ISSN: 1017-060X (Print)

ISSN: 1735-8515 (Online)

Bulletin of the

Iranian Mathematical Society

Vol. 43 (2017), No. 5, pp. 1357–1375

Title:

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Author(s):

L. Shiri and M. Zarebnia

Published by the Iranian Mathematical Society
http://bims.ims.ir

Bull. Iranian Math. Soc.
Vol. 43 (2017), No. 5, pp. 1357–1375
Online ISSN: 1735-8515

CONVERGENCE OF THE SINC METHOD APPLIED TO DELAY VOLTERRA INTEGRAL EQUATIONS

L. SHIRI AND M. ZAREBNIA*

(Communicated by Davod Khojasteh Salkuyeh)

Abstract. In this paper, the numerical solutions of linear and nonlinear Volterra integral equations with nonvanishing delay are considered by two methods. The methods are developed by means of the sinc approximation with the single exponential (SE) and double exponential (DE) transformations. The existence and uniqueness of sinc-collocation solutions for these equations are provided. These methods improve conventional results and achieve exponential convergence. Numerical results are included to confirm the efficiency and accuracy of the methods.
Keywords: Volterra integral equations, nonvanishing delay, sinc-collocation.
MSC(2010): Primary: 45D05; Secondary: 65R20.

1. Introduction

Delay Volterra integral equations arise widely in scientific fields such as physics, biology, ecology, control theory, etc. Due to the practical application of these equations, they must be solved successfully with efficient numerical approaches. In recent years, there have been extensive studies in convergence properties and stability analyses of numerical methods for them. Some numerical methods for the delay Volterra integral equation of vanishing and nonvanishing types have been studied. Piecewise collocation methods for VIE with vanishing delay is investigated in [8–10,18] and also for the case of non-vanishing delay, the papers [6, 7] investigate the piecewise collocation and Runge-Kutta methods.

Sinc methods for approximating the solutions of Volterra integral equations have received considerable attention mainly due to their high accuracy. These approximations converge rapidly to the exact solutions as the number sinc points increases. Systematic introduction of these methods can be found in [15]. In [17] sinc-collocation method is emplyed to solve Hammerstein Volterra

Article electronically published on 31 October, 2017.
Received: 16 June 2015, Accepted: 10 June 2016.
*Corresponding author.

integral equations. The analytical and numerical techniques used in this work can be extended to linear and nonlinear delay integral equations.

The main objective of the current study is to implement the sinc-collocation method for linear and nonlinear Volterra integral equation of the form

$$y(t) = g(t) + \int_0^t K_1(t,s)y(s)ds + \int_0^{\theta(t)} K_2(t,s)y(s)ds, \tag{1.1}$$

$$y(t) = g(t) + \int_0^t K_1(t,s,y(s))ds + \int_0^{\theta(t)} K_2(t,s,y(s))ds, \tag{1.2}$$

where the functions g, K_1 and K_2 are countiuous on their domains. The delay function θ is subject to the following conditions:

(1) (D1) $\theta(t) = t - \tau(t)$, and $\tau \in C^d(I := [0,T])$ for some $d \geqslant 0$;
(2) (D2) $\tau(t) \geqslant \tau_0 > 0$ for $t \in I$;
(3) (D3) θ is strictly increasing on I.

In the above equation, $\theta(t)$ is a continuous delay function, such that $\theta(t) < t$ for all $t \in (0,T)$. It is called vanishing delay if $t - \theta(t)$ vanishes at $t = 0$, otherwise it is called non-vanishing. $\theta(t) = t - \tau$, $\tau > 0$ is an important example of non-vanishing delay. In this paper we consider non-vanishing delays that it is more general than constant delay. Existence and uniqueness results for (1.2) can be found in [4].

The layout of this paper is as follows. Section 2 outlines some of the main properties of sinc function that is necessary for the formulation of the delay integral equation. Sinc-collocation method for linear delay integral equation is considered in Section 3. In Section 4, we analyze the existence and uniqueness of numerical solutions for nonlinear delay integral equations. Also, in Sections 3 and 4, the orders of scheme convergence using the new approach are described. Finally, Section 5 contains the numerical experiments.

