

DYNAMICAL SYSTEMS ON HILBERT C^* -MODULES

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ABSTRACT. We investigate the generalized derivations and show that every generalized derivation on a simple Hilbert C^* -module is either closable or has a dense range. We also describe dynamical systems on a full Hilbert C^* -module $\mathcal{M}$ over a C^* -algebra $\mathcal{A}$ as a one-parameter group of unitaries on $\mathcal{M}$ and prove that if $\alpha : \mathbb{R} \rightarrow U(\mathcal{M})$ is a dynamical system, where $U(\mathcal{M})$ denotes the set of all unitary operator on $\mathcal{M}$, then we can correspond a C^* -dynamical system α' on $\mathcal{A}$ such that if δ and d are the infinitesimal generators of α and α' respectively, then δ is a d -derivation.

1. Introduction

A Hilbert C^* -module over a C^* -algebra $\mathcal{A}$ is an algebraic left $\mathcal{A}$ -module $\mathcal{M}$ equipped with an $\mathcal{A}$ -valued inner product $\langle \cdot, \cdot \rangle$ which

MSC(2000): Primary 47D03; Secondary 46L08, 46L05

Keywords: Hilbert C^* -module, Semigroup of operators, Dynamical system, Generalized derivation

Received: 24 September 2005 , Revised: 17 October 2005

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is $\mathcal{A}$ -linear in the first and conjugate linear in the second variable such that $\mathcal{M}$ is a Banach space with respect to the norm $\|x\| = \|\langle x, x \rangle\|^{1/2}$. The Hilbert module $\mathcal{M}$ is called full if the closed linear span $\langle \mathcal{M}, \mathcal{M} \rangle$ of all elements of the form $\langle x, y \rangle$ ($x, y \in \mathcal{M}$) is equal to $\mathcal{A}$.

Hilbert C^* -modules are first introduced and investigated by I. Kaplansky [5], M. Rieffel [13] and W. Paschke [12]. These are a generalization of Hilbert spaces, but there are some differences between these two classes. For example, each operator on a Hilbert space has an adjoint, but a bounded $\mathcal{A}$ -module map on a Hilbert $\mathcal{A}$ -module is not adjointable in general [6].

In this paper, we investigate the generalized derivations. This notion was first appeared in the context of operator algebras [7]. Later, it was introduced in the framework of pure algebra [4]. We shall show that every generalized derivation δ on a simple Hilbert C^* -module $\mathcal{M}$ is either closable or has a dense range in $\mathcal{M}$. This is a generalization of a result of A. Niknam [10].

We also describe dynamical systems on a full Hilbert C^* -module $\mathcal{M}$ over a C^* -algebra $\mathcal{A}$ as a one-parameter group of unitaries on $\mathcal{M}$ and prove that if $\alpha : \mathbb{R} \rightarrow U(\mathcal{M})$ is a dynamical system, then we can correspond a C^* -dynamical system α' on $\mathcal{A}$ such that if δ and d are the infinitesimal generators of α and α' respectively, then δ is a d -generalized derivation.

The reader is referred to [6] for more details on Hilbert C^* -modules and to [14] for more information on C^* -dynamical systems.

2. Preliminaries.

Throughout this section $\mathcal{M}$ and $\mathcal{N}$ are assumed to be Hilbert modules over C^* -algebras $\mathcal{A}$ and $\mathcal{B}$ respectively, and $\varphi : \mathcal{A} \rightarrow \mathcal{B}$ is a morphism of C^* -algebras.

A map $\Phi : \mathcal{M} \rightarrow \mathcal{N}$ is said to be a φ -morphism of Hilbert C^* -modules if

$$\langle \Phi(x), \Phi(y) \rangle = \varphi(\langle x, y \rangle) \quad (x, y \in \mathcal{M})$$

