Bulletin of the Iranian Mathematical Society Vol. 31 *No. 1* (*2005*), *pp 13-23.*

n-HOMOMORPHISMS

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Communicated by Heydar Radjavi

ABSTRACT. Let $\mathcal{A}$ and $\mathcal{B}$ be two (complex) algebras. A linear map $\varphi : \mathcal{A} \to \mathcal{B}$ is called an n-homomorphism if $\varphi(a_1 \dots a_n) = \varphi(a_1) \dots \varphi(a_n)$ for each $a_1, \dots, a_n \in \mathcal{A}$. In this paper, we investigate n-homomorphisms and their relation to homomorphisms. We characterize n-homomorphisms in terms of homomorphisms under certain conditions. Some results related to continuity and commutativity are given as well.

1. Introduction

Let $\mathcal{A}$ and $\mathcal{B}$ be two algebras. A linear mapping $\varphi : \mathcal{A} \to \mathcal{B}$ is called an n-homomorphism if $\varphi(a_1 \dots a_n) = \varphi(a_1) \dots \varphi(a_n)$ for each $a_1, \dots, a_n \in \mathcal{A}$. A 2-homomorphism is then a homomorphism, in the usual sense, between algebras.

MSC(2000): Primary 47B48; Secondary 16N60, 46L05

Keywords: n-homomorphism, Semiprime algebra, Commutator, Commutativity, Continuity, C^*-algebra, Second dual, Partial isometry

Received: 03 June 2005 , Revised: 03 October 2005

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For a homomorphism $\varphi : \mathcal{A} \to \mathcal{B}$ we can see that $\varphi(a_1 \dots a_n) = \varphi(a_1) \dots \varphi(a_n)$ for each $a_1, \dots, a_n \in \mathcal{A}$ and for each n. The converse is not true (see Example 2.1).

In this paper we examine the relationship between notions of n-homomorphism and homomorphism. We investigate n-homomorphisms which preserve commutativity under some conditions and study n-homomorphisms on Banach algebras.

Throughout the paper, all Banach algebras are assumed to be over the complex field $\mathbb{C}$.

2. Relationship Between n-Homomorphisms and Homomorphisms

We begin this section with a typical example:

Example 2.1. Let $\mathcal{A}$ be a unital algebra, a_0 be a central element of $\mathcal{A}$ with $a_0^n = a_0$ for some natural number n (for example an $(n-1)$-root of the unit in $\mathbb{C}$) and let $\theta : \mathcal{A} \to \mathcal{A}$ be a homomorphism. Define $\varphi : \mathcal{A} \to \mathcal{A}$ by $\varphi(a) = a_0\theta(a)$. Then we have

$$\begin{aligned} \varphi(a_1 \dots a_n) &= a_0\theta(a_1 \dots a_n) \\ &= a_0^n\theta(a_1) \dots \theta(a_n) \\ &= a_0\theta(a_1) \dots a_0\theta(a_n) \\ &= \varphi(a_1) \dots \varphi(a_n). \end{aligned}$$

Hence φ is an n-homomorphism. In addition, $a_0 = \varphi(1_{\mathcal{A}})$ whenever θ is onto.

The above example gives us an n-homomorphism as a multiple of a homomorphism. Indeed, if $\mathcal{A}$ has the identity $1_{\mathcal{A}}$ then each n-homomorphism is of this form, where $a_0 = \varphi(1_{\mathcal{A}})$ as the following proposition shows.