2. Review of the sinc approximation

In this section, we will review the sinc function properties, sinc quadrature rule, and the sinc method. These are discussed thoroughly in [15]. The sinc basis functions are given by

$$S(j,h)(z) = \text{sinc}(\frac{z - jh}{h}), \qquad j = 0, \pm 1, \pm 2, \ldots, \tag{2.1}$$

where

$$\text{sinc}(z) = \begin{cases} \frac{\sin(\pi z)}{\pi z}, & z \neq 0, \\ 1, & z = 0, \end{cases}$$

and h is a step size appropriately chosen depending on a given positive integer N, and j is an integer.

Originally, the sinc approximation for a function u is expressed as

$$u(t) \approx \sum_{j=-N}^{N} u(jh)S(j,h)(t), \qquad t \in \mathbb{R}. \tag{2.2}$$

The above approximation is valid on $\mathbb{R}$, whereas equations (1.1) and (1.2) are defined on finite interval $[0,T]$. Equation (2.2) can be adapted to approximate on general intervals with the aid of appropriate variable transformations $t = \phi(x)$. The transformation function $\phi(x)$ appropriate single exponential (SE) and double exponential (DE) transformations are applied. The single exponential transformation and its inverse can be introduced, respectively, as below

$$\psi_{SE}(x) = \frac{Te^x}{1+e^x},$$
$$\phi_{SE}(t) = \ln(\frac{t}{T-t}).$$

In order to define a convenient function space, the strip domain $D_d = \{z \in \mathcal{C} : |Imz| < d\}$ for some $d > 0$ is introduced.

The following definitions and theorems are considered for further details of the procedure.

Definition 2.1. Let D be a simply connected domain which satisfies $(0,T) \subset D$ and α and c_1 be a positive constant. Then $\mathcal{L}_\alpha(D)$ denotes the family of all functions $u \in \mathbf{Hol}(D)$ which satisfy

$$|u(z)| \leqslant c_1 |Q(z)|^\alpha, \tag{2.3}$$

for all z in D where $Q(z) = z(T-z)$.

In what follow, D is $\psi_{SE}(D_d)$, where

$$\psi_{SE}(D_d) = \big\{z \in \mathcal{C} : |\arg(\frac{z}{T-z})| < d\big\}.$$

The next theorem shows the exponential convergence of the SE-sinc approximation, the proof of this theorem was first given by Stenger [15], and then it was improved by Okayama et al. [14].

Theorem 2.2. *Let $u \in \mathcal{L}_\alpha(D)$, N be a positive integer, and h be selected by the formula $h = \sqrt{\frac{\pi d}{\alpha N}}$. Then there exists a positive constant c_2, independent of N, such that*

$$\sup_{t\in(a,b)} \big|u(t) - \sum_{j=-N}^{N} u(\psi_{SE}(jh))S(j,h)(\phi_{SE}(t))\big| \leqslant c_2(b-a)^{2\alpha}\sqrt{N}e^{-\sqrt{\pi d\alpha N}}.$$

Stenger [15] had given the error analysis of the sinc indefinite integration, and after that Okayama et al. [14] gave error estimate with explicit constant for this approximation.

Theorem 2.3. *Let $uQ \in \mathcal{L}_\alpha(D)$ for d with $0 < d < \pi$. Let also $h = \sqrt{\frac{\pi d}{\alpha N}}$. Then there exists a constant c_3, which is independent of N, such that*

$$\sup_{t\in(a,b)} \left|\int_a^t u(s)ds - h\sum_{j=-N}^{N} u(\psi_{SE}(jh))\psi'_{SE}(jh)J(j,h)(\phi_{SE}(t))\right| \leqslant \tag{2.4}$$

$$c_3(b-a)^{2\alpha-1}e^{-\sqrt{\pi d\alpha N}} \tag{2.5}$$

where

$$J(j,h)(x) = \frac{1}{2} + \int_0^{\frac{x}{h}-j} \frac{\sin(\pi t)}{\pi t} dt.$$

The double exponential transformation can be used instead of the single exponential transformation. DE-transformation and its inverse are

$$\begin{aligned}\psi_{DE}(x) &= \frac{T}{2}\big(\tanh(\frac{\pi}{2}\sinh(x)) + 1\big),\\ \phi_{DE}(t) &= \ln\left[\frac{1}{\pi}\ln(\frac{t}{T-t}) + \sqrt{1 + \left\{\frac{1}{\pi}\ln\left(\frac{t}{T-t}\right)\right\}^2}\,\right].\end{aligned}$$