Applying the polarization, we can immediately conclude that Φ is a φ -morphism if and only if $\langle \Phi(x), \Phi(x) \rangle = \varphi(\langle x, x \rangle)$ for all $x \in \mathcal{M}$. Each φ -morphism is necessarily a linear operator since for any $x, y, z \in \mathcal{M}$ and $\alpha \in \mathbb{C}$ we have

$$\begin{aligned} \langle \Phi(\alpha x + y), \Phi(z) \rangle &= \alpha \varphi(\langle x, z \rangle) + \varphi(\langle y, z \rangle) \\ &= \alpha \langle \Phi(x), \Phi(z) \rangle + \langle \Phi(y), \Phi(z) \rangle, \end{aligned}$$

and hence $\langle \Phi(\alpha x + y) - \alpha \Phi(x) - \Phi(y), \Phi(z) \rangle = 0$ for all $z \in \mathcal{M}$. If we replace z with x, y and $\alpha x + y$ then we can infer that

$$\langle \Phi(\alpha x + y) - \alpha \Phi(x) - \Phi(y), \Phi(\alpha x + y) - \alpha \Phi(x) - \Phi(y) \rangle = 0.$$

Thus $\Phi(\alpha x + y) - \alpha \Phi(x) - \Phi(y) = 0$ and therefore $\Phi(\alpha x + y) = \alpha \Phi(x) + \Phi(y)$. Similarly every φ -morphism is necessarily a module map in the sense that $\Phi(ax) = \varphi(a)\Phi(x)$ for all $x \in \mathcal{M}$ and for all $a \in \mathcal{A}$.

Further, let $\psi : \mathcal{B} \rightarrow C$ be a morphism of C^* -algebras and let $\mathcal{Q}$ be a Hilbert C^* -module over C . If $\Phi : \mathcal{M} \rightarrow \mathcal{N}$ is a φ -morphism and $\Psi : \mathcal{N} \rightarrow \mathcal{Q}$ is a ψ -morphism, then obviously $\Psi\Phi : \mathcal{M} \rightarrow \mathcal{Q}$ is a $\psi\varphi$ -morphism of Hilbert C^* -modules.

Following [2] we call a map $\Phi : \mathcal{M} \rightarrow \mathcal{N}$ a unitary operator if there exists an injective morphism of C^* -algebras $\varphi : \mathcal{A} \rightarrow \mathcal{B}$ such that Φ is a surjection φ -morphism.

Remark 2.1.

- (i) If $\Phi : \mathcal{M} \rightarrow \mathcal{N}$ be a φ -morphism of Hilbert C^* -modules and φ is injective then Φ is an isometry; cf. [2]. Thus each unitary operator of Hilbert C^* -modules is necessarily an isometry.
- (ii) It is known [2] that if $\Phi : \mathcal{M} \rightarrow \mathcal{N}$ is a φ -morphism of Hilbert C^* -modules then $\text{Im}\Phi$ is a closed subspace of $\mathcal{N}$ and a Hilbert C^* -module over the C^* -algebra $\text{Im}\varphi \subset \mathcal{B}$ such that $\langle \text{Im}\Phi, \text{Im}\Phi \rangle = \varphi(\langle \mathcal{M}, \mathcal{M} \rangle)$. Moreover, if Φ is surjective, and if $\mathcal{N}$ is a full $\mathcal{B}$ -module then φ is also surjective. Thus if Φ is a unitary then it is surjective and hence $\langle \mathcal{N}, \mathcal{N} \rangle = \varphi(\langle \mathcal{M}, \mathcal{M} \rangle) \simeq \langle \mathcal{M}, \mathcal{M} \rangle$. Moreover, if $\mathcal{N}$ is a full Hilbert $\mathcal{B}$ -module then φ is surjective and so it is an isomorphism of C^* -algebras.

Example 2.2. Let $\mathcal{H}$ be a Hilbert space. Then $\mathcal{H}$ can be regarded as a $K(\mathcal{H})$ -module via $T \cdot x = T(x)$ ($T \in K(\mathcal{H}), x \in \mathcal{H}$). If we define a $K(\mathcal{H})$ -inner product on $\mathcal{H}$ via $\langle x, y \rangle = x \otimes y$, $x, y \in \mathcal{H}$, where $x \otimes y(z) = (z, y)x$, $z \in \mathcal{H}$ and $(\cdot, \cdot)$ denotes the complex inner product on $\mathcal{H}$, then $\mathcal{H}$ can be regarded as a Hilbert C^* -module over $K(\mathcal{H})$. In this case, $U : \mathcal{H} \rightarrow \mathcal{H}$ is a unitary as an operator on Hilbert space $\mathcal{H}$ if and only if U is a unitary operator on Hilbert $K(\mathcal{H})$ -module $\mathcal{H}$ in the above sense. This is a consequence of the facts that $\langle U(x), U(y) \rangle = U(x) \otimes U(y) = U(x \otimes y)U^*$, and that $\text{Ad}U : K(\mathcal{H}) \rightarrow K(\mathcal{H})$ defined by $\text{Ad}U(V) = UVU^*$, $V \in K(\mathcal{H})$ is a $*$ -isomorphism and each $*$ -isomorphism on $K(\mathcal{H})$ is of this form; cf. [9].

