i/\mathcal{L}_i] : a \in S\Big\}.$$

The structure

$$\bigotimes_{i \in I} \mathfrak{M}_i = \langle S, \mathcal{L} \rangle$$

will be called *the Segre product* of $\mathfrak{M}_i$. We say that a line $L$ in this product arises as a line $l$ of $\mathfrak{M}_i$ if $L = a[i/l]$ for some $a \in S$, $i \in I$. Based on [15] let us recall some simple facts.

**Fact 2.6.** *Let $\mathfrak{M} = \bigotimes_{i \in I} \mathfrak{M}_i$ be the Segre product of partial linear spaces $\mathfrak{M}_i$.*

(i) *The product $\mathfrak{M}$ is a partial linear space. The connected component of a point $a$ of $\mathfrak{M}$ is the set $\big\{x \in S : \big|i : x_i \neq a_i\big| < \infty\big\}$. Consequently, $\mathfrak{M}$ is connected whenever $I$ is finite.*

(ii) *A triangle in $\mathfrak{M}$ has the form $a[i/T_i]$ for some $a \in S$, $i \in I$, and a triangle $T_i$ in $\mathfrak{M}_i$.*

(iii) *A strong subspace of $\mathfrak{M}$ has the form $a[i/X_i]$ for some $a \in S$, $i \in I$, and a strong subspace $X_i$ in $\mathfrak{M}_i$.*

(iv) *If all the $\mathfrak{M}_i$ are gamma spaces, then $\mathfrak{M}$ is a gamma space.*

(v) *If all the $\mathfrak{M}_i$ are Veblenian, then $\mathfrak{M}$ is Veblenian.*

If $\mathfrak{A}_i = \langle S_i, \mathcal{L}_i, \|_i \rangle$ is a partially affine partial linear space for $i \in I$ we define the Segre product $\mathfrak{A} = \bigotimes_{i \in I} \mathfrak{A}_i = \langle S, \mathcal{L}, \| \rangle$ so that $\langle S, \mathcal{L} \rangle = \bigotimes_{i \in I} \langle S_i, \mathcal{L}_i \rangle$ and for lines $L, K$ of $\mathfrak{A}$ we have

$$L \parallel K \quad : \text{ if and only if } \quad (\exists i \in I)[\, L_i \parallel_i K_i \,] \tag{2.3}$$

$L_i, K_i$ being $i$-th coordinate of $L, K$ respectively. For further applications let us define a parallelism $\|^{\sim}$ on $\mathcal{L}$ by the following formula, much more, in fact, in the spirit of a product

$$L \|^{\sim} K \quad :\iff \quad (\forall i \in I)[\, L_i = K_i \vee L_i \parallel_i K_i\,]. \tag{2.4}$$

**Fact 2.7** (cf. [25]). (i) *The Segre product $\mathfrak{A}$ is a partially affine partial linear space. Its connected components are as in Fact 2.6(i).*

(ii) *If the $\mathfrak{A}_i$'s are affine partial linear spaces, then $\mathfrak{A}$ is also an affine partial linear space.*

(iii) *If $\mathfrak{A}_i$ satisfies the Tamaschke Bedingung for all $i \in I$, then $\mathfrak{A}$ also satisfies this condition.*

(iv) *If $\mathfrak{A}_i$ satisfies the parallelogram completion condition for all $i \in I$, then $\mathfrak{A}$ also satisfies this condition.*

*Remark* 2.8. The structure $\mathfrak{A} = \langle S, \mathcal{L}, \|^{\sim}\rangle$ is a partially affine partial linear space, but it is not an affine partial linear space. To show this, note that, clearly $\mathfrak{A}$ is a partial linear space and $\|^{\sim}$ is an equivalence relation on $\mathcal{L}$. Let $i \in I$, $a \in S$, $l \in \mathcal{L}_i$ and $L = a[i/l]$. Suppose that there is $K \in \mathcal{L}$ such that $L \sim K$ and $L \|^{\sim} K$. Then $K = a[i/k]$ for some line $k \in \mathcal{L}_i$, $k \parallel_i l$ and lines $k$, $l$ share a point. It follows that $k = l$ and consequently $K = L$, so $\mathfrak{A}$ is partially affine.

There is however $b \in S$ such that for some $i_1, i_2 \in I$ we have $b_{i_1}, b_{i_2} \neq a_i$ for all $i \in I$. Hence no line through $b$ is parallel to $L$, and thus $\mathfrak{A}$ is not affine.

## 3. Affinization of partial linear spaces

Let $\mathfrak{M} = \langle S, \mathcal{L}\rangle$ be a partial linear space and $\mathcal{H}$ its hyperplane. We write $\mathcal{L}^{\propto} = \{L \in \mathcal{L} \colon L \nsubseteq \mathcal{H}\}$. For each $L \in \mathcal{L}^{\propto}$ there is a unique point $L^{\infty} \in \mathcal{H}$ with $L^{\infty} \in L$. This enables us to define a natural parallelism $\parallel_{\mathcal{H}}$ on $\mathcal{L}^{\propto} \times \mathcal{L}^{\propto}$ by the following condition

$$L \parallel_{\mathcal{H}} K \iff L^{\infty} = K^{\infty}. \tag{3.1}$$

The complement of $\mathcal{H}$ in $\mathfrak{M}$ is

$$\mathfrak{M} \setminus \mathcal{H} = \langle S \setminus \mathcal{H}, \mathcal{L}^{\propto}, \parallel_{\mathcal{H}}\rangle. \tag{3.2}$$

With straightforward reasoning we get the following.

**Fact 3.1.** *Let $\mathfrak{M}$ be a partial linear space and let $\mathcal{H}$ be its hyperplane.*

(i) *The complement $\mathfrak{A} = \mathfrak{M} \setminus \mathcal{H}$ is a partially affine partial linear space.*

(ii) *If $\mathcal{H}$ is spiky, then the complement $\mathfrak{A}$ is an affine partial linear space if and only if $a \sim b$ for all $a \in \mathcal{H}$ and $b \notin \mathcal{H}$.*

(iii) *If $\mathfrak{M}$ is a linear space, then $\mathfrak{A}$ is an affine linear space.*

(iv) *If $\mathfrak{M}$ is Veblenian, then $\mathfrak{A}$ satisfies the parallelogram completion condition and the Tamaschke Bedingung.*

*Remark* 3.2. The converse of Fact 3.1(iii) is false, in general. To see this, take any linear space $\mathfrak{M} = \langle S, \mathcal{L} \rangle$ with a hyperplane $\mathcal{H}$ (e.g. let $\mathfrak{M}$ be a classical projective space). Consider the set $\mathcal{L}(\mathcal{H}) = \{L \in \mathcal{L} \colon L \subseteq \mathcal{H}\}$, then $\mathfrak{M}' = \langle S, \mathcal{L} \setminus \mathcal{L}(\mathcal{H}) \rangle$ is not a linear space. Nevertheless, $\mathcal{H}$ is a hyperplane of $\mathfrak{M}'$ and $\mathfrak{M}' \setminus \mathcal{H} = \mathfrak{M} \setminus \mathcal{H}$ is a linear space.

Affinization may break vital properties like connectedness.

**Example 3.3.** Let $Y_1, Y_2$ be two projective $k$-subspaces of a projective space $\mathfrak{P}$ such that $\mathcal{H} = Y_1 \cap Y_2$ has dimension $k - 1 \geq 1$. Let $\mathfrak{M}$ be the restriction of $\mathfrak{P}$ to $Y_1 \cup Y_2$. Then $\mathfrak{M}$ is a strongly connected Veblenian gamma space and $\mathcal{H}$ is a flappy hyperplane in $\mathfrak{M}$. However, $\mathfrak{M} \setminus \mathcal{H}$ is not connected, and thus not strongly connected.

It may also break the property of being spiky or being flappy. That is, if $\mathcal{H}$ is a flappy hyperplane in $\mathfrak{M}$, then its restriction to a substructure $\langle S_0, \mathcal{L}_0 \rangle$, in the sense of Lemma 2.5, may be non-spiky, and consequently non-flappy, hyperplane in that substructure.

There is a natural correspondence between strong subspaces of hyperplane complements in Veblenian gamma spaces and strong subspaces of the respective ambient spaces.

**Lemma 3.4.** *Let $\mathfrak{M}$ be a Veblenian gamma space with lines of size at least 4 and let $\mathcal{H}$ be a hyperplane in $\mathfrak{M}$. A set $X \subseteq S$ is a strong subspace in $\mathfrak{M} \setminus \mathcal{H}$ if and only if there is a strong subspace $Y$ in $\mathfrak{M}$ such that $X = Y \setminus \mathcal{H}$.*

*Proof.* ($\Rightarrow$) Set $\mathcal{L}^{\propto}(X) := \{L \in \mathcal{L}^{\propto} \colon |L \cap X| \geq 2\}$. Let us begin proving that

$(*)$ given $a \in X$ and $L \in \mathcal{L}^{\propto}(X)$ there is a line $K \in \mathcal{L}^{\propto}(X)$ such that $a \in K \parallel_{\mathcal{H}} L$.

We drop the trivial case where $a \in L$ and assume that $a \notin L$. Take two distinct points $p, q \in L \cap X$. As $X$ is strong we have $a \sim p, q$. Hence by none-one-or-all axiom $a \sim L^\infty$ in $\mathfrak{M}$. Set $K := \overline{a, L^\infty}$. Now take a point $r \in \overline{a, p} \cap X$ distinct from $a, p$. The line $\overline{q, r}$ intersects two sides of the triangle $a, p, L^\infty$ in $\mathfrak{M}$ so, it intersects $K$ in some point $b$ by the Veblen condition. Note that $q, r \in X$ and $L^\infty \notin \overline{q, r}$ thus $b \in X$. Finally, $K = \overline{a, b} \in \mathcal{L}^{\propto}(X)$, and then $a \in K \parallel_{\mathcal{H}} L$. Thereby the condition $(*)$ is true.

Now, set $X^\infty := \big\{L^\infty \colon L \in \mathcal{L}^{\propto}(X)\big\}$ and $Y := X \cup X^\infty$. We will show that $Y$ is a strong subspace in $\mathfrak{M}$.

Let $u \in X^\infty$ and $a \in X$. Note that $u = L^\infty$ for some $L \in \mathcal{L}^{\propto}(X)$. Then by $(*)$ we have $a \sim u$ and $\overline{a, u} \subseteq Y$.