This transformation maps D_d onto the domain

$$\psi_{DE}(D_d) = \left\{z \in \mathcal{C} : \left|\arg\left[\frac{1}{\pi}\ln(\frac{t}{T-t}) + \sqrt{1 + \left\{\frac{1}{\pi}\ln\left(\frac{t}{T-t}\right)\right\}^2}\,\right]\right| < d\right\}.$$

The following theorem describes the extreme accuracy of DE-sinc approximation when $u \in \mathcal{L}_\alpha(\psi_{DE}(D_d))$.

Theorem 2.4. *Let $u \in \mathcal{L}_\alpha(\psi_{DE}(D_d))$ for d with $0 < d < \frac{\pi}{2}$, N be a positive integer and h be selected by the formula $h = \frac{\ln(2dN/\alpha)}{N}$. Then there exists a constant c_4 which is independent of N, such that*

$$\sup_{t\in(a,b)} \left|u(t) - \sum_{j=-N}^{N} u(\psi_{DE}(jh))S(j,h)(\phi_{DE}(t))\right| \leqslant c_4(b-a)^{2\alpha}e^{-\pi dN/\ln(2dN/\alpha)}.$$

If we use the DE transformation instead of the SE transformation, the DE-sinc quadrature is achieved. The rate of convergence is accelerated as the next theorem states.

Theorem 2.5 ([14])**.** *Let $uQ \in \mathcal{L}_\alpha(\psi_{DE}(D_d))$ for d with $0 < d < \frac{\pi}{2}$. Let also $\alpha' = \alpha - \epsilon$ for $0 < \epsilon < \alpha$, N be a positive integer with $N > \alpha'/(2d)$, and h be selected by the formula*

$$h = \frac{\ln(2dN/\alpha')}{N}.$$

Then there exists a constant c_5 which is independent of N, such that

$$\sup_{t\in(a,b)}\left|\int_a^t u(s)ds - h\sum_{j=-N}^{N} u(\psi_{DE}(jh))\psi'_{DE}(jh)J(j,h)(\phi_{DE}(t))\right|$$
$$\leqslant c_5(b-a)^{2\alpha-1}e^{-\pi dN/\ln(2dN/\alpha')}.$$

3. **Linear delay integral equation**

In the present section, we apply the sinc-collocation method to solve (1.1) which we state again for the convenience of the reader:

$$y(t) = g(t) + \int_0^t K_1(t,s)y(s)ds + \int_0^{\theta(t)} K_2(t,s)y(s)ds, \qquad t \in (0,T].$$

Notice that for t that $\theta(t) \leqslant 0$ we have

$$y(t) = g(t) + \int_0^t K_1(t,s)y(s)ds + \int_0^{\theta(t)} K_2(t,s)\varphi(s)ds, \quad t \in (0,\theta^{-1}(0)].$$

This is a Volterra integral equation.

If $t = 0$, we have $y(0) = \varphi(0)$. For ease of calculation, we employ the transformation

$$u(t) = y(t) - \frac{T-t}{T}\varphi(0).$$

In this case $u(0) = 0$. Then the above problem becomes

$$u(t) = f(t) + \int_0^t K_1(t,s)u(s)ds + \int_0^{\theta(t)} K_2(t,s)u(s)ds, \tag{3.1}$$

where

$$\begin{aligned} f(t) \quad &:= \quad g(t) - \frac{1}{T}(T-t)\varphi(0) \\ &\quad + \frac{1}{T}\varphi(0)\left\{\int_0^t K_1(t,s)(T-s)ds + \int_0^{\theta(t)} K_2(t,s)(T-s)ds\right\}. \end{aligned}$$

Equation (3.1) may be written in the form

$$u = f + \mathcal{V}u + \mathcal{V}_\theta u, \tag{3.2}$$

where

$$\begin{aligned} \mathcal{V}u(t) \quad &= \quad \int_0^t K_1(t,s)u(s)ds, \\ \mathcal{V}_\theta u(t) \quad &= \quad \int_0^{\theta(t)} K_2(t,s)u(s)ds. \end{aligned}$$