We denote by $U(\mathcal{M})$ the group of all unitary operators of $\mathcal{M}$ onto $\mathcal{M}$. If $\mathcal{M}$ is full and $\alpha : \mathcal{M} \rightarrow \mathcal{M}$ is a unitary operator then by Remark 2.1(ii) there is a $*$ -isomorphism $\alpha' : \mathcal{A} \rightarrow \mathcal{A}$ such that α is an α' -morphism.

We end this section with the following useful lemma which can be found in [1] and [8].

Lemma 2.3. *Let $\mathcal{M}$ be a full Hilbert module over the C^* -algebra $\mathcal{A}$ and let $a \in \mathcal{A}$. Then $a = 0$ if and only if $ax = 0$ for all $x \in \mathcal{M}$.*

3. Generalized Derivation

This section is devoted to study of generalized derivations. Our aim is to show that every generalized derivation δ on a simple Hilbert C^* -module is either closable or has a dense range.

Definition 3.1. Let $\mathcal{M}$ be a full Hilbert $\mathcal{A}$ -module. A linear map $\delta : D(\delta) \subseteq \mathcal{M} \rightarrow \mathcal{M}$, where $D(\delta)$ is a dense subspace of $\mathcal{M}$, is called a *generalized derivation* if there exists a mapping $d : D(d) \rightarrow \mathcal{A}$, where $D(d)$ is a dense subalgebra of $\mathcal{A}$ such that $D(\delta)$ is an algebraic left $D(d)$ -module, and $\delta(ax) = a\delta(x) + d(a)x$ for all $x \in D(\delta)$ and all $a \in D(d)$.

In this case d must be a derivation since for any $a, b \in D(d)$ and $x \in D(\delta)$ we have

$$\delta(abx) = ab\delta(x) + d(ab)x.$$

On the other hand,

$$\delta(abx) = \delta(a(bx)) = a\delta(bx) + d(a)bx = ab\delta(x) + ad(b)x + d(a)bx,$$

whence

$$(d(ab) - (ad(b) + d(a)b))x = 0$$

for all $x \in D(\delta)$. Thus by Lemma 2.3 we obtain $d(ab) = ad(b) + d(a)b$ since $D(\delta)$ is dense in $\mathcal{M}$.

Similarly we can show that d is linear, so $d : D(d) \subseteq \mathcal{A} \rightarrow \mathcal{A}$ is a derivation. We call δ a *d-derivation*.

Denote by $GDer(\mathcal{M})$ the set of all generalized derivations on $\mathcal{M}$. Then it is easy to see that $GDer(\mathcal{M})$ is a linear space. In fact if $\delta_1, \delta_2 \in GDer(\mathcal{M})$, δ_1 is a d_1 -derivation and δ_2 is a d_2 -derivation, then $\alpha\delta_1 + \beta\delta_2$ is a $\alpha d_1 + \beta d_2$ -derivation and so $\alpha\delta_1 + \beta\delta_2 \in GDer(\mathcal{M})$. Also the Lie product $[\delta_1, \delta_2]$ is $[d_1, d_2]$ -derivation and so $[\delta_1, \delta_2] \in GDer(\mathcal{M})$.

Now we show a similar result as in [10] for generalized derivations:

Theorem 3.2. *Let $\mathcal{M}$ be a simple full Hilbert C^* -module in the sense that it has no trivial left $\mathcal{A}$ -submodule and let $\delta : D(\delta) \subseteq \mathcal{M} \rightarrow \mathcal{M}$ be a d -derivation. Then either δ is closable or the range of δ is dense in $\mathcal{M}$.*

Proof. Let $\sigma(\delta)$ be the separating space of δ , that is,

$$\sigma(\delta) = \{x \in \mathcal{M}; \text{there is a sequence } (x_n) \text{ in } D(\delta) \text{ with } x_n \rightarrow 0, \\ \delta(x_n) \rightarrow x\}.$$

It is obvious that $\sigma(\delta)$ is a closed subspace of $\mathcal{M}$. We show that $\sigma(\delta)$ is a left submodule of $\mathcal{M}$. Let $a \in \mathcal{A}, x \in \sigma(\delta)$. Thus there exists a sequence $(x_n) \subseteq D(\delta)$ such that $x_n \rightarrow 0$ and $\delta(x_n) \rightarrow x$, so we have $ax_n \rightarrow 0$ and $\delta(ax_n) = a\delta(x_n) + d(a)x_n \rightarrow ax$.