Now, let $u, w \in X^\infty$. There are $L, M \in \mathcal{L}^{\propto}(X)$ with $u = L^\infty$ and $w = M^\infty$. Take $a, b \in M \cap X$. By $(*)$ we get $u \sim a, b$. Then, by none-one-or-all axiom $u \sim w$. What is left is to show that $\overline{u, w} \subseteq Y$. Take a point $v \in \overline{u, w}$ distinct from $u, w$. Hence $v \in \mathcal{H}$. As $a \sim u, w$ we get $a \sim v$ by none-one-or-all axiom. Take a point $p \in \overline{u, a}$ distinct from $u, a$. The line $K := \overline{v, a}$ intersects two sides: $\overline{u,w}$ and $\overline{u,p}$, of the triangle $u,w,p$. Hence, by the Veblen condition, it intersects $\overline{w,p}$ but not in $w$ or $p$ as otherwise we would have $v=w$ or $v=u$, respectively, which is impossible. Therefore $|K\cap X|\geq 2$ and thus $v\in K\subseteq Y$.

($\Leftarrow$) Immediate by the definition of a strong subspace. □

3.1. **Recovering.** Right from the definitions (3.1), (3.2), assuming that $\mathcal{H}$ is spiky, the deleted points on $\mathcal{H}$ can be identified with the equivalence classes of $\|_{\mathcal{H}}$ i.e. with the elements of $\mathcal{L}^{\propto}/\|_{\mathcal{H}}$. To recover $\mathfrak{M}$ from its affine reduct $\mathfrak{A}=\mathfrak{M}\setminus\mathcal{H}$ we need also to determine in terms of $\mathfrak{A}$ the (ternary) collinearity relation on $\mathcal{L}^{\propto}/\|_{\mathcal{H}}$. In short, this is usually achieved by use of planes in $\mathfrak{A}$ that intersect $\mathcal{H}$. This is a very rough approximation of what should be done. The root of the problem is to determine assumptions under which this recovering procedure can be implemented.

The points $a_1,a_2,\dots$ of $\mathfrak{M}$ are said to be *collinear* when they all are on a line of $\mathfrak{M}$, in symbols $\mathbf{L}(a_1,a_2,\dots)$.

**Proposition 3.5.** *Let $\mathfrak{M}$ be a Veblenian gamma space with lines of size at least 3. If $\mathcal{H}$ is a flappy hyperplane in $\mathfrak{M}$, then for all pairwise distinct points $p_1,p_2,p_3\in\mathcal{H}$ we have*

(3.3)
$$\mathbf{L}(p_1,p_2,p_3)\iff(\exists a_1,a_2,a_3\in S\setminus\mathcal{H})\big[\wedge_{\neq(i,j,k)}\big(a_i\sim a_j\wedge p_k\in\overline{a_i,a_j}\big)\big].$$

*The lines of $\mathcal{H}$ are defined in an abstract way as the equivalence classes of the relation*

(3.4)
$$\mathbf{L}\big([L_1]_\|,[L_2]_\|,[L_3]_\|\big)\iff\begin{array}{l}\textit{there is a triangle with the sides } K_1,K_2,K_3\\ \textit{such that } K_i\parallel L_i \textit{ for all } i=1,2,3 \textit{ in } \mathfrak{M}\setminus\mathcal{H}.\end{array}$$

*Proof.* ($\Rightarrow$) Let $L$ be the line through $p_1,p_2,p_3\in\mathcal{H}$. Since $\mathcal{H}$ is flappy there is a point $a\notin\mathcal{H}$ such that $a\sim p_i$ for every $i=1,2,3$. Take a point $b\in\overline{a,p_1}$ distinct from $a,p_1$. Since $a\notin\mathcal{H}$ we get $b\notin\mathcal{H}$ as well. We have $p_2\sim a,p_1$ and thus $p_2\sim b$. Let $K=\overline{b,p_2}$. From the Veblen condition we get a point $c\in K\cap\overline{a,p_3}$ of $\mathfrak{M}\setminus\mathcal{H}$. Finally $a,b,c\in S\setminus\mathcal{H}$ form the required triangle.

($\Leftarrow$) As $\mathfrak{M}$ is a gamma space, from $a_2\sim a_3,a_1$ we get $a_2\sim p_2$, and then $p_2\sim a_3,a_2$ yields $p_2\sim p_1$. The line $L=\overline{p_1,p_2}$ meets $\overline{a_1,a_2}$ in a point $q$, that follows from the Veblen condition. Furthermore $L\subseteq\mathcal{H}$, thus $q\in\mathcal{H}$. If $q\neq p_3$ then $\overline{q,p_3}\subseteq\mathcal{H}$ and $a_1,a_2\in\mathcal{H}$ in particular, that contradicts the assumptions. Therefore $q=p_3$ and points $p_1,p_2,p_3$ are collinear. □

It is more than likely that the above method that relies on flappy property of hyperplanes to recover the ambient space is not unique and there are other procedures that could be applied. We do not want however to go any deeper into discussion of possible methods.

3.2. **Automorphisms.** Let $\mathcal{H}$ be a hyperplane of a partial linear space $\mathfrak{M} = \langle S, \mathcal{L} \rangle$. For $p \in S$, $K \in \mathcal{L}$ we define

$$\Pi(p, K) := \bigcup \big\{ M \in \mathcal{L} \colon p \in M,\ M \cap K \neq \emptyset \big\}.$$

Two possibilities arise: either $p \in K$ and then $\Pi(p, K)$ is the set of all points on lines through $p$, or $p \notin K$ and then $\Pi(p, K) = \emptyset$ or $\Pi(p, K)$ is the set of all points on lines through $p$ that cross $K$ not in $p$. We call $\Pi(p, K)$ a *near-plane* of $\mathfrak{M}$ if $p \notin K$, $\Pi(p, K) \neq \emptyset$, and $\Pi(p, K)$ is not a single line.

The following is just a standard exercise.

**Fact 3.6.** *Let $\mathcal{H}$ be a hyperplane of a partial linear space $\mathfrak{M} = \langle S, \mathcal{L} \rangle$ and let $\mathfrak{A} = \mathfrak{M} \setminus \mathcal{H}$.*

(i) *If $F$ is an automorphism of $\mathfrak{M}$ that preserves $\mathcal{H}$, then $F \restriction (S \setminus \mathcal{H})$ is an automorphism of $\mathfrak{A}$.*

(ii) *Let $f \in \mathrm{Aut}(\mathfrak{A})$.*

  (a) *If $\mathcal{H}$ is spiky, then $f$ extends to a bijection $F$ of $S$ determined by the conditions:*

  $F(x) = f(x)$ *for* $x \in S \setminus \mathcal{H}$, $F(L^\infty) = f(L)^\infty$ *for every line $L$ of $\mathfrak{A}$.*

  (b) *If $\mathcal{H}$ satisfies the condition:*

(3.5) *for every point $p \notin \mathcal{H}$ and every line $K \not\subseteq \mathcal{H}$ the near-plane $\Pi(p, K)$ of $\mathfrak{M}$ meets $\mathcal{H}$ in a line,*

*and the following analogue of flappy condition:*

(3.6) *for every line $L \subseteq \mathcal{H}$ there is a near-plane $\Pi(p, K)$ of $\mathfrak{M}$ containing $L$ such that $p \notin \mathcal{H}$ and $K \not\subseteq \mathcal{H}$*

*then $F$ from* (a) *is an automorphism of $\mathfrak{M}$ preserving $\mathcal{H}$.*

**Lemma 3.7.** *Let $\mathfrak{M}$ be a Veblenian gamma space with lines of size at least* 3.

(i) *Hyperplanes of $\mathfrak{M}$ satisfy* (3.5).

(ii) *Flappy hyperplanes of $\mathfrak{M}$ satisfy* (3.6).

*Proof.* (i) Let $\mathcal{H}$ be a hyperplane in $\mathfrak{M}$. Let $\Pi(p, K)$ be a near-plane of $\mathfrak{M}$ with $p \notin \mathcal{H}$ and $K \not\subseteq \mathcal{H}$. Since $\mathfrak{M}$ is a gamma space and the size of $K$ is at least 3, by definition the near-plane $\Pi(p, K)$ contains 3 distinct lines $L_1, L_2, L_3$ such that $p \in L_1, L_2, L_3$. Let $a_i \in L_i \cap K$, $b_i \in L_i \cap \mathcal{H}$ for $i = 1, 2, 3$. As $\mathfrak{M}$ is a gamma space we get $a_1 \sim b_2$, and next $b_1 \sim b_2$. Let $\overline{b_1, b_2} = L$. Assume also $b_3 \notin L$. None-one-or-all axiom gives $b_2 \sim b_3$ and $b_3 \sim b_1$. Denote $\overline{b_2, b_3} = L'$ and $\overline{b_3, b_1} = L''$. Then, from the Veblen condition, $K$ meets lines $L, L', L''$ in at least two distinct points. These points are on $\mathcal{H}$, that contradicts $K \not\subseteq \mathcal{H}$. Thus $b_3 \in L$, and consequently $\mathcal{H}$ satisfies (3.5).

(ii) Assume that the hyperplane $\mathcal{H}$ is flappy, and let $L$ be a line contained in $\mathcal{H}$. There is a point $p \notin \mathcal{H}$ such that $L \subseteq [p]_\sim$. Since $|L| > 2$ there are two distinct lines $L_1, L_2$ such that $p \in L_1, L_2$ and $L_i \cap L \neq \emptyset$ for $i = 1, 2$. Let $a_i \in L_i \cap L$ and take $b_i \in L_i$ with $b_i \neq a_i, p$ for $i = 1, 2$. From none-one-or-all axiom $a_1 \sim b_2$, and then $b_2 \sim b_1$ as well. Denote $\overline{b_1, b_2}$ by $K$. Note, that $K \nsubseteq \mathcal{H}$ as $b_1, b_2 \notin \mathcal{H}$. As $\mathfrak{M}$ is Veblenian, we have $K \cap L \neq \emptyset$. Let us consider the near-plane $\Pi(p, K)$ and a point $a_3 \in L$. If $a_3 = a_1$, $a_3 = a_2$ or $a_3 \in K \cap L$, then immediately $a_3 \in \Pi(p, K)$. Otherwise we take a line $L_3 := \overline{a_3, p}$. The lines $K, L_3$ intersect $L, L_1$ so that there are 4 distinct points of intersection. Hence, the Veblen condition yields $K \cap L_3 \neq \emptyset$. So, we get $a_3 \in \Pi(p, K)$, and thus $L \subseteq \Pi(p, K)$. □

From Fact 3.6 and Lemma 3.7 we obtain

**Theorem 3.8.** *If $\mathfrak{M}$ is a Veblenian gamma space with lines of size at least 3 and $\mathcal{H}$ is a flappyhyperplane of $\mathfrak{M}$, then every automorphism of $\mathfrak{M} \setminus \mathcal{H}$ can be uniquely extended to an automorphism of $\mathfrak{M}$.*

Observe that, if $\mathfrak{M}$ has lines of size 2 and we remove a haperplane from $\mathfrak{M}$, then there are no lines in the remainder. This means that $\mathfrak{M}\setminus\mathcal{H}$ is not a partial linear space and there is no way to reconstruct $\mathfrak{M}$ from $\mathfrak{M}\setminus\mathcal{H}$, that is to extend an automorphism of $\mathfrak{M} \setminus \mathcal{H}$ to $\mathfrak{M}$. In view of Example 3.3, connectedness of $\mathfrak{M}$ need not to imply connectedness of $\mathfrak{M} \setminus \mathcal{H}$. Therefore, an essential tool to redefine $\mathcal{H}$ in terms of its complement in $\mathfrak{M}$ is the parallelism $\parallel_{\mathcal{H}}$. That is why one needs to be aware that Theorem 3.8, as well as forthcoming Proposition 4.6 and Theorem 4.18, are false for a hyperplane complement considered as a point-line incidence structure without parallelism.