Proposition 2.2. *Let $\mathcal{A}$ be a unital algebra with identity $1_{\mathcal{A}}$, $\mathcal{B}$ be an algebra and $\varphi : \mathcal{A} \to \mathcal{B}$ be an n-homomorphism. If $\psi : \mathcal{A} \to \mathcal{B}$ is defined by $\psi(a) = (\varphi(1_{\mathcal{A}}))^{n-2}\varphi(a)$ then ψ is a homomorphism and $\varphi(a) = \varphi(1_{\mathcal{A}})\psi(a)$.*

Proof. We have

$$\varphi(1_{\mathcal{A}}) = \varphi(1_{\mathcal{A}}^n) = (\varphi(1_{\mathcal{A}}))^n,$$

and

$$\begin{aligned}
\psi(ab) &= (\varphi(1_{\mathcal{A}}))^{n-2}\varphi(ab) \\
&= (\varphi(1_{\mathcal{A}}))^{n-2}\varphi(a1_{\mathcal{A}}^{n-2}b) \\
&= (\varphi(1_{\mathcal{A}}))^{n-2}\varphi(a)(\varphi(1_{\mathcal{A}}))^{n-2}\varphi(b) \\
&= \psi(a)\psi(b).
\end{aligned}$$

It follows from $(\varphi(1_{\mathcal{A}}))^{n-1}\varphi(a) = \varphi(1_{\mathcal{A}}^{n-1}a) = \varphi(a)$ that $(\varphi(1_{\mathcal{A}}))^{n-1}$ is an identity for $\varphi(\mathcal{A})$. Thus

$$\begin{aligned}
\varphi(1_{\mathcal{A}})\psi(a) &= \varphi(1_{\mathcal{A}})((\varphi(1_{\mathcal{A}}))^{n-2}\varphi(a)) \\
&= (\varphi(1_{\mathcal{A}}))^{n-1}\varphi(a) \\
&= \varphi(a). \qquad \square
\end{aligned}$$

Whence we characterized all n-homomorphisms on a unital algebra. For a non-unital algebra $\mathcal{A}$ we use the unitization and some other useful constructions. Recall that for an algebra $\mathcal{A}$, the linear space $\mathcal{A}_1 = \mathcal{A} \oplus \mathbb{C} = \{(a, \alpha) | a \in \mathcal{A}, \alpha \in \mathbb{C}\}$ equipped with the multiplication $(a, \alpha)(b, \beta) = (ab + \alpha b + \beta a, \alpha\beta)$, so-called the unitization of $\mathcal{A}$, is a unital algebra with identity $(0, 1)$ containing $\mathcal{A}$ as a two-sided ideal.

Now we shall prove that each n-homomorphism is a multiple of a homomorphism under some conditions.

Definition 2.3. An algebra $\mathcal{A}$ is called a factorizable algebra if for each $a \in \mathcal{A}$ there are $b, c \in \mathcal{A}$ such that $a = bc$.

Theorem 2.4. *Let $\mathcal{A}$ and $\mathcal{B}$ be two factorizable algebras, $lan(\mathcal{B}) = \{b \in \mathcal{B}; b\mathcal{B} = 0\} = \{0\}$ and $\varphi : \mathcal{A} \to \mathcal{B}$ an onto n-homomorphism. Then $\ker\varphi$ is a two-sided ideal of $\mathcal{A}$ and there is a unital algebra $\tilde{\mathcal{B}} \supseteq \mathcal{B}$ and an $x \in \tilde{\mathcal{B}}$ with $x^{n-1} = 1_{\tilde{\mathcal{B}}}$ such that $\psi : \mathcal{A} \to \tilde{\mathcal{B}}$ defined by $\psi(a) = x^{n-2}\varphi(a)$ is a homomorphism.*

Proof. Suppose that $a \in \ker\varphi$ and $u \in \mathcal{A}$. Since $\mathcal{A}$ is a factorizable algebra there are $u_1, \ldots u_{n-1} \in \mathcal{A}$ such that $u = u_1 \ldots u_{n-1}$. Hence

$$\varphi(au) = \varphi(au_1 \ldots u_{n-1}) = \varphi(a)\varphi(u_1) \ldots \varphi(u_{n-1}) = 0.$$

Therefore $au \in \ker\varphi$. Similarly $ua \in \ker\varphi$.