Hence $ax \in \sigma(\delta)$. By the hypothesis $\sigma(\delta) = \{0\}$ or $\sigma(\delta) = \mathcal{M}$. Therefore, δ is closable or range δ is dense in $\mathcal{M}$. $\square$

4. Dynamical Systems On Full Hilbert C^* -Modules

We start this section with a basic definition.

Definition 4.1. Let $\mathcal{M}$ be a full Hilbert $\mathcal{A}$ -module. A map α from the real line $\mathbb{R}$ to $U(\mathcal{M})$ which maps t to α_t is said to be a *one-parameter group of unitaries* if

- (i) $\alpha_0 = I$,
- (ii) $\alpha_{t+s} = \alpha_t \alpha_s$ ($t, s \in \mathbb{R}$).

Further, α is said to be a *strongly continuous one-parameter group of unitaries* if, in addition, $\lim_{t \rightarrow 0} \alpha_t(x) = x$ in the norm of $\mathcal{M}$ for all $x \in \mathcal{M}$. In this case we call α a *dynamical system* on $\mathcal{M}$.

We can define the infinitesimal generator of a dynamical system as follows:

Definition 4.2. Let $\alpha : \mathbb{R} \rightarrow U(\mathcal{H})$ be a dynamical system on $\mathcal{M}$, we define the *infinitesimal generator* δ of α as a mapping $\delta : D(\delta) \subseteq \mathcal{M} \rightarrow \mathcal{M}$, where

$$D(\delta) = \{x \in \mathcal{M} : \lim_{t \rightarrow 0} \frac{\alpha_t(x) - x}{t} \text{ exists}\},$$

and

$$\delta(x) = \lim_{t \rightarrow 0} \frac{\alpha_t(x) - x}{t}, \quad x \in D(\delta).$$

Now we are ready to prove the main theorem of this paper.

Theorem 4.3. Let $\mathcal{M}$ be a full Hilbert $\mathcal{A}$ -module, α be a dynamical system on $\mathcal{M}$ and δ be the infinitesimal generator of α . Then $D(\delta)$ is a dense subspace of $\mathcal{M}$ and there exists a derivation $d : D(d) \subseteq$

$\mathcal{A} \rightarrow \mathcal{A}$ such that $D(\delta)$ is a left $D(d)$ -module and $\delta(ax) = a\delta(x) + d(a)x$, $a \in D(d), x \in D(\delta)$.

Proof. By Hille-Yosida theorem [3], $D(\delta)$ is a dense subspace of $\mathcal{M}$. Since α is a dynamical system on $\mathcal{M}$, for each $t \in \mathbb{R}$, the mapping $\alpha_t : \mathcal{M} \rightarrow \mathcal{M}$ is a unitary. So there exists $*$ -isomorphism $\alpha'_t : \mathcal{A} \rightarrow \mathcal{A}$ such that $\langle \alpha_t(x), \alpha_t(y) \rangle = \alpha'_t(\langle x, y \rangle)$, and hence $\alpha_t(ax) = \alpha'_t(a)\alpha_t(x)$ ($a \in \mathcal{A}, x \in \mathcal{M}$). Now we show that $\alpha' : \mathbb{R} \rightarrow \text{Aut}(\mathcal{A})$ is a C^* -dynamical system. For each $a \in \mathcal{A}, x \in \mathcal{M}$ we have $ax = \alpha_0(ax) = \alpha'_0(a)\alpha_0(x) = \alpha'_0(a)x$, thus by Lemma 2.3 $\alpha'_0(a) = a$ for all $a \in \mathcal{A}$. Therefore $\alpha'_0 = I$. Also for all $t, s \in \mathbb{R}$ we have