3.1. **SE-sinc scheme.** A Sinc approximation $\mathcal{U}_N^{SE}$ to the solution u of equation. (3.1) is described in this part. Let us define the operator $\mathcal{P}_N^{SE}$ as follows

$$\mathcal{P}_N^{SE} u(t) = \sum_{j=-N}^{N} u_j^{SE} S(j,h)(\phi_{SE}(t)) + u_{N+1}^{SE} w_{SE}(t), \tag{3.3}$$

in which $u_j^{SE} = u(t_j^{SE})$. We choose $w_{SE}(t)$ so that $\mathcal{P}_N^{SE}$ interpolate the function u at the points

$$t_j^{SE} := \begin{cases} \psi_{SE}(jh), & j = -N, \dots, N; \\ T, & j = N+1. \end{cases}$$

Then

$$w_{SE}(t) := \frac{1}{T}\left(t - \sum_{j=-N}^{N} t_j^{SE} S(j,h)(\phi_{SE}(t))\right).$$

$\mathcal{P}_N^{SE}$ is called the collocation operator. The approximate solution $\mathcal{U}_N^{SE}$ is considered that has the form

$$\mathcal{U}_N^{SE}(t) = \sum_{j=-N}^{N} u_j^{SE} S(j,h)(\phi_{SE}(t)) + u_{N+1}^{SE} w_{SE}(t). \tag{3.4}$$

Applying the operator $\mathcal{P}_N^{SE}$ to both sides of (3.2) at $t = t_k^{SE}$, $k = -N, \dots, N+1$ and using the sinc quadrature formula give us the following approximate equation in operator form

$$\mathcal{Z}_N^{SE} = \mathcal{P}_N^{SE} \mathcal{V} \mathcal{Z}_N^{SE} + \mathcal{P}_N^{SE} \mathcal{V}_\theta \mathcal{Z}_N^{SE} + \mathcal{P}_N^{SE} f. \tag{3.5}$$

So the collocation method for solving (3.1) amounts to solving (3.5) for N sufficiently large. We are interested in approximating the integral operator in (3.5) by the quadrature formula presented in (2.4). So the following discrete SE operators can be defined

$$\begin{aligned} \mathcal{V}_N^{SE} u(t) &:= h \sum_{j=-N}^{N} \psi'_{SE}(jh) K_1(t, t_j^{SE}) J(j,h)(\phi_{SE}(t)) u_j^{SE}, \\ \mathcal{V}_{\theta N}^{SE} u(t) &:= h \sum_{j=-N}^{N} \psi'_{SE}(jh) K_2(t, t_j^{SE}) J(j,h)(\phi_{SE}(\theta(t))) u_j^{SE}. \end{aligned}$$

This numerical procedure leads us to replace (3.5) by

$$\mathcal{U}_N^{SE} = \mathcal{P}_N^{SE} \mathcal{V}_N^{SE} \mathcal{U}_N^{SE} + \mathcal{P}_N^{SE} \mathcal{V}_{\theta N}^{SE} \mathcal{U}_N^{SE} + \mathcal{P}_N^{SE} f.$$

By substituting $\mathcal{U}_N^{SE}$ into (3.1) and approximating the integral by means of the sinc quadrature formula and considering its collocation on $2N+2$ sampling points at $t = t_k^{SE}$, for $k = -N, \dots, N, N+1$, the following linear system of equations is obtained

$$u_k^{SE} = h \sum_{j=-N}^{N} \psi'_{SE}(jh) \Big\{ K_1(t_k^{SE}, t_j^{SE}) J(j,h)(\phi_{SE}(t_k^{SE})) + K_2(t_k^{SE}, t_j^{SE}) J(j,h)(\phi_k^{SE}) \Big\} u_j^{SE} + f(t_k^{SE}),$$

in which $\phi_k^{SE} := \phi(\theta(t_k^{SE}))$. From definition of t_k^{SE} we can write

$$J(j,h)(\phi_{SE}(t_k^{SE})) = \begin{cases} J(j,h)(kh), & k = -N, \dots, N; \\ 1, & k = N+1. \end{cases}$$