Let $\tilde{\mathcal{B}} = \{b_\circ + \beta_1 x + \ldots + \beta_{n-2}x^{n-2}; b_\circ \in \mathcal{B}_1, \text{ and } \beta_1, \ldots, \beta_{n-2} \in \mathbb{C}\}$ as a subset of the algebra $\mathcal{B}_1[x]$ of all polynomials in x with coefficients in the unitization $\mathcal{B}_1$ of $\mathcal{B}$. Using the ordinary multiplication of polynomials, we define a multiplication on $\tilde{\mathcal{B}}$ by $x^{n-1} = 1$ and $bx = \varphi(a_1)\varphi(a_2)$ where $b = \varphi(a) = \varphi(a_1a_2)$ and $a = a_1a_2 \in \mathcal{A}$. We show that the multiplication is well-defined.

Let $b = d \in \mathcal{B}$ and $b = \varphi(a) = \varphi(a_1a_2), d = \varphi(c) = \varphi(c_1c_2)$ with $a = a_1a_2, c = c_1c_2 \in \mathcal{A}$. Then we have $\varphi(a_1a_2) = \varphi(c_1c_2)$. So $\varphi(a_1a_2)b_2 \ldots b_n = \varphi(c_1c_2)b_2 \ldots b_n$ for all $b_2 \ldots b_n \in \mathcal{B}$. Since φ is onto, there exist $u_2 \ldots u_n \in \mathcal{A}$ such that $\varphi(u_i) = b_i$. We can then write

$$\begin{aligned}
& \varphi(a_1)\varphi(a_2)\varphi(u_2)\dots\varphi(u_{n-2})\varphi(u_{n-1}u_n) \\
= \; & \varphi(a_1a_2u_2\dots u_{n-1}u_n) \\
= \; & \varphi(a_1a_2)\varphi(u_2)\dots\varphi(u_{n-1})\varphi(u_n) \\
= \; & \varphi(c_1c_2)\varphi(u_2)\dots\varphi(u_{n-1})\varphi(u_n) \\
= \; & \varphi(c_1c_2u_2\dots u_{n-1}u_n) \\
= \; & \varphi(c_1)\varphi(c_2)\varphi(u_2)\dots\varphi(u_{n-2})\varphi(u_{n-1}u_n).
\end{aligned}$$

This implies that $\varphi(a_1)\varphi(a_2)b = \varphi(c_1)\varphi(c_2)b$ for each $b \in \mathcal{B}$, since $\mathcal{B}$ is a factorizable algebra. Hence $(\varphi(a_1)\varphi(a_2) - \varphi(c_1)\varphi(c_2))\mathcal{B} = 0$. Since $lan(\mathcal{B}) = \{0\}$, we conclude that $\varphi(a_1)\varphi(a_2) = \varphi(c_1)\varphi(c_2)$. In particular, $\varphi(a)\varphi(b)x^{n-2} = \varphi(ab)$ for all $a, b \in \mathcal{A}$. Note that associativity of our multiplication is inherited from that of multiplication of polynomials.

We can inductively prove that $\varphi(a_1)\dots\varphi(a_m)x^{n-m} = \varphi(a_1\dots a_m)$ for all $m \geq 2$. To show this, suppose that it holds for $m \geq 2$ and $a_{m+1} \in \mathcal{A}$. Then

$$\begin{aligned}
& \varphi(a_1)\dots\varphi(a_{m-1})\varphi(a_m)\varphi(a_{m+1})x^{n-m-1} \\
= \; & \varphi(a_1)\dots\varphi(a_{m-1})\varphi(a_m)\varphi(a_{m+1})x^{n-(m+1)}x^{n-1} \\
= \; & \varphi(a_1)\dots\varphi(a_{m-1})(\varphi(a_m)\varphi(a_{m+1})x^{n-2})x^{n-m} \\
= \; & \varphi(a_1)\dots\varphi(a_{m-1})\varphi(a_ma_{m+1})x^{n-m} \\
= \; & \varphi(a_1\dots a_{m-1}a_ma_{m+1}).
\end{aligned}$$