$$\begin{aligned} \alpha'_{t+s}(a)\alpha_{t+s}(x) &= \alpha_{t+s}(ax) \\ &= \alpha_t(\alpha_s(ax)) \\ &= \alpha_t(\alpha'_s(a)\alpha_s(x)) \\ &= \alpha'_t(\alpha'_s(a))\alpha_t(\alpha_s(x)) \\ &= \alpha'_t(\alpha'_s(a))\alpha_{t+s}(x), \end{aligned}$$

and so $\alpha'_{t+s}(a) = \alpha'_t(\alpha'_s(a))$. Thus $\alpha'_{t+s} = \alpha'_t\alpha'_s$. Since for each $x \in \mathcal{M}$, $\alpha_t(x) \rightarrow x$ as $t \rightarrow 0$, we have

$$\|\alpha'_t(a)x - ax\| \leq \|\alpha'_t(a)x - \alpha'_t(a)\alpha_t(x)\| + \|\alpha'_t(a)\alpha_t(x) - ax\|.$$

Thus $\lim_{t \rightarrow 0} \alpha'_t(a)x = ax$ for all $x \in \mathcal{M}$, whence $\lim_{t \rightarrow 0} \alpha'_t(a) = a$ for all $a \in \mathcal{A}$. Therefore $\alpha' : \mathbb{R} \rightarrow \text{Aut}(\mathcal{A})$ is a C^* -dynamical system on $\mathcal{A}$. If d is the infinitesimal generator of α' then for each $a \in D(d), x \in D(\delta)$ we have

$$\begin{aligned}
\lim_{t \rightarrow 0} \frac{\alpha_t(ax) - ax}{t} &= \lim_{t \rightarrow 0} \frac{a\alpha_t(x) - ax}{t} + \lim_{t \rightarrow 0} \frac{\alpha'_t(a)\alpha_t(x) - a\alpha_t(x)}{t} \\
&= a \lim_{t \rightarrow 0} \frac{\alpha_t(x) - x}{t} + \lim_{t \rightarrow 0} \frac{\alpha'_t(a) - a}{t} \alpha_t(x) \\
&= a\delta(x) + d(a)x.
\end{aligned}$$

Hence $ax \in D(\delta)$ and $\delta(ax) = a\delta(x) + d(a)x$. Furthermore, $D(\delta)$ is a left $D(d)$ -module. $\square$

Example 4.4. Let $\mathcal{H}$ be a Hilbert space and $T \in B(\mathcal{H})$ be a self adjoint operator. By Stone's theorem there is a one-parameter group of unitaries $t \mapsto \alpha_t (\alpha_t \in B(\mathcal{H}))$ which its infinitesimal generator is iT . By Example 2.2, this is a dynamical system on Hilbert $K(\mathcal{H})$ -module $\mathcal{H}$ with corresponding dynamical system $t \mapsto \alpha'_t$, where $\alpha'_t(V) = \alpha_t V \alpha_t^*$. Here α_t^* denotes the adjoint of α_t in $B(\mathcal{H})$. If δ and δ' are infinitesimal generators of α and α' respectively, then $\delta = iT$ and by Theorem 4.3 we have $\delta(Vx) = V\delta(x) + \delta'(V)x$, and hence $\delta(V(x)) = V(\delta(x)) + \delta'(V)(x)$. Thus

$$\delta'(V)(x) = \delta(V(x)) - V(\delta(x)) = iT(V(x)) - iV(T(x)).$$

So

$$\delta'(V) = i(TV - VT) = i[T, V], \quad V \in D(\delta').$$

The density of $D(\delta')$ in $K(\mathcal{H})$ and the fact that δ' has no proper extension [14] implies that $\delta'(V) = i[T, V]$ and $V \in K(\mathcal{H})$.

We have shown that the generalized derivations, like ordinary derivations, behave well under some operations. In particular, closed derivations can be regarded as the infinitesimal generators of some dynamical systems on Hilbert C^* -modules. Since Hilbert

C^* -modules naturally generalize C^* -algebras, we may extend the study of approximately dynamical systems to Hilbert C^* -modules [11]. We close the paper with a relevant question which may be of special interest in the theory of approximately inner dynamical systems [14].

Question. Is a generalized derivation approximately inner?

Acknowledgment

The authors would like to thank the referee for his/her useful comments and suggestions.

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