## 4. **Affinization of Segre products**

4.1. **Hyperplanes in Segre products.** Let $\mathfrak{M}_i = \langle S_i, \mathcal{L}_i \rangle$ be a partial linear space, and let $\mathcal{H}_i$ be the family of all hyperplanes in $\mathfrak{M}_i$ and the point set $S_i$ for $i \in I$. Set $S := \times_{i \in I} S_i$ and $\mathfrak{M} := \bigotimes_{i \in I} \mathfrak{M}_i$. Consider a hyperplane $\mathcal{H}$ in $\mathfrak{M}$. We will write

$$\mathcal{H}_i^{[a]} := \{ x \in S_i \colon a[i/x] \in \mathcal{H} \}. \tag{4.1}$$

for a point $a \in S$ and $i \in I$.

**Theorem 4.1.** *For $\mathcal{H} \subseteq S$ the following conditions are equivalent:*

(i) *$\mathcal{H}$ is a hyperplane in $\mathfrak{M}$.*
(ii) *For all $a \in S$ and $i \in I$ we have $\mathcal{H}_i^{[a]} \in \mathcal{H}_i$ but $\mathcal{H}_i^{[a]} \neq S_i$ for some $a \in S$ and $i \in I$.*

*Proof.* To justify the equivalence of (i) and (ii) it suffices to consider the sets $a[i/S_i]$ for arbitrary $a \in S$ and $i \in I$, which are subspaces of $\mathfrak{M}$. Note that either $\mathcal{H} \cap a[i/S_i]$ is the whole of $a[i/S_i]$ or a hyperplane in it. Clearly, for fixed $a \in S$ and $i \in I$ the map $S_i \ni x_i \mapsto a[i/x_i]$ is an isomorphism of $\mathfrak{M}_i$ onto

$a[i/S_i]$. Therefore, there is $X \in \mathcal{H}_i$ such that $a[i/S_i] \cap \mathcal{H} = a[i/X]$. It is seen that $X = \mathcal{H}_i^{[a]}$. □

In particular case of a product of two spaces, Theorem 4.1 can be worded in terms of a correlation.

*Remark* 4.2. Let $I = \{1, 2\}$. The set $\mathcal{H} \subseteq S$ is a hyperplane in $\mathfrak{M}$ if and only if there are two maps: $\delta_i \colon S_i \longrightarrow \mathcal{H}_{3-i}$ such that

$$\delta_i(a_i) = \mathcal{H}_{3-i}^{[a]} \quad \text{and} \quad \delta_{3-i}(a_{3-i}) = \big\{a_i \in S_i \colon a_{3-i} \in \delta_i(a_i)\big\}$$

for all $a = (a_1, a_2) \in S_i \times S_2$, $i \in I$ and there is $a_i \in S_i$ with $\delta_i(a_i) \neq S_{3-i}$ for some $i \in I$. Moreover, if $\mathcal{H}$ is a hyperplane then

$$\mathcal{H} = \big\{(a_1, a_2) \colon a_1 \in S_1, a_2 \in \delta_1(a_1)\big\} = \big\{(a_1, a_2) \colon a_2 \in S_2, a_1 \in \delta_2(a_2)\big\}.$$

*Remark* 4.3. If $\mathcal{H}$ is a spiky hyperplane in the product $\mathfrak{M}$ of partial linear spaces on at least three points each, then $\mathfrak{M} \setminus \mathcal{H}$ *is not* an affine partial linear space (cf. Fact 3.1(i)).

*Proof.* Note first that for a point $a$ in $\mathfrak{M}$ we have $[a]_\sim \subseteq \bigcup_{i\in I} a[i/S_i]$. Suppose to the contrary that $\mathfrak{M} \setminus \mathcal{H}$ is an affine partial linear space. Let $p \in \mathcal{H}$. In view of Fact 3.1(ii), all the points non-collinear with $p$ lie on $\mathcal{H}$. In that case $x_i \neq p_i$ for all $i$ yields $x \in \mathcal{H}$, for every $x \in S$. From the assumptions, there exists $p' \in S$ such that $p'_j \neq p_j$ for all $j \in I$. Let $i \in I$. There exists $x \in S_i$ with $x \neq p_i, p'_i$. Take $q \in S$ with $q_i = x$, $q_j = p_j$ for $j \neq i$. We have, consecutively, $p' \in \mathcal{H}$ (as $p$ and $p'$ differ on all of the coordinates), and $q \in \mathcal{H}$ (as $p'$ and $q$ differ on all of the coordinates). So, we get that: if $p \in \mathcal{H}$ and $\big|\{i \in I \colon p_i = p'_i\}\big| \leq m$ with $m = 1$, then $p' \in \mathcal{H}$. Inductively, we can enlarge $m$ and finally we get $\mathcal{H} = S$, a contradiction. □

Although complements of spiky hyperplanes are not affine partial linear spaces these hyperplanes remain beneficial for affinization: all the points of a spiky hyperplane $\mathcal{H}$ are directions of the parallelism $\parallel_{\mathcal{H}}$ in $\mathfrak{M} \setminus \mathcal{H}$.

4.1.1. *Non-degenerate hyperplanes.* In recovering the Segre product from the complement of its hyperplane we heavily rely on the flappy property of that hyperplane. It will be shown later that this property is related to another intrinsic property of hyperplanes.

A hyperplane $\mathcal{H}$ of a Segre product of partial linear spaces is called *non-degenerate* when $\mathcal{H}_i^{[a]}$ is a hyperplane for all $a \in S$ and $i \in I$. In the context of Remark 4.2 we can say that $\mathcal{H}$ is non-degenerate if both $\delta_1$ and $\delta_2$ take hyperplanes as their values.

Main properties of non-degenerate hyperplanes of the Segre product come into hyperplanes of its components.

**Proposition 4.4.** *Let $\mathcal{H}$ be a hyperplane of $\mathfrak{M}$. The following assertions are fulfilled:*

(i) *Assume that $\mathcal{H}$ is non-degenerate. Then $\mathcal{H}$ is flappy if and only if $\mathcal{H}_i^{[a]}$ is a flappy hyperplane in $\mathfrak{M}_i$ for all $a \in S$ and $i \in I$.*
(ii) *The hyperplane $\mathcal{H}$ is spiky if and only if for every $a \in S$ there is $i \in I$ such that $\mathcal{H}_i^{[a]}$ is a spiky hyperplane in $\mathfrak{M}_i$.*

*Proof.* Only the right-hand part of the equivalence in (ii) seems to be not evident. Suppose that there is a point $a \in S$ such that either $\mathcal{H}_i^{[a]} = S_i$ or $\mathcal{H}_i^{[a]}$ is non-spiky for all $i \in I$. Thus, for every $i \in I$ there is a point $b_i \in \mathcal{H}_i^{[a]}$, which is collinear with no point in $\mathfrak{M}_i \setminus \mathcal{H}_i^{[a]}$. Then $b = (b_1, b_2, \dots) \in \mathcal{H}$ is collinear with no point in $\mathfrak{M} \setminus \mathcal{H}$, so $\mathcal{H}$ is non-spiky. □

Immediately from Proposition 4.4 we obtain

**Corollary 4.5.** *Let $\mathfrak{M}_i$ be a linear space for all $i \in I$ and let $\mathcal{H}$ be a hyperplane in $\mathfrak{M}$.*

(i) *The hyperplane $\mathcal{H}$ is flappy if and only if $\mathcal{H}$ is non-degenerate.*
(ii) *The hyperplane $\mathcal{H}$ is spiky if and only if for every $a \in S$ there is $i \in I$ such that $\mathcal{H}_i^{[a]}$ is a hyperplane in $\mathfrak{M}_i$.*

Now, by Proposition 3.5, we can state the following.

**Proposition 4.6.** *If $\mathfrak{M}$ is a Veblenian gamma space with lines of size at least 3 and $\mathcal{H}$ is a non-degenerate flappy hyperplane in $\mathfrak{M}$, then $\mathfrak{M}$ can be defined in terms of $\mathfrak{M} \setminus \mathcal{H}$.*

4.1.2. *Degenerate hyperplanes.* Degenerate hyperplanes are indeed defective from our view.

**Lemma 4.7.** *Degenerate hyperplanes of $\mathfrak{M}$ are not flappy.*

*Proof.* Let $\mathcal{H}$ be a degenerate hyperplane of a Segre product. So, there are $a$ and $i$ such that $\mathcal{H}_i^{[a]} = S_i$, which means that $a[i/S_i] \subseteq \mathcal{H}$. Let $l \in \mathcal{L}_i$, then $L = a[i/l]$ is a line of the product contained in $\mathcal{H}$. In view of Fact 2.6(ii) a triangle in the product with $L$ as one of its sides is contained in $a[i/S_i]$, so no point outside $\mathcal{H}$ can be a vertex of such a triangle. □

There is quite natural construction of a hyperplane in the Segre product as long as there are hyperplanes in all of the components. The outcome, however, is degenerate. For hyperplanes $\mathcal{H}_i$ in $\mathfrak{M}_i$, $i \in I$, we write

$$\otimes_{i\in I} \mathcal{H}_i := \bigcup_{i\in I} (S_1 \times \cdots \times S_{i-1} \times \mathcal{H}_i \times S_{i+1} \dots). \tag{4.2}$$

To shorten notation let us set $\mathcal{H} := \otimes_{i\in I}\mathcal{H}_i$.