Now define $\tilde{\varphi} : \mathcal{A}_1 \to \tilde{\mathcal{B}}$ by $\tilde{\varphi}(a,\alpha) = \varphi(a) + \alpha x$ for each $(a,\alpha) \in \mathcal{A}_1$. Then for each $(a_1,\alpha_1),\ldots,(a_n,\alpha_n) \in \mathcal{A}_1$ we have

$$\tilde{\varphi}(\prod_{i=1}^{n}(a_i,\alpha_i)) = \tilde{\varphi}(\sum \alpha_{j_1}\ldots\alpha_{j_k}a_{i_1}\ldots a_{i_l}),$$

where the summation is taken over all $i_1,\ldots,i_l,j_1,\ldots,j_k$ with $i_1 < \ldots < i_l, j_1 < \ldots < j_k, 0 \leq k,l \leq n, \{i_1,\ldots,i_l\} \cap \{j_1,\ldots,j_k\} = \emptyset$ and $\{i_1,\ldots,i_l\} \cup \{j_1,\ldots,j_k\} = \{1,\ldots,n\}$. Thus if $\varphi()$ denotes $1 \in \mathbb{C}$ then we can write

$$\begin{aligned}\tilde{\varphi}(\prod_{i=1}^{n}(a_i,\alpha_i)) &= \sum \alpha_{j_1}\ldots\alpha_{j_k}\varphi(a_{i_1}\ldots a_{i_l})\\ &= \sum \alpha_{j_1}\ldots\alpha_{j_k}\varphi(a_{i_1})\ldots\varphi(a_{i_l})x^k\\ &= \prod_{i=1}^{n}(\varphi(a_i)+\alpha_i x) = \prod_{i=1}^{n}\tilde{\varphi}(a_i,\alpha_i).\end{aligned}$$

This shows that $\tilde{\varphi}$ is an n-homomorphism on $\mathcal{A}_1$. Now Proposition 2.3 implies that $\tilde{\psi} : \mathcal{A}_1 \to \tilde{\mathcal{B}}$ defined by $\tilde{\psi}(a,\alpha) = (\tilde{\varphi}(1_{\mathcal{A}_1}))^{n-2}\tilde{\varphi}(a,\alpha) = (\tilde{\varphi}(0,1))^{n-2}(\varphi(a)+\alpha x) = x^{n-2}(\varphi(a)+\alpha x)$ is a homomorphism on $\mathcal{A}_1$. Thus $\psi : \mathcal{A} \to \tilde{\mathcal{B}}$ defined by $\psi(a) = x^{n-2}\varphi(a)$ is a homomorphism on $\mathcal{A}$. □

Example 2.5. In general, the kernel of an n-homomorphism may not be an ideal. As an example, take the algebra $\mathcal{A}$ of all 3×3 matrices having 0 on and below the diagonal. In this algebra product of any 3 elements is equal to 0, so any linear map from $\mathcal{A}$ into itself is a 3-homomorphism but its kernel does not need to be an ideal.

3. **Commutativity**

Recall that an algebra $\mathcal{A}$ is called semiprime if $a\mathcal{A}a = \{0\}$ implies that $a = 0$ for each $a \in \mathcal{A}$.

Lemma 3.1. *If $\mathcal{A}$ is a semiprime algebra with center $\mathcal{Z}$, and $a \in \mathcal{A}$ is such that $[a, \mathcal{A}] \subseteq \mathcal{Z}$, then $a \in \mathcal{Z}$.*

Proof. For any $x \in \mathcal{A}$ we have $a[a, x] = [a, ax] \in \mathcal{Z}$ and $[a, x] \in \mathcal{Z}$, and hence $[a, x]^2 = [a[a, x], x] = 0$. Since the center of a semiprime ring cannot contain nonzero nilpotents, it follows that $[a, x] = 0$, and so $a \in \mathcal{Z}$. □