For t_k that $\theta(t_k) \leqslant 0$ we set

$$K_2(t_k^{SE}, t_j^{SE}) := 0, \qquad j = -N, \dots, N,$$

and

$$f(t_k^{SE}) := g(t_k^{SE}) - \frac{1}{T}(T - t_k^{SE})\varphi(0) + \frac{1}{T}\varphi(0) \int_0^{t_k^{SE}} K_1(t_k^{SE}, s)(T - s) ds + \int_0^{\theta(t_k^{SE})} K_2(t_k^{SE}, s)\varphi(s) ds.$$

Linear system (3.6) of equations is equivalent to (3.1). By solving this system, the unknown coefficients u_j^{SE} are determined. We rewrite the linear system in matrix form

$$[I - \mathcal{A}^{SE}]\mathbf{U}_N^{SE} = \mathbf{F}^{SE}, \tag{3.6}$$

where

$$\mathbf{U}_N^{SE} = [u_{-N}^{SE}, \dots, u_N^{SE}]^t, \qquad \mathbf{F}^{SE} = [f(t_{-N}^{SE}), \dots, f(t_N^{SE})]^t,$$

and for $k, j = -N, \dots N$,

$$\mathcal{A}_{k,j}^{SE} = h\psi'_{SE}(jh) \left\{ K_1(t_k^{SE}, t_j^{SE}) J(j,h)(kh) + K_2(t_k^{SE}, t_j^{SE}) J(j,h)(\phi_k^{SE}) \right\}.$$

By solving equation (3.6) we obtain $u_{-N}^{SE}, \dots, u_N^{SE}$ and then by using (3.6) u_{N+1}^{SE} is determained.

3.2. **DE-sinc scheme.** In this subsection we apply the DE-sinc method for solving (1.2). The approximate solution $\mathcal{U}_N^{DE}$ and the operator $\mathcal{P}_N^{DE}$ can be defined similar to the SE-sinc method,

$$\mathcal{P}_N^{DE} u(t) := \sum_{j=-N}^{N} u_j^{DE} S(j,h)(\phi_{DE}(t)) + u_{N+1}^{DE} w_{DE}(t). \tag{3.7}$$

The discrete DE operators are introduced

$$\mathcal{V}_N^{DE} u(t) := h \sum_{j=-N}^{N} \psi'_{DE}(jh) K_1(t, t_j^{DE}) J(j,h)(\phi_{DE}(t)) u_j^{DE}, \tag{3.8}$$

$$\mathcal{V}_{\theta N}^{DE}u(t) := h\sum_{j=-N}^{N}\psi'_{DE}(jh)K_2(t,t_j^{DE})J(j,h)(\phi_{DE}(\theta(t)))u_j^{DE}.$$

By applying (3.7)-(3.9) and setting its collocation on $2N+2$ sampling points at $t = t_k^{DE}$, for $k = -N,\dots,N+1$, in equation (3.1), the linear system

$$\begin{aligned} u_k^{DE} &= h\sum_{j=-N}^{N}\psi'_{DE}(jh)\Big\{K_1(t_k^{DE},t_j^{DE})J(j,h)(\phi_{DE}(t_k^{DE})) \\ &\quad +K_2(t_k^{DE},t_j^{DE})J(j,h)(\phi_k^{DE})\Big\}u_j^{DE} + f(t_k^{DE}), \end{aligned}$$

is achieved. This linear system can be stated in matrix form as follows

$$[I-\mathcal{A}^{DE}]\mathbf{U}_N^{DE} = \mathbf{F}^{DE}. \tag{3.9}$$

By solving this system, the unknown coefficients in $\mathbf{U}_N^{DE}$ are found.

3.3. **Existence and uniqueness of the sinc-collocation solution.** We now use the computational forms of the collocation equation derived in Section 4.2 to show that the existence of a unique sinc-collocation solutions of (3.6) and (3.9).

Lemma 3.1 ([15])**.** *For $x \in \mathbb{R}$, the function $J(j,h)(x)$ is bounded by*

$$|J(j,h)(x)| \leqslant 1.1.$$

Theorem 3.2. *Assume that K_1, K_2 and f in the Volterra integral equation (3.1) are continuous on their respective domains D, D_θ and I. Then there exists a $\overline{h}>0$ so that for any $h\in(0,\overline{h})$ the linear algebraic systems (3.6) and (3.9) have a unique solution $\boldsymbol{U}_N$.*