**Proposition 4.8.** *The set $\mathcal{H}$ is a degenerate and non-spiky hyperplane in $\mathfrak{M}$.*

*Proof.* Let $a$ be a point of $\mathfrak{M}$ and $L_i$ a line in $\mathfrak{M}_i$ for some $i \in I$. Then $L = a[i/L_i]$ is a line in $\mathfrak{M}$. There is a point $a'_i \in L_i \cap \mathcal{H}_i$ and thus $a[i/a'_i]$ is a common point of $L$ and $\mathcal{H}$. Hence $\mathcal{H}$ is a hyperplane of $\mathfrak{M}$.

Take a point $b$ with $b_i \in \mathcal{H}_i$. Then $\mathcal{H}_j^{[b]} = S_j$ for any $j \neq i$ and thus $\mathcal{H}$ is degenerate. Let $d$ be a point of $\mathfrak{M}$ with $d_i \in \mathcal{H}_i$ for some $i = i_1, i_2 \in I$, $i_1 \neq i_2$. Clearly $d \in \mathcal{H}$. Let $d'$ be a point of the product collinear with $d$. So, $d' = d[i/d'_i]$ for some $i \in I$, $d'_i \sim_i d_i$. It is easy to note, that $d' \in \mathcal{H}$ for any $i \in I$. Consequently, $\mathcal{H}$ is not spiky. □

Observe that the points and the lines of the complement $\mathfrak{M} \setminus \mathcal{H}$ coincide with the points and the lines of the product $\bigotimes_{i\in I} \langle \mathfrak{M}_i \setminus \mathcal{H}_i \rangle$. Since all complements $\mathfrak{M}_i \setminus \mathcal{H}_i$ for $i \in I$ are partially affine partial linear spaces by Fact 3.1(i), we can apply (2.3) to define parallelism $\parallel$ on their product $\bigotimes_{i\in I} \langle S_i \setminus \mathcal{H}_i, \mathcal{L}_i^{\propto} \rangle$. This parallelism however, is not compatible with the parallelism $\parallel_{\mathcal{H}}$ in the complement $\mathfrak{M} \setminus \mathcal{H}$. As $\mathcal{H}$ is a hyperplane introduced in (4.2) the parallelism $\parallel_{\mathcal{H}}$ in $\mathfrak{M} \setminus \mathcal{H}$ is the relation $\parallel^{\sim}$ given by (2.4) with $\parallel_i = \parallel_{\mathcal{H}_i}$. This is the subject of the following statement.

**Proposition 4.9.** *The following assretions hold.*

(i) $\langle S \setminus \mathcal{H}, \mathcal{L}^{\propto} \rangle \cong \bigotimes_{i\in I} \langle S_i \setminus \mathcal{H}_i, \mathcal{L}_i^{\propto} \rangle$.
(ii) $\mathfrak{M} \setminus \mathcal{H} = \langle S \setminus \mathcal{H}, \mathcal{L}^{\propto}, \parallel_{\mathcal{H}} \rangle \cong \big(\bigotimes_{i\in I} \langle S_i \setminus \mathcal{H}_i, \mathcal{L}_i^{\propto} \rangle, \parallel^{\sim}\big)$.

*Proof.* (i) It suffices to note that

$$S \setminus \mathcal{H} = \times_{i\in I} S_i \setminus \bigcup_{i\in I} (S_1 \times \cdots \times S_{i-1} \times \mathcal{H}_i \times S_{i+1} \dots) = \times_{i\in I} (S_i \setminus \mathcal{H}_i).$$

(ii) Take $L, K \in \mathcal{L}^{\propto}$ such that $L \parallel_{\mathcal{H}} K$. Then $L = a[i/L_i]$, $K = b[j/K_j]$ for some $a, b \in S \setminus \mathcal{H}$, $L_i \in \mathcal{L}_i^{\propto}$, $K_j \in \mathcal{L}_j^{\propto}$, and $i, j \in I$. It is seen that

$$a[i/L_i^{\infty}] = L^{\infty} = K^{\infty} = b[j/K_j{}^{\infty}]$$

which means that $i = j$, $a_s = b_s$ for all $s \neq i$, and $L_i \parallel_{\mathcal{H}_i} K_i$. This reasoning can be easily reversed. □

4.2. **Strong subspaces.** Directly from Fact 2.6(iv)–(v) and Lemma 3.4 we get the following.

**Lemma 4.10.** *Let $\mathfrak{M}_i$ be a Veblenian gamma space with lines of size at least 4 for all $i \in I$ and let $\mathcal{H}$ be a hyperplane in $\mathfrak{M}$. A set $X \subseteq S$ is a strong subspace in $\mathfrak{M} \setminus \mathcal{H}$ if and only if there is a strong subspace $Y$ in $\mathfrak{M}$ such that $X = Y \setminus \mathcal{H}$.*

This lets us get a more detailed characterization of strong subspaces in the complement of a product.

**Proposition 4.11.** *Let $\mathfrak{M}_i$ be a Veblenian gamma space with lines of size at least 4 for all $i \in I$ and let $\mathcal{H}$ be a hyperplane in $\mathfrak{M}$. For $X \subseteq S$ the following conditions are equivalent:*

(i) *$X$ is a strong subspace of the complement $\mathfrak{M} \setminus \mathcal{H}$.*
(ii) *$X = a[i/Y_i] \setminus \mathcal{H}$ for some $a \in S$, $i \in I$, and a strong subspace $Y_i$ in $\mathfrak{M}_i$.*
(iii) *$X = a[i/X_i]$ for some $a \in S$, $i \in I$, and a strong subspace $X_i$ in $\mathfrak{M}_i \setminus \mathcal{H}_i^{[a]}$.*

*Proof.* (i)⇒(ii) According to Lemma 4.10 we have a strong subspace $Y$ in $\mathfrak{M}$ such that $X = Y \setminus \mathcal{H}$. By Fact 2.6(iii), $Y = a[i/Y_i]$ for some $a \in S$, $i \in I$ and a strong subspace $Y_i$ in $\mathfrak{M}_i$. Thus $X = a[i/Y_i] \setminus \mathcal{H}$.

(ii)⇒(iii) For $X = \emptyset$ it suffices to take $X_i = \emptyset$, so assume that $X \neq \emptyset$. Hence $\mathcal{H}_i^{[a]}$ is a hyperplane in $\mathfrak{M}_i$. Taking $X_i := Y_i \setminus \mathcal{H}_i^{[a]}$ we are through by Lemma 3.4.

(iii)⇒(i) Immediate by the definition of a strong subspace. □

By Facts 2.6(v) and 3.1, we infer the following.

**Corollary 4.12.** *Let $\mathfrak{M}_i$ be a Veblenian gamma space with lines of size at least 4 for all $i \in I$. If $\mathcal{H}$ is a hyperplane in $\mathfrak{M}$, then the complement $\mathfrak{M} \setminus \mathcal{H}$ satisfies the parallelogram completion condition and the Tamaschke Bedingung. Consequently, its strong subspaces are affine spaces.*

4.3. **Parallelism in terms of incidence.** Most of the time, also in a hyperplane complement of a Segre product, parallelism can be defined in terms of incidence using the Veblen configuration as follows. Let $L_1, L_2 \in \mathcal{L}$.

$$L_1 \parallel^\circ L_2 \iff L_1 = L_2 \vee \text{ there are lines } K_1, K_2 \text{ through a point } p \text{ such that } p \notin L_1, L_2 \wedge L_1, L_2 \sim K_1, K_2 \wedge L_1 \not\sim L_2. \tag{4.3}$$

There is however parallelism typical to a Segre product.

$$L_1 \parallel^* L_2 \iff \text{there exists a quadrangle } p, q, r, s \text{ without diagonals such that } L_1 = \overline{p, q} \wedge L_2 = \overline{r, s}. \tag{4.4}$$

It is based on the fact that four lines $L_1, L_2, K_1, K_2$ form a quadrangle without diagonals in $\mathfrak{M}$ if they all arise as lines $l_1, l_2, k_1, k_2$ of some component $\mathfrak{M}_i$ where $l_1, l_2, k_1, k_2$ form a quadrangle without diagonals, or they are from different components. If the latter is the case, inspecting coordinates carefully one can easily see that the opposite sides are always disjoint. Note that if $\mathfrak{M}_i$ are line spaces, then only the latter holds true.

In an affine space the parallelism can be defined in terms of the incidence. The same can be done in the Segre product of affine spaces.

**Theorem 4.13.** *Let $\mathfrak{A}_i = \langle S_i, \mathcal{L}_i, \parallel_i \rangle$ be an affine space and $\mathfrak{A}_i' = \langle S_i, \mathcal{L}_i \rangle$ for all $i = 1, 2$. The parallelism $\parallel$ of the Segre product $\mathfrak{A}_1 \otimes \mathfrak{A}_2$ can be defined in terms of the point-line incidence, i.e. in $\mathfrak{A}_1' \otimes \mathfrak{A}_2'$.*

*Proof.* Let $L_1, L_2$ be lines of the product $\mathfrak{A} := \mathfrak{A}_1 \otimes \mathfrak{A}_2$. It suffices to observe that in view of (4.3) and (4.4) both $\parallel^\circ$ and $\parallel^*$ are definable in $\mathfrak{A}_1' \otimes \mathfrak{A}_2'$, and that the following three facts hold true:

(a) $L_1 \parallel^{\sim} L_2$ if and only if $L_1 \parallel^{\circ} L_2$.
(b) If $L_1 \parallel^{*} L_2$, then $L_1 \parallel L_2$.
(c) $L_1, L_2$ are parallel in $\mathfrak{A}$ if and only if there is a line $L_3$ with $L_1 \parallel^{\sim} L_3 \parallel^{*} L_2$. □

To characterize $\parallel_{\mathcal{H}}$ a new parallelism comes in handy. Let $L_1, L_2 \in \mathcal{L}^{\propto}$, then

$$\text{(4.5)}\quad L_1 \parallel_{\circ} L_2 \iff \text{there are lines } K_1, K_2, M_1, M_2 \text{ such that } K_1 \parallel^{*} K_2 \wedge M_1, M_2, L_1 \sim K_1, K_2 \wedge L_2 \sim M_1, M_2 \wedge L_1 \not\sim L_2.$$

In plain words: $L_1 \parallel_{\circ} L_2$ if and only if $L_1, L_2$ are non-adjacent in $\mathfrak{M} \setminus \mathcal{H}$, there is a quadrangle $Q$ without diagonals such that $L_1$ intersects the lines in one pair of the opposite sides of $Q$, and $L_2$ intersects the other pair of sides.