Theorem 3.2. *Suppose that $\mathcal{A}$ and $\mathcal{B}$ are two algebras, $\mathcal{B}$ is semiprime and $\varphi : \mathcal{A} \to \mathcal{B}$ is a surjective n-homomorphism. If $\mathcal{A}$ is commutative, then so is $\mathcal{B}$.*

Proof. Let a be an arbitrary element of the commutative algebra $\mathcal{A}$. Then

$$\underbrace{[\cdots[}_{n-2}[a, c_1], c_2], \cdots], c_{n-1}] = \underbrace{[\cdots[}_{n-3}[0, c_2], \cdots], c_{n-1}] = 0$$

for all $c_1, \cdots, c_{n-1} \in \mathcal{A}$. Since φ is n-homomorphism, we get

$$\underbrace{[\cdots[}_{n-2}[\varphi(a), \varphi(c_1)], \varphi(c_2)], \cdots], \varphi(c_{n-1})] = 0 \in \mathcal{Z}_{\mathcal{B}}$$

for all $c_1, \cdots, c_{n-1} \in \mathcal{A}$, where $\mathcal{Z}_{\mathcal{B}}$ denotes the center of $\mathcal{B}$. Repeatedly applying Lemma 3.1 and applying the surjectivity of φ we conclude that $\varphi(a) \in \mathcal{Z}_{\mathcal{B}}$. Hence $B = \mathcal{Z}_{\mathcal{B}}$ is commutative. □

4. n-Homomorphisms on Banach Algebras

Recall that the second dual $\mathcal{A}^{**}$ of a Banach algebra $\mathcal{A}$ equipped with the first Arens product is a Banach algebra. The first Arens product is indeed characterized as the unique extension to $\mathcal{A}^{**}\times\mathcal{A}^{**}$ of the mapping $(a,b)\mapsto ab$ from $\mathcal{A}\times\mathcal{A}$ into $\mathcal{A}$ with the following properties :
(i) for each $G\in\mathcal{A}^{**}$, the mapping $F\mapsto FG$ is weak*-continuous on $\mathcal{A}^{**}$;
(ii) for each $a\in\mathcal{A}$, the mapping $G\mapsto aG$ is weak*-continuous on $\mathcal{A}^{**}$.
The second Arens product can be defined in a similar way. If the first and the second Arens products coincide on $\mathcal{A}^{**}$, then $\mathcal{A}$ is called regular.
We identify $\mathcal{A}$ with its image under the canonical embedding $i:\mathcal{A}\longrightarrow\mathcal{A}^{**}$.

Theorem 4.1. *Suppose that $\mathcal{A}$ and $\mathcal{B}$ are two Banach algebras and $\varphi:\mathcal{A}\to\mathcal{B}$ is a continuous n-homomorphism. Then the second adjoint $\varphi^{**}:\mathcal{A}^{**}\to\mathcal{B}^{**}$ of φ is also an n-homomorphism.*
If, in addition, $\mathcal{A}$ is Arens regular and has a bounded approximate identity, then ϕ is a certain multiple of a homomorphism.

Proof. Let $F_1,\ldots,F_n\in\mathcal{A}^{**}$. By Goldstine's theorem (cf. [3]), there are nets $(a_i^1),\ldots,(a_j^n)$ in $\mathcal{A}$ such that

$$\text{weak}^*\!-\lim_i a_i^1=F_1,\ldots,\text{weak}^*\!-\lim_j a_j^n=F_n.$$

Since φ^{**} is weak$^*-$continuous, we have

$$\begin{aligned}
\varphi^{**}(F_1 \dots F_n) &= \varphi^{**}(\text{weak}^*-\lim_i \dots \text{weak}^*-\lim_j a_i^1 \dots a_j^n) \\
&= \text{weak}^*-\lim_i \dots \text{weak}^*-\lim_j \varphi^{**}(a_i^1 \dots a_j^n) \\
&= \text{weak}^*-\lim_i \dots \text{weak}^*-\lim_j \varphi^(a_i^1 \dots a_j^n) \\
&= \text{weak}^*-\lim_i \dots \text{weak}^*-\lim_j(\varphi(a_i^1) \dots \varphi(a_j^n)) \\
&= \text{weak}^*-\lim_i \varphi(a_i^1) \dots \text{weak}^*-\lim_j \varphi(a_j^n) \\
&= \text{weak}^*-\lim_i \varphi^{**}(a_i^1) \dots \text{weak}^*-\lim_j \varphi^{**}(a_j^n) \\
&= \varphi^{**}(F_1) \dots \varphi^{**}(F_n).
\end{aligned}$$