Proof. Using Lemma 3.1, Theorem 2.3, Theorem 2.5 and continuity of K_1 and K_2, there exists $c>0$ so that $|\mathcal{A}_{k,j}| < ch$ for $k,j=-N,\dots N$. Then $\|\mathcal{A}\|_\infty < 1$ whenever h is sufficiently small. In other words, there is a $\overline{h}>0$ so that for any $h<\overline{h}$ the matrix $(I-\mathcal{A})$ has a uniformly bounded inverse. The assertion of Theorem 3.2 now follows. ☐

4. Nonlinear delay integral equation

In the present section, the solution of the functional equation (1.2) will be approximated by the sinc-collocation sloution. Equation (1.2) is stated again for the convenience of the reader:

$$y(t) = g(t) + \int_0^t K_1(t,s,u(s)) + \int_0^{\theta(t)} K_2(t,s,y(s))ds, \quad t\in I := (0,T].$$

If $t=0$ we have $y(0)=\varphi(0)$. Like Section 3, we use the following transformation

$$u(t) = y(t) - \frac{T-t}{T}\varphi(0).$$

In this case $u(0) = 0$. Then the above problem can be rewritten as

$$u(t) = f(t) + \int_0^t \mathcal{K}_1(t, s, u(s)) + \int_0^{\theta(t)} \mathcal{K}_2(t, s, u(s))ds \tag{4.1}$$

with

$$\begin{aligned} f(t) &:= g(t) - \frac{T-t}{T}\varphi(0), \\ \mathcal{K}_1(t, s, u(s)) &:= K_1(t, s, u(s) + \frac{T-t}{T}\varphi(0)), \\ \mathcal{K}_2(t, s, u(s)) &:= K_2(t, s, u(s) + \frac{T-t}{T}\varphi(0)). \end{aligned}$$

Now, let $u(t)$ be the exact solution of equation (4.1).

4.1. **SE-sinc scheme.** We replace the approximate solution (3.4) in (4.1). Substituting $t = t_k^{SE}$, $k = -N, \ldots, N+1$ we can obtain

$$\begin{aligned} u_k^{SE} = f(t_k^{SE}) + \int_0^{t_k^{SE}} \mathcal{K}_1(t_k^{SE}, s, \sum_{j=-N}^{N} u_j^{SE} S(j,h)(\phi_{SE}(s)) + u_{N+1}^{SE} w_{SE}(s))ds + \\ \int_0^{\theta(t_k^{SE})} \mathcal{K}_2(t_k^{SE}, s, \sum_{j=-N}^{N} u_j^{SE} S(j,h)(\phi_{SE}(s)) + u_{N+1}^{SE} w_{SE}(s))ds. \end{aligned} \tag{4.2}$$

We approximate the integral in above equation by the quadrature formula presented in (2.4)

$$\begin{aligned} &\int_0^{t_k^{SE}} \mathcal{K}_1(t_k^{SE}, s, \sum_{j=-N}^{N} u_j^{SE} S(j,h)(\phi_{SE}(s)) + u_{N+1}^{SE} w_{SE}(s))ds \\ &\qquad = h \sum_{l=-N}^{N} \psi'_{SE}(lh) J(l,h)(\phi_{SE}(t_k^{SE}))\mathcal{K}_1(t_k^{SE}, t_l^{SE}, u_l^{SE}), \end{aligned}$$

$$\begin{aligned} &\int_0^{\theta(t_k^{SE})} \mathcal{K}_2(t_k^{SE}, s, \sum_{j=-N}^{N} u_j^{SE} S(j,h)(\phi_{SE}(s)) + u_{N+1}^{SE} w_{SE}(s))ds \\ &\qquad = h \sum_{l=-N}^{N} \psi'_{SE}(lh) J(l,h)(\phi_k^{SE})\mathcal{K}_2(t_k^{SE}, t_l^{SE}, u_l^{SE}), \end{aligned}$$

where

$$\phi_k^{SE} := \phi_{SE}(\theta(t_k^{SE})).$$

Thus (4.2) is written as

$$u_k^{SE} = f(t_k^{SE}) + h\sum_{l=-N}^{N}\psi'_{SE}(lh)\Big\{J(l,h)(\phi_{SE}(t_k^{SE}))\mathcal{K}_1(t_k^{SE},t_l^{SE},u_l^{SE}) + J(l,h)(\phi_k^{SE})\mathcal{K}_2(t_k^{SE},t_l^{SE},u_l^{SE})\Big\}, \tag{4.3}$$