**Lemma 4.14.** *Assume that all the lines in $\mathcal{L}_i$ are of size at least 4 for all $i \in I$ and $\mathcal{H}$ is non-degenerate. If $L_1, L_2 \in \mathcal{L}^{\propto}$ are distinct lines through a point on $\mathcal{H}$ and there is no $b \in S$, $i \in I$ with $L_1, L_2 \subseteq b[i/S_i]$, then $L_1 \parallel_{\circ} L_2$.*

*Proof.* Let $a \in \mathcal{H}$ and $a \in L_1 \cap L_2$. Up to an order of variables we can assume that $L_1 = (l_1, a_2, a_3, \dots)$ and $L_2 = (a_1, l_2, a_3, \dots)$ for some $l_1 \in \mathcal{L}_1$, $l_2 \in \mathcal{L}_2$. For brevity, we omit the coordinates $a_3, a_4, \dots$, which does not affect our reasoning. Every point $(a_1, x)$ with $x \neq a_2$ is on $L_2 \setminus \mathcal{H}$. Let $l_2 \ni x \neq a_2$. Then $(l_1, x)$ is a line through $(a_1, x)$, which is not contained in $\mathcal{H}$. If $(y', x) \in (l_1, x) \cap \mathcal{H}$ then all points $(y, x)$ on $(l_1, x)$ with $y \neq y'$ are outside $\mathcal{H}$. Take $y_1, y_2 \in l_1$ such that $y_1, y_2 \neq y'$. Clearly the intersection points $(y_i, a_2) \in L_1 \cap (y_i, l_2)$ are outside $\mathcal{H}$ for $i = 1, 2$. There is exactly one point in $\mathcal{H}$ on the line $(y_i, l_2)$, so there is $z \in l_2$ such that $z \neq a_2, x$ and $(y_i, z) \notin \mathcal{H}$. The line through the points $(y_1, z)$, $(y_2, z)$ intersects $L_2$ in a point $(a_1, z)$. Finally, the points $(y_1, x)$, $(y_1, z)$, $(y_2, x)$, $(y_2, z)$ give a required quadrangle without diagonals and $L_1, L_2$ both cross their sides, as required. □

Two lines in $\mathfrak{M} \setminus \mathcal{H}$ are parallel if they share a point on $\mathcal{H}$, and there are two possibilities: they arise as lines of one hyperplane complement, in one variable, or of two distinct hyperplane complements, in two distinct variables. This observation makes the following fact immediate.

**Fact 4.15.** *Let $\mathcal{H}$ be non-degenerate and $L_1, L_2 \in \mathcal{L}^{\propto}$. Then $L_1 \parallel_{\mathcal{H}} L_2$ if and only if one of the following holds*

(i) *there is $a \in S$, $i \in I$ such that $L_j = a[i/l_j]$ for some $l_j \in \mathcal{L}_i$, $j = 1, 2$, and $l_1 \parallel_{\mathcal{H}_i^{[a]}} l_2$, or*
(ii) *there is no $a \in S$, $i \in I$ as in (i) and $L_1 \parallel_{\circ} L_2$.*

The parallelism $\parallel_{\circ}$ can be expressed in terms of the point-line incidence of $\mathfrak{M} \setminus \mathcal{H}$ via (4.5) and Lemma 4.14. To be able to express $\parallel_{\mathcal{H}}$ in terms of incidence we need to do so with $\|_{\mathcal{H}_i^{[a]}}$. The problem is that it depends not only on the variable $i$ but also on $a \in S$. So, we need to distinguish those products $\mathfrak{M}$ where every parallelism of any hyperplane complement in $\mathfrak{M}_i$ can be defined by a single uniform formula in terms of the point-line incidence of that complement for all $i \in I$. If that is the case, Fact 4.15 is a half-way to express $\|_{\mathcal{H}}$ in terms of incidence. What is still missing is an incidence formula for two lines being in one component of the product. This is addressed by the next fact which follows from Proposition 4.11 and [15].

**Fact 4.16.** *Let $\mathfrak{M}_i$ be a Veblenian gamma space with lines of size at least 4 and let $\mathfrak{M}_i \setminus \mathcal{H}_i^{[a]}$ be strongly connected for all $a \in S$, $i \in I$. Given two lines $L_1, L_2 \in \mathcal{L}^{\propto}$ the following conditions are equivalent:*

(i) *there is $a \in S$, $i \in I$ such that $L_1, L_2 \subseteq a[i/S_i]$,*

(ii) *there is a sequence $Y_0, \ldots, Y_m$ of strong subspaces in $\mathfrak{M} \setminus \mathcal{H}$ such that $L_1 \subseteq Y_0$, $L_2 \subseteq Y_m$, and $Y_{i-1}, Y_i$ share a line for $i = 1, \ldots, m$.*

This together with Fact 4.15 and the remarks below it gives

**Proposition 4.17.** *Let $\mathfrak{M}_i$ be a Veblenian gamma space with lines of size at least 4 such that the parallelism of every hyperplane complement in $\mathfrak{M}_i$ can be uniformly defined in terms of the point-line incidence of that complement for all $i \in I$. Assume that $\mathcal{H}$ is non-degenerate and $\mathfrak{M}_i \setminus \mathcal{H}_i^{[a]}$ is strongly connected for all $a \in S$, $i \in I$. Then the parallelism $\|_{\mathcal{H}}$ in $\mathfrak{M} \setminus \mathcal{H}$ can be characterized in terms of the point-line incidence of the product $\mathfrak{M}$.*

Example 3.3 shows that a strongly connected space could turn out to be not connected after affinization. So, the assumption that hyperplane complements in the components of the product are all strongly connected is indispensable in Fact 4.16 and Proposition 4.17.

Proposition 4.17 is applicable in case of projective and polar spaces as the parallelism in question is uniformly definable in affine spaces (a folklore) and in affine polar spaces (cf. [5]). We do not know however, if parallelism is uniformly definable in affine Grassmann spaces (we guess so) and in affine polar Grassmann spaces.

4.4. **Automorphisms.** Whether an automorphism of a hyperplane complement can be extended to an automorphism of the ambient space is one of the most common questions when it comes to affinization. We have discussed that for partial linear spaces in Section 3.2 and now we are doing so for the Segre product.

**Theorem 4.18.** *Let $\mathfrak{M}_i$ be a strongly connected partial linear space for all $i \in I$, and let $\mathcal{H}$ be a flappy hyperplane in $\mathfrak{M} = \bigotimes_{i \in I} \mathfrak{M}_i$. Set $\mathfrak{A} := \mathfrak{M} \setminus \mathcal{H}$. The automorphisms of $\mathfrak{A}$ are the the automorphisms of $\mathfrak{M}$ that preserve $\mathcal{H}$, restricted to the point set of $\mathfrak{A}$. More precisely, a map $f$ is an automorphism of $\mathfrak{A}$ if and* *only if there is a permutation $\sigma$ of $I$ and a family of isomorphisms $f_i$ that map $\mathfrak{M}_i$ onto $\mathfrak{M}_{\sigma(i)}$ such that the map $F$ defined by the condition $F(a)_i = f_i(a_i)$ preserves $\mathcal{H}$ and $f$ is the restriction of $F$ to the point set of $\mathfrak{A}$.*

*Proof.* Let $f \in \mathrm{Aut}(\mathfrak{A})$. By Lemma 2.1 the hyperplane $\mathcal{H}$ is spiky, so according to Fact 3.6(ii-a) we have the extension $F$ of $f$ to the point set of $\mathfrak{N}$ such that $F(L^\infty) = f(L)^\infty$ for every line $L$ of $\mathfrak{A}$. As $\mathcal{H}$ is flappy in view of Fact 2.6(iv)–(v) and Lemma 3.7 we can apply Fact 3.6(ii-b). Hence $F$ is a collineation of $\mathfrak{N}$ that preserves $\mathcal{H}$. Now, from [15, Proposition 1.10] the required $\sigma$ and $f_i$ exist. □

## 5. Applications and examples: products of partial linear spaces embeddable into projective spaces and their affinizations

Let $\mathfrak{M} := \bigotimes_{i\in I} \mathfrak{M}_i = \langle S, \mathcal{L}\rangle$ be the Segre product of partial linear spaces with a hyperplane $\mathcal{H}$. If $\mathcal{H}$ is given by (4.2), then in view of Proposition 4.9(i) the complement $\mathfrak{M}\setminus\mathcal{H}$ is isomorphic to the product of hyperplane complements, when both of them are considered as incidence structures without parallelism. It need not to be true however, in case of other affinizations. From Remark 4.3 we know that if $\mathcal{H}$ is spiky, then the complement $\mathfrak{M} \setminus \mathcal{H}$ is not an affine space.

The family of strong subspaces in the hyperplane complement $\mathfrak{M} \setminus \mathcal{H}$ will be written as

(5.1)
$$\mathrm{SC}(\mathfrak{M} \setminus \mathcal{H}) := \Big\{a[i/X_i] \colon a \in S,\ i \in I,\ X_i \text{ is a strong subspace of } \mathfrak{M}_i \setminus \mathcal{H}_i^{[a]}\Big\}.$$

A straightforward outcome of Proposition 4.11 and Corollary 4.12 is as follows

**Fact 5.1.** *If all $\mathfrak{M}_i$ are Veblenian gamma spaces with lines of size at least 4, then $\mathrm{SC}(\mathfrak{M} \setminus \mathcal{H})$ is a covering of the hyperplane complement $\mathfrak{M} \setminus \mathcal{H}$ by affine spaces, i.e. $S \setminus \mathcal{H} = \bigcup \mathrm{SC}(\mathfrak{M} \setminus \mathcal{H})$.*

As we are interested in affine-like Segre products, due to Fact 5.1 we will investigate products of some analytical Veblenian gamma spaces: projective spaces, polar spaces, Grassmann spaces, and polar Grassmann spaces. All of them are strongly connected. Thus, Proposition 4.17 and Theorem 4.18 can be applied as far as there are non-degenerate or flappy hyperplanes in these spaces. Constructions of hyperplanes with such properties will be established for products of spaces that are embeddable into a projective space. We focus on geometries of common types, although hyperplanes are also known in many other embeddable spaces (cf. [26]).