If $\mathcal{A}$ is Arens regular and has a bounded approximate identity , it follows from and Proposition 28.7 of [1] that $\mathcal{A}^{**}$ has an identity. By proposition 2.3, there exists a homomorphism $\psi : \mathcal{A}^{**} \to \mathcal{B}^{**}$ such that $\varphi(a) = \varphi^{**}|_{\mathcal{A}}(a) = \varphi^{**}(1_{\mathcal{A}^{**}})\psi(a)$ for all $a \in \mathcal{A}$. □

Remark 4.2. A computational proof similar to that of Theorem 6.1 of [2] may be used in extending n-homomorphisms to the second duals.

Now suppose that φ is a non-zero 3-homomorphism from a unital algebra $\mathcal{A}$ to $\mathbb{C}$. Then $\varphi(1) = 1$ or -1. Hence either φ or $-\varphi$ is a character on $\mathcal{A}$. If $\mathcal{A}$ is a Banach algebra, then φ is automatically continuous; cf. Theorem 16.3 of [1]. It may however happen that a 3-homomorphism is not continuous.

Example 4.3. Let $\mathcal{A}$ be the algebra of all 3 by 3 matrices having 0 on and below the diagonal and $\mathcal{B}$ be the algebra of all $\mathcal{A}$-valued continuous functions from $[0, 1]$ into $\mathcal{A}$ with sup norm. Then $\mathcal{B}$ is an infinite dimensional Banach algebra $\mathcal{B}$ in which the product of any three elements is 0. Since $\mathcal{B}$ is infinite dimensional, there are linear discontinuous maps (as discontinuous 3-homomorphisms) from $\mathcal{B}$ into itself.

Theorem 4.4. *Let $\mathcal{A}$ be a W^*-algebra and $\mathcal{B}$ a C^*-algebra. If $\varphi : \mathcal{A} \to \mathcal{B}$ is a weakly-norm continuous 3-homomorphism preserving the involution, then $\|\varphi\| \leq 1$.*

Proof. The closed unit ball of $\mathcal{A}$ is compact in weak topology. By the Krein-Milman theorem this convex set is the closed convex hull of its extreme points. On the other hand, the extreme points of the closed unit ball of $\mathcal{A}$ are the partial isometries x such that $(1 - xx^*)\mathcal{A}(1 - x^*x) = \{0\}$, cf. Problem 107 of [4], and Theorem I.10.2 of [5]. Since $\varphi(xx^*x) = \varphi(x)\varphi(x)^*\varphi(x)$, the mapping φ preserves the partial isometries. Since every partial isometry x has norm $\|x\| \leq 1$, we conclude that

$$\|\varphi(\sum_{i=1}^{n}\lambda_i x_i)\| = \|\sum_{i=1}^{n}\lambda_i\varphi(x_i)\| \leq \sum_{i=1}^{n}\lambda_i\|\varphi(x_i)\| \leq 1,$$

where $x_1, \ldots, x_n$ are partial isometries, $\lambda_1 \ldots \lambda_n > 0$ and $\sum_{i=1}^{n}\lambda_i = 1$. It follows from weak continuity of φ that $\|\varphi\| \leq 1$. □

Question. Is every $*$-preserving n-homomorphism between C^*-algebras continuous?

Acknowledgment

The authors would like to thank Professor Krzysztof Jarosz for providing Examples 2.5 and 4.3. They also thank the referee for useful suggestions.

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