where $u_k^{SE} = u(t_k^{SE})$. This nonlinear system of equations is equivalent to (4.1). By solving this system, the unknown coefficients u_k^{SE} are determined. We rewrite the nonlinear system (4.3) in matrix form

$$\mathcal{A}^{SE}(\mathbf{U}_N^{SE}) = \mathbf{U}_N^{SE}, \tag{4.4}$$

where

$$\begin{aligned}
[\mathcal{A}^{SE}(\mathbf{U}_N^{SE})]_{k,l} \;&:=\; f(t_k^{SE}) + h\psi'_{SE}(lh)\Big\{J(l,h)(kh)\mathcal{K}_1(t_k^{SE},t_l^{SE},u_l^{SE}) \\
&\qquad +J(l,h)(\phi_k^{SE})\mathcal{K}_2(t_k^{SE},t_l^{SE},u_l^{SE})\Big\}, \quad k,l=-N,\dots,N,\\
\mathbf{U}_N^{SE} \;&:=\; [u_{-N}^{SE},\dots,u_N^{SE}]^t.
\end{aligned}$$

4.2. **DE-sinc scheme.** The main consideration of this subsection is on DE-sinc case. The approximate solution $\mathcal{U}_N^{DE}$, analogous to the SE-sinc method, can be presented in the following form

$$\mathcal{U}_N^{DE}(t) = \sum_{j=-N}^{N} u_j^{DE} S(j,h)(\phi_{DE}(t)) + u_{N+1}^{DE} w_{DE}(t). \tag{4.5}$$

By applying (4.5) and setting collocation points $t = t_k^{DE}$, for $k = N,\dots,N+1$, in equation (4.1), the following nonlinear system

$$\mathcal{A}^{DE}(\mathbf{U}_N^{DE}) = \mathbf{U}_N^{DE}, \tag{4.6}$$

is achieved. By solving this system, the unknown coefficients in $\mathbf{U}_N^{DE}$ are found.

4.3. **Existence and uniqueness of the sinc-collocation solution.** In this section, we study the existence and uniqueness of the solution to (4.4) and (4.6).

Theorem 4.1. *Assume that $\mathcal{K}_1$, $\mathcal{K}_2$ and f in the nonlinear Volterra equation (4.1) are continuous and*

$$\begin{aligned}
|\mathcal{K}_1(t,s,u(t)) - \mathcal{K}_1(t,s,v(t))| < L_1|u(t)-v(t)|,\\
|\mathcal{K}_2(t,s,u(t)) - \mathcal{K}_2(t,s,v(t))| < L_2|u(t)-v(t)|.
\end{aligned}$$

Then the nonlinear algebraic systems (4.4) and (4.6) have a unique solution.

Proof. Using Lemma 3.1 and continuity of $\mathcal{K}_1$ and $\mathcal{K}_2$ we have

$$\begin{aligned}
\|\mathcal{A}^{SE}(\mathbf{U}_N^{SE})-\mathbf{F}^{SE}\|_\infty &= \max_k \Big| h\sum_{l=-N}^{N}\psi'_{SE}(lh)[J(l,h)(\phi_{SE}(t_k^{SE}))\mathcal{K}_1(t_k^{SE},t_l^{SE},u_l^{SE}) \\
&\quad +J(l,h)(\phi_k^{SE})\mathcal{K}_2(t_k^{SE},t_l^{SE},u_l^{SE})]\Big| \\
&\leqslant 1.1h\sup_x|\psi'(x)|\sum_{l=-N}^{N}\Big|\mathcal{K}_1(t_k^{SE},t_l^{SE},u_l^{SE})+\mathcal{K}_2(t_k^{SE},t_l^{SE},u_l^{SE})\Big| \\
&\leqslant 1.1he^{-Nh}\sum_{l=-N}^{N}|\mathcal{K}_1(t_k^{SE},t_l^{SE},u_l^{SE})|+|\mathcal{K}_2(t_k^{SE},t_l^{SE},u_l^{SE})| \\
&\leqslant 1.1he^{-Nh}(2N+1)M,
\end{aligned}$$

where M is an upper bound of $|\mathcal{K}_1(t,s,u(t))|+|\mathcal{K}_2(t,s,u(t))|$ and $\mathbf{F}^{SE}:=[f(t_{-N}^{SE}),\dots,f(t_N^{SE})]^t$. By using the fixed point theorem, this proves that the nonlinear system has a solution in the closed ball with center $\mathbf{F}^{SE}$ and radius $1.1hT(2N+1)M$.