### 5.1. Algebraic background.

Let $V$ be a (left) vector space over a division ring $D$. The set of all subspaces of $V$ will be written as $\mathrm{Sub}(V)$ and the set of all $k$-dimensional subspaces as $\mathrm{Sub}_k(V)$. For $H \in \mathrm{Sub}_{k-1}(V)$ and $B \in \mathrm{Sub}_{k+1}(V)$ with $H \subseteq B$ a *$k$-pencil* is the set

$$\mathbf{p}(H, B) := \big\{U \in \mathrm{Sub}_k(V) \colon H \subseteq U \subseteq B\big\}.$$

Taking $k$-subspaces as points and $k$-pencils as lines we get a *Grassmann space* (cf. [17, 28])

$$\mathbf{P}_k(V) := \langle \mathrm{Sub}_k(V), \mathcal{P}_k(V) \rangle.$$

For $k = 1$, and dually for $k = n - 1$ when $V$ is of finite dimension $n$, $\mathbf{P}_k(V)$ is a projective space, while for $1 < k < n - 1$ there are non-collinear points in $\mathbf{P}_k(V)$, so it is a proper partial linear space. It is worth to mention that $\mathbf{P}_k(V) = \mathbf{P}_{k-1}(\mathbf{P}_1(V))$.

Given a reflexive bilinear form $\xi$ on $V$, we write $Q_k(\xi)$ for the set of all isotropic $k$-subspaces of $V$ w.r.t. $\xi$. If $H \in \mathrm{Sub}_{k-1}(V)$, $B \in Q_{k+1}(\xi)$, and $H \subseteq B$ (actually we have $H \in Q_{k-1}(\xi)$), then we get an *isotropic $k$-pencil*

$$\mathbf{p}_\xi(H, B) := \mathbf{p}(H, B) \cap Q_k(\xi).$$

Taking isotropic $k$-subspaces as points and isotropic $k$-pencils as lines we get a *polar Grassmann space* (cf. [17, 18, 28])

$$\mathbf{P}_k(\xi) := \langle Q_k(\xi), \mathcal{G}_k(\xi) \rangle.$$

It is embedded in the Grassmann space $\mathbf{P}_k(V)$ in a natural way, so that the points and lines of $\mathbf{P}_k(\xi)$ are the points and lines of $\mathbf{P}_k(V)$ respectively. Note that $\mathbf{P}_1(\xi)$ is a polar space and $\mathbf{P}_k(\xi) = \mathbf{P}_{k-1}(\mathbf{P}_1(\xi))$.

Recall that the map

$$\mathbf{g}\colon \langle u_1, \dots, u_k \rangle \mapsto \langle u_1 \wedge \cdots \wedge u_k \rangle$$

provided that $D$ is a field, is the well known *Grassmann embedding* (sometimes called also the Plücker embedding) of the Grassmann space $\mathbf{P}_k(V)$ into the projective space $\mathbf{P}_1(\bigwedge^k V)$.

5.2. **Hyperplanes arising from Segre embeddings.** Let $V_i$ be a vector space over a field $D$ of characteristic not 2 for $i = 1, \dots, n$ and let $k = k_1 + \cdots + k_n$ for some positive integers $k_1, \dots, k_n$. For brevity of notation we apply a convention that $u^i = [u^i_1, \dots, u^i_{k_i}] \in V_i^{k_i}$ and $u = (u^1, \dots, u^n)$ for $u \in \times_{i=1}^n V_i^{k_i} =: V$. Here, we investigate the Segre product

$$\mathfrak{M} = \mathfrak{M}_{k_1,\dots,k_n}(V_1, \dots, V_n) := \mathbf{P}_{k_1}(V_1) \otimes \cdots \otimes \mathbf{P}_{k_n}(V_n). \tag{5.2}$$

Consider a mapping $\mu\colon V \longrightarrow D$ that is semilinear and alternating on every of $n$ segments w.r.t. $k_1, \dots, k_n$, i.e. with the property that

$$\mu(u_1, \dots, \alpha u_i, \dots, u_k) = \alpha^{\sigma_i} \mu(u_1, \dots, u_i, \dots, u_k)$$

for some automorphism $\sigma_i$ of $D$ and any $\alpha \in D$, and

$$\mu(u_1, \dots, u_{j_1}, \dots, u_{j_2}, \dots, u_k) = -\mu(u_1, \dots, u_{j_2}, \dots, u_{j_1}, \dots, u_k)$$

for all $i = 1, \dots, n$ and $j_1, j_2$ such that $k_1 + \cdots + k_{i-1} < j_1 < j_2 \le k_1 + \cdots + k_i$. We shall say that $\mu$ is *segment-wise* semilinear and alternating. Note that $\sigma_{j_1} = \sigma_{j_2}$ for $j_1, j_2$ within one segment like above. Thus there could be up to $n$ field automorphisms $\sigma_i$ associated with $\mu$.

For $u \in V$ define a map $\mu_i^{[u]}\colon V_i^{k_i} \longrightarrow D$ by setting

$$\mu_i^{[u]}(x^i) := \mu(u^1, \dots, u^{i-1}, x^i, u^{i+1}, \dots, u^n).$$

It is an alternating $k_i$-semilinear form on $V_i$ associated with some field automorphism $\sigma_i$. For every map $\mu_i^{[u]}$ there is an alternating $k_i$-linear form $\eta$ on $V_i$ with its zero-set equal to that of $\mu_i^{[u]}$, i.e. $\mu_i^{[u]}(x_1, \dots, x_{k_i}) = 0$ if and only if $\eta(x_1, \dots, x_{k_i}) = 0$ for all $x_1, \dots, x_{k_i} \in V_i$. A $k$-linear form $\mu'$ such that the zero-sets of $\mu$ and $\mu'$ coincide exists only if $\sigma_1 = \cdots = \sigma_n$. This justifies not taking $\mu$ to be simply $k$-linear.

In case $\sigma_1 = \cdots = \sigma_n = \mathrm{id}$, that is when $\mu$ is $k$-linear, it determines an $n$-linear form $\mu^*$ on $\big(\bigwedge^{k_1} V_1\big) \otimes \cdots \otimes \big(\bigwedge^{k_n} V_n\big)$ in a standard way as follows

$$\mu^*(u_1^1 \wedge \cdots \wedge u_{k_1}^1 \otimes \cdots \otimes u_1^n \wedge \cdots \wedge u_{k_n}^n) := \mu(u^1, \dots, u^n), \tag{5.3}$$

where $u^i \in V_i^{k_i}$ for $i = 1, \dots, n$.

It is known that every hyperplane in the projective space $\mathbf{P}_1(V)$ is of the form $\mathrm{Ker}(\eta)$ for some linear form or a covector $\eta \in V^*$, and indeed for $n = k = 1$ we have $\mu \in V^*$. A standard embedding $\mathbf{s}$ of the product $\bigotimes_{i=1}^n \mathbf{P}_1(V_i)$ into the projective space $\mathbf{P}_1(\bigotimes_{i=1}^n V_i)$ given by

$$\mathbf{s}\colon (\langle w_1\rangle, \dots, \langle w_n\rangle) \longmapsto \langle w_1 \otimes \cdots \otimes w_n\rangle. \tag{5.4}$$

is called a *Segre embedding*. Let us define

$$\mathcal{H}_{k_1,\dots,k_n}(\mu) := \big\{\big(\langle u^1\rangle, \dots, \langle u^n\rangle\big) \in \mathrm{Sub}_{k_1}(V_1) \times \cdots \times \mathrm{Sub}_{k_n}(V_n)\colon \mu(u^1, \dots, u^n) = 0\big\}. \tag{5.5}$$

Since $n, k, k_1, \dots, k_n$ are all fixed we will abbreviate $\mathcal{H}_{k_1,\dots,k_n}(\mu) = \mathcal{H}(\mu)$ as it should cause no confusion. For $k$-linear $\mu$ we have

$$\mathcal{H}(\mu) = \mathbf{s}^{-1}(\mathbf{g}_1^{-1} \times \cdots \times \mathbf{g}_n^{-1})(\mathrm{Ker}(\mu^*)).$$

For all $u \in V$ such that $U := \big(\langle u^1\rangle, \dots, \langle u^n\rangle\big) \in \mathrm{Sub}_{k_1}(V_1) \times \cdots \times \mathrm{Sub}_{k_n}(V_n)$ by (4.1) and (5.5) we have

$$\mathcal{H}\big(\mu_i^{[u]}\big) = \big\{\langle x^i\rangle \in \mathrm{Sub}_{k_i}(V_i)\colon \mu_i^{[u]}(x^i) = 0\big\} = \big\{X \in \mathrm{Sub}_{k_i}(V_i)\colon U[i/X] \in \mathcal{H}(\mu)\big\} = \mathcal{H}(\mu)_i^{[U]}, \tag{5.6}$$

so by Theorem 4.1 the following proposition is evident.

**Proposition 5.2.** *The set $\mathcal{H}(\mu)$ is either a hyperplane in $\mathfrak{M}$ or all of $\mathfrak{M}$.*

We say that $\mu$ is *non-zero on i-th segment* when for all $u \in V$ such that

$(*_i)$ $u^j$ is a linearly independent system in $V_j$, where $1 \le j \le n,\ j \ne i$

there is $x^i \in V_i^{k_i}$ with $\mu_i^{[u]}(x^i) \ne 0$. Note that $u^j$ is linearly independent if and only if $u_1^j \wedge \cdots \wedge u_{k_j}^j \ne 0$ in (5.3). Moreover, the system $u^j$ must be linearly independent to have $\langle u^j\rangle \in \mathrm{Sub}_{k_j}(V_j)$ in (5.5).

**Proposition 5.3.** *If the form $\mu$ is non-zero on at least one of $n$ segments, then $\mathcal{H}(\mu)$ is a hyperplane in $\mathfrak{M}$. If $\mu$ is non-zero on all $n$ segments, then $\mathcal{H}(\mu)$ is a non-degenerate hyperplane in $\mathfrak{M}$.*

Immediately, from Proposition 5.3 we get

**Corollary 5.4.** *There is a (non-degenerate) hyperplane in $\mathfrak{M}$.*

Following [10, Chapter 14.1] the form $\mu$ is *GKZ non-degenerate* if for all $u \in V$ there is $i \in \{1, \dots, k\}$ and $v \in V$ such that $\mu(u_1, \dots, u_{i-1}, v_i, u_{i+1} \dots, u_k) \neq 0$.

*Remark* 5.5. If $k_1 = \dots = k_n = 1$, i.e., if $\mathfrak{M}$ is the Segre product of projective spaces, then the form $\mu$ is GKZ non-degenerate if and only if $\mathcal{H}(\mu)$ is spiky.

According to [10, Chapter 14] the form $\mu$ is GKZ non-degenerate if and only if the hyperdeterminant of the multidimensional matrix associated with $\mu$ is non-zero. This let us interpret non-zero hyperdeterminants as those corresponding to spiky hyperplanes in suitable Segre products.

Two papers [12] and [27] (see also [6,7]) provide an exhaustive characterization of hyperplanes in Grassmann spaces. Let us recall the embeddable case.

**Fact 5.6.** *Let $n = 1$, so $\mathfrak{M}$ is a Grassmann space embeddable into a projective space. Then $\mathcal{H}$ is a hyperplane in $\mathfrak{M}$ if and only if $\mathcal{H} = \mathcal{H}(\mu)$ for some non-zero $k$-linear alternating form $\mu$.*

In view of [5], by Fact 5.6 we get the following result.

**Fact 5.7.** *Let $n = k = 1$, so $\mathfrak{M}$ is a projective space, and let $\xi$ be a bilinear reflexive form on $V$. Then $\mathcal{H}$ is a hyperplane in the polar space $\mathbf{P}_1(\xi)$ if and only if $\mathcal{H} = \mathcal{H}(\mu) \cap Q_1(\xi)$ for some non-zero $\mu \in V^*$ such that $Q_1(\xi) \nsubseteq \mathcal{H}(\mu)$.*

Then, as a natural generalization of Fact 5.7, we obtain a formula for hyperplanes in the Segre product of polar Grassmann spaces.

**Proposition 5.8.** *Let $\xi_i$ be a bilinear reflexive form on $V_i$ for all $i = 1, \dots, n$. Assume that $\mu$ and $\xi_1, \dots, \xi_n$ satisfy the following condition: if $\langle u^j \rangle \in Q_{k_j}(\xi_j)$ for $j \neq i$, then $\mathcal{H}(\mu_i^{[u]}) \cap Q_{k_i}(\xi_i)$ is neither empty nor a single point for all $u^i \in V_i^{k_i}$, $i = 1, \dots, n$. If $\mu$ is non-zero on all $n$ segments and $Q_{k_1}(\xi_1) \times \dots \times Q_{k_n}(\xi_n) \nsubseteq \mathcal{H}(\mu)$, then*

$$\mathcal{H}(\mu, \xi_1, \dots, \xi_n) = \mathcal{H}(\mu) \cap \big(Q_{k_1}(\xi_1) \times \dots \times Q_{k_n}(\xi_n)\big)$$

*is a non-degenerate hyperplane in $\mathbf{P}_{k_1}(\xi_1) \otimes \dots \otimes \mathbf{P}_{k_n}(\xi_n)$.*

*Proof.* Set $\mathcal{H} := \mathcal{H}(\mu, \xi_1, \dots, \xi_n)$ and take

$$U = \big(\langle u^1 \rangle, \dots, \langle u^n \rangle\big) \in Q_{k_1}(\xi_1) \times \dots \times Q_{k_n}(\xi_n).$$

Note that $\mathcal{H}_i^{[U]} = \mathcal{H}(\mu_i^{[u]}) \cap Q_{k_i}(\xi_i)$. By the assumed condition and Lemma 2.5 the set $\mathcal{H}_i^{[U]}$ is a hyperplane in $\mathbf{P}_{k_i}(\xi_i)$ as the intersection of a hyperplane and the point set $Q_{k_i}(\xi_i)$ of the polar space $\mathbf{P}_{k_i}(\xi_i)$ embedded into $\mathbf{P}_{k_i}(V_i)$ for $i = 1, \ldots, n$. Therefore, $\mathcal{H}$ is non-degenerate. $\square$

Some families of non-degenerate hyperplanes were presented so far, but in view of Theorem 4.18 flappy hyperplanes are needed.

We say that $\mu$ is *non-degenerate on $i$-th segment* when for all $u \in V$ satisfying $(*_i)$ any linearly independent system $x^i_1, \ldots, x^i_{k_i-1} \in V_i$ can be completed with $x^i_{k_i} \in V_i$ so that $\mu^{[u]}_i(x^i) \neq 0$. This notion is a strengthening of a corresponding notion for alternating $k$-linear forms in [11]. More precisely, in case $n = 1$, i.e. for Grassmann spaces, if $\mu$ is non-degenerate, then $\mu$ is non-degenerate in the sense of [11], while the inverse is true only for $k \leq 2$. Obviously, if $\mu$ is non-degenerate on $i$-th segment, then it is non-zero on $i$-th segment.

**Lemma 5.9.** *If $\mu$ is non-degenerate on $i$-th segment, then $\mathcal{H}\big(\mu^{[u]}_i\big)$ is a flappy hyperplane in $\mathbf{P}_{k_i}(V_i)$ for all $u \in V$ satisfying $(*_i)$.*

*Proof.* Let us fix $u \in V$ that satisfies $(*_i)$ and let $L = \mathbf{p}(H, B)$ be a line of $\mathbf{P}_{k_i}(V_i)$ contained in $\mathcal{H}\big(\mu^{[u]}_i\big)$. Assume that $H = \langle u_1, \ldots, u_{k_i-1}\rangle$ for some $u_1, \ldots, u_{k_i-1} \in V_i$. Note that $B = H \oplus \langle w_1, w_2\rangle$ for some $w_1, w_2 \in V_i$, $U_j := H \oplus \langle w_j\rangle$ are points on $L$, and $\mu^{[u]}_i(u_1, \ldots, u_{k_i-1}, w_j) = 0$ for $j = 1, 2$. As $\mu^{[u]}_i$ is non-degenerate there is $v \in V_i$ such that $\mu^{[u]}_i(u_1, \ldots, u_{k_i-1}, v) \neq 0$. This means that $u_1, \ldots, u_{k_i-1}, w_1, w_2, v$ are linearly independent, in other words we have a point $U := H \oplus \langle v\rangle$ in $\mathbf{P}_{k_i}(V_i)$ which together with $U_1, U_2$ forms a triangle, i.e. spans a plane, and $U \notin \mathcal{H}\big(\mu^{[u]}_i\big)$. $\square$

**Proposition 5.10.** *If $\mu$ is non-degenerate on all $n$ segments, then $\mathcal{H}(\mu)$ is a flappy hyperplane in $\mathfrak{M}$.*

*Proof.* Note that $\mathcal{H}(\mu)$ is a non-degenerate hyperplane. Let us take $U = \big(\langle u^1\rangle, \ldots, \langle u^n\rangle\big) \in \mathrm{Sub}_{k_1}(V_1) \times \cdots \times \mathrm{Sub}_{k_n}(V_n)$. By Lemma 5.9 $\mathcal{H}(\mu^{[u]}_i) = \mathcal{H}(\mu)^{[U]}_i$ is a flappy hyperplane in $\mathbf{P}_{k_i}(V_i)$ for all $i = 1, \ldots, n$. By Proposition 4.4(i) the hyperplane $\mathcal{H}(\mu)$ is flappy. $\square$

In particular cases, combining Propositions 5.8 and 5.10 yields the formula for flappy hyperplanes in the Segre product of polar spaces.

**Proposition 5.11.** *Assume that $k = n$ i.e. $k_1 = \cdots = k_n = 1$ or $\mathfrak{M}$ is the Segre product of projective spaces. If $\xi_i$ is a non-degenerate symplectic bilinear form on $V_i$ for all $i = 1, \ldots, n$ and $\mu$ is non-zero (or equivalently non-degenerate in this case) on all $n$ segments, then $\mathcal{H}(\mu, \xi_1, \ldots, \xi_n)$ is a flappy hyperplane in the Segre product $\mathbf{P}_1(\xi_1) \otimes \cdots \otimes \mathbf{P}_1(\xi_n)$.*

*Proof.* Set $\mathcal{H} := \mathcal{H}(\mu, \xi_1, \ldots, \xi_n)$, take $U \in Q_1(\xi_1) \times \cdots \times Q_1(\xi_n)$, $i \in I$, and consider the set $\mathcal{H}^{[U]}_i$. Let $L \in Q_2(\xi_i)$ such that $L \subseteq \mathcal{H}^{[U]}_i$. We take any two points $U_1, U_2$ on $L$ and set $A := U_1^{\perp_{\xi_i}} \cap U_2^{\perp_{\xi_i}}$. It is impossible that $A \subseteq \mathcal{H}^{[U]}_i$.

Recall that $Q_1(\xi_i)$ and the point set of $\mathbf{P}_1(V_i)$ coincide, so $\mathcal{H}_i^{[U]}$ is not all of $\mathbf{P}_1(V_i)$. We are through by Corollary 4.5(i). □

When $n = 1$ and $k = 2$ the form $\mu$ turns out to be a bilinear symplectic form. Hence, in view of Lemma 5.9, we can state the following corollary.

**Corollary 5.12.** *If $\mu$ is a non-degenerate symplectic bilinear form on $V$ (i.e., $n = 1$, $k = 2$), then the set $Q_2(\mu)$ of all isotropic 2-subspaces of $V$ w.r.t. $\mu$ is a flappy hyperplane in $\mathfrak{M}_2(V)$.*

Let us consider the Segre product of two projective spaces and its hyperplanes.

**Proposition 5.13.** *Let $V_1, V_2$ be vector spaces over a division ring $D$. Then the following conditions are equivalent:*

(i) *$\mathcal{H}$ is a hyperplane in $\mathfrak{M}_{1,1}(V_1, V_2)$.*

(ii) *There is a sesquilinear form $\xi\colon V_1 \times V_2 \longrightarrow D$ which determines a conjugacy $\perp$ by the condition that $\langle u_1\rangle \perp \langle u_2\rangle$ if and only if $\xi(u_1, u_2) = 0$ for all $u_i \in V_i$ and we have $\mathcal{H} = \{(p, q)\colon p \perp q\}$ (actually $\mathcal{H} = \perp$).*

*Proof.* (i)⇒(ii) The hyperplane $\mathcal{H}$ determines a relation $\perp$ on $V_1 \times V_2$ by the condition $u_1 \perp u_2$ if and only if $u_1$ is null or $u_2$ is null or $\big(\langle u_1\rangle, \langle u_2\rangle\big) \in \mathcal{H}$. In view of [14, Theorem 32.6] there is a required sesquilinear form $\xi$ such that $\perp = \perp_\xi$.

(ii)⇒(i) Straightforward. □

Note that Proposition 5.13 provides examples of hyperplanes corresponding to forms that are essentially segment-wise semilinear.

**Corollary 5.14.** *There are hyperplanes in $\mathfrak{M}_{1,1}(V_1, V_2)$ that do not arise from a Segre embedding.*

*Remark* 5.15. If $k = n > 2$ and $\mu$ is alternating, then $\mathcal{H}(\mu)$ is non-spiky and thus non-flappy. This, together with Proposition 5.13, means that in the Segre product of two projective spaces a hyperplane is flappy if and only if it is given by some non-degenerate 2-semilinear form $\mu$ (i.e. $\mathcal{H}(\mu)$ is non-degenerate in view of Corollary 4.5(i), but it is no longer true if the number of factors is more than two.

Moreover, when $k = n > 2$ and $D$ is algebraically closed there are no forms $\mu$ that are non-zero (or equivalently non-degenerate in this case) on all $n$ segments. So, it follows that there are no non-degenerate hyperplanes that arise from a form (cf. Proposition 5.3), and thus there are no flappy hyperplanes by Lemma 4.7.

The theory of multilinear forms is definitely complex in general. Even for 3-linear forms there is no complete classification (cf. [9]). Therefore, it should not be expected that the classification of hyperplanes in Segre products of many factors is possible at the moment.

Clearly, there are non-flappy hyperplanes in $\mathbf{P}_k(V)$ as well. An interesting example of such hyperplane can be established without the form $\mu$. Let $\mathcal{H}(W)$ be the set of those $k$-subspaces of $V$ that non-trivially intersect some fixed subspace $W$ of codimension $k$ in $V$ (cf. [6]). Note that $\mathcal{H}(W)$ is a hyperplane in $\mathbf{P}_k(V)$ regardless of whether it is embeddable or non-embeddable, while $\mathcal{H}(\mu)$ occurs only in embeddable case.

**Lemma 5.16.** *All hyperplanes of the form $\mathcal{H}(W)$ in $\mathbf{P}_k(V)$ are non-spiky.*

*Proof.* Consider a hyperplane $\mathcal{H} = \mathcal{H}(W)$ in $\mathbf{P}_k(V)$. Take a point $U_1$ on $\mathcal{H}$ such that $\dim(U_1 \cap W) \geq 2$. Any point $U_2$ not on $\mathcal{H}$ is complementary to $W$. Suppose that $U_1 \sim U_2 \notin \mathcal{H}$, i.e. $\dim(U_1 \cap U_2) = k - 1$. Then the subspace $U_1 \cap U_2$ is a hyperplane in $U_1$ and as such non-trivially intersects at least 2-dimensional subspace $U_1 \cap W$ of $U_1$. Hence, there is a non-zero $w \in U_1 \cap U_2 \cap W$, a contradiction as $U_2 \cap W$ should be trivial. □

Immediately from Lemmas 2.1 and 5.16 none of hyperplanes of the form $\mathcal{H}(W)$ is flappy. Nevertheless, hyperplanes of this type are used to assemble hyperplanes in specific Segre products of Grassmann spaces.

**Proposition 5.17.** *Let $V$ be a a finite-dimensional vector space. For integers $k_1, k_2$ such that $1 < k_1 < \dim(V) - 1$ and $k_1 + k_2 = \dim(V)$*

$$\mathcal{H}_{k_1,k_2}(V) := \left\{(U_1, U_2) \in \mathrm{Sub}_{k_1}(V) \times \mathrm{Sub}_{k_2}(V) \colon 0 < \dim(U_1 \cap U_2)\right\}$$

*is a non-degenerate non-spiky hyperplane in $\mathfrak{M}_{k_1,k_2}(V, V)$.*

*Proof.* It suffices to note that for all $U = (U_1, U_2) \in \mathrm{Sub}_{k_1}(V) \times \mathrm{Sub}_{k_2}(V)$ and $i = 1, 2$ the set $(\mathcal{H}_{k_1,k_2}(V))_i^{[U]} = \mathcal{H}(U_{3-i})$ is, by Lemma 5.16, a non-spiky hyperplane in $\mathbf{P}_{k_i}(V)$. Hence $\mathcal{H}_{k_1,k_2}(V)$ is a hyperplane in our product by Theorem 4.1. It is clear that this hyperplane is non-degenerate. So, $\mathcal{H}_{k_1,k_2}(V)$ is non-spiky by Proposition 4.4(ii). □

The example above is interesting in that the complement

$$\mathfrak{M}_{k_1,k_2}(V, V) \setminus \mathcal{H}_{k_1,k_2}(V)$$

is a pretty well known structure of linear complements i.e. the substructure of the respective product defined on the set

$$\left\{(U_1, U_2) \in \mathrm{Sub}_{k_1}(V) \times \mathrm{Sub}_{k_2}(V) \colon V = U_1 \oplus U_2\right\} \tag{5.7}$$

(cf. [16, 23]). Such structures however are investigated with no parallelism involved in the mentioned papers. The set (5.7) is also the geometric interpretation of $(k, n-k)$-involutions in $\mathrm{GL}(V)$, and it was used in the description of automorphisms of $\mathrm{GL}(V)$ (cf. [8, 17]).

5.3. **Affinization of the product of projective spaces vs the product of affine spaces.** According to Fact 5.1 most of affinizations of Segre products of partial linear spaces are covered by affine spaces. Obviously it does not mean that these affinizations are, up to an isomorphism, products of affine spaces in general. Nevertheless, one could suppose that the complement of some hyperplane in the product of projective spaces will be isomorphic to the product of affine spaces, as affinizations of all components of this product are exactly affine spaces. The complement of a degenerate hyperplane given by (4.2) is, loosely speaking, very close to be that kind of (cf. Proposition 4.9), and the only barrier is a parallelism. In view of Proposition 4.4(ii) all non-degenerate and a lot of degenerate hyperplanes in the product of projective spaces are spiky.

Let us stress on that the complements of spiky hyperplanes in products of projective spaces and products of affine spaces are essentially distinct.

**Proposition 5.18.** *The complement of a spiky hyperplane in a product of projective spaces is isomorphic to no product of affine spaces.*

*Proof.* The sufficient reason is that according to Remark 4.3 the complement in question is not an affine partial linear space, while the product of any affine spaces is an affine partial linear space by Fact 2.7. □

The Segre product of affine polar spaces, or affine Grassmann spaces, or affine polar Grassmann spaces seems to be also an affine partial linear space, although it is not straightforward by Fact 2.7 and may require specific reasoning. Thus we believe that the following analogy of Proposition 5.18 is true.

*Conjecture* 5.19. Let $\mathfrak{A}_1$, $\mathfrak{A}_2$, $\mathfrak{A}_3$ be respectively the complement of spiky hyperplane in the product of polar spaces, Grassmann spaces, and polar Grassmann spaces. Then

(i) $\mathfrak{A}_1$ is isomorphic to no product of affine polar spaces,
(ii) $\mathfrak{A}_2$ is isomorphic to no product of affine Grassmann spaces,
(iii) $\mathfrak{A}_3$ is isomorphic to no product of affine polar Grassmann spaces.

We presume that for the product of projective spaces even more can be said.

*Conjecture* 5.20. If $\mathfrak{A}$ is the complement of a spiky hyperplane in a product of projective spaces, then the family of lines of $\mathfrak{A}$ cannot be completed so that the arising structure is a product of affine spaces.

To justify Conjecture 5.20 let us think through the following example.

**Example 5.21.** Let $V$, $W_i$ be vector spaces over a field $D$ for all $i = 1, \dots, n$ and let $\mathcal{H}$ be a spiky hyperplane of $\mathbf{P}_1(V) \otimes \mathbf{P}_1(V)$. Set $\mathfrak{A}_{\text{proj}} := \mathbf{P}_1(V) \otimes \mathbf{P}_1(V) \setminus \mathcal{H}$. Assume that $\mathfrak{A}_{\text{proj}}$ can be completed by adding some new lines to an affine partial linear space which is the product of affine spaces $\mathfrak{A}_{\text{aff}} :=$

$\bigotimes_{i=1}^{n} \mathbf{A}(W_i)$. Then the maximal strong subspaces of $\mathfrak{A}_{\text{proj}}$ and $\mathfrak{A}_{\text{aff}}$ should be isomorphic. Moreover

$$\operatorname{Aut}(\mathfrak{A}_{\text{proj}}) \cong \operatorname{Aut}(\mathfrak{A}_{\text{aff}}).$$

Let us compute the size of the corresponding automorphism groups. All the calculations will be done under the assumption that $D = \operatorname{GF}(p)$ for a prime $p$, $\dim(V) = m'$, and $\dim(W_i) = m$ for all $i = 1, \dots, n$. Let $\sigma$ be a permutation of $\{1, \dots, n\}$ Then any collineation of $\mathfrak{A}_{\text{aff}}$ is a map $f_\sigma = (f_{\sigma 1}, \dots, f_{\sigma n})$, where $f_{\sigma i}$ is an isomorphism that maps $\mathbf{A}(W_i)$ onto $\mathbf{A}(W_{\sigma(i)})$. Thus

$$|\operatorname{Aut}(\mathfrak{A}_{\text{aff}})| = n!\, p^m |\operatorname{GL}(p, m)|, \text{ where}$$
$$|\operatorname{GL}(p, m)| = (p^m - p^0)(p^m - p^1) \dots (p^m - p^{m-1}).$$

By Proposition 5.13 the hyperplane $\mathcal{H}$ is determined by some non-degenerate bilinear form $\xi$. The automorphisms of $\mathfrak{A}_{\text{proj}}$ are exactly the automorphisms of the relation $\perp_\xi$ (cf. [23]) and each of these is uniquely determined by a collineation $f$ of $\mathbf{P}_1(V)$ and a permutation in $S_2$. Thus

$$|\operatorname{Aut}(\mathfrak{A}_{\text{proj}})| = 2\frac{|\operatorname{GL}(p, m')|}{p-1}.$$

The sizes of both respective automorphism groups are polynomials in the variable $p$. Their degrees and leading coefficients are, respectively, $m^2 + m$, $n!$, and $m'^2 - 1$, 2. From the isomorphism assumed we get $n = 2$, $m' = m$ and then $m^2 + m = m^2 - 1$ yields $m = -1$, a contradiction.

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