It may be shown that, if $\mathcal{K}_1$ and $\mathcal{K}_2$ are Lipschitz with respect to $u(t)$, the solution is unique. Suppose that $\mathbf{U}_N^{SE}$ and $\mathbf{V}_N^{SE}$ are two possible solutions. Then

$$\begin{aligned}
\|\mathbf{U}_N^{SE}-\mathbf{V}_N^{SE}\|_\infty &= \|\mathcal{A}^{SE}(\mathbf{U}_N^{SE})-\mathcal{A}^{SE}(\mathbf{V}_N^{SE})\|_\infty \\
&= \max_k\Big|h\sum_{l=-N}^{N}\psi'_{SE}(lh)J(l,h)(\phi_k^{SE})\left[\mathcal{K}_1(t_k^{SE},t_l^{SE},u_l)-\mathcal{K}_1(t_k^{SE},t_l^{SE},v_l)\right] \\
&\quad +\psi'_{SE}(lh)J(l,h)(\phi_{SE}(t_k^{SE}))\left[\mathcal{K}_2(t_k^{SE},t_l^{SE},u_l)-\mathcal{K}_2(t_k^{SE},t_l^{SE},v_l)\right]\Big| \\
&\leqslant 1.1he^{-Nh}\max_k\sum_{l=-N}^{N}\Big[\left|\mathcal{K}_1(t_k^{SE},t_l^{SE},u_l)-\mathcal{K}_1(t_k^{SE},t_l^{SE},v_l)\right| \\
&\quad +\left|\mathcal{K}_2(t_k^{SE},t_l^{SE},u_l)-\mathcal{K}_2(t_k^{SE},t_l^{SE},v_l)\right|\Big] \\
&\leqslant 1.1he^{-Nh}\max_k\sum_{l=-N}^{N}L_1|u_l-v_l|+L_2|u_l-v_l| \\
&\leqslant 2.2he^{-Nh}(2N+1)L\|\mathbf{U}_N^{SE}-\mathbf{V}_N^{SE}\|_\infty \\
&< \|\mathbf{U}_N^{SE}-\mathbf{V}_N^{SE}\|_\infty,
\end{aligned}$$

because $\lim_{N\to\infty}he^{-Nh}(2N+1)=0$, we can write the last inequality for some N. It follows that $\|\mathbf{U}_N^{SE}-\mathbf{V}_N^{SE}\|_\infty$ vanishes and there is thus uniqueness. The similar conclusions are achieved for DE case. □

5. Convergence analysis

The convergence of the sinc-collocation method which was introduced in the previous subsections is discussed in the present subsection. It is assumed that u is the exact solution of equation (3.1) and $\mathcal{U}_N^{SE}$ and $\mathcal{U}_N^{DE}$ are the approximations of the sinc method. Firstly, we state the following lemma which is used subsequently.

Lemma 5.1. *Assume that the equation (1.2) has a unique solution. Then for all sufficiently large N , $N \geqslant n$, the operator $(\mathcal{I}-\mathcal{P}_N(\mathcal{V}_N+\mathcal{V}_{\theta N}))^{-1}$ exists and it is uniformly bounded:*

$$\sup_{N\geqslant n} \|(\mathcal{I}-\mathcal{P}_N(\mathcal{V}_N+\mathcal{V}_{\theta N}))^{-1}\| < \infty. \tag{5.1}$$

Proof. From Theorem 2.2 we can write

$$\begin{aligned} \|\mathcal{V}_N - \mathcal{P}_N^{SE}\mathcal{V}_N\| &\leqslant C_1\sqrt{N}e^{-\sqrt{\pi d\alpha N}}, \\ \|\mathcal{V}_{\theta N} - \mathcal{P}_N^{SE}\mathcal{V}_{\theta N}\| &\leqslant C_2\sqrt{N}e^{-\sqrt{\pi d\alpha N}}, \end{aligned}$$

so

$$\|\mathcal{V}_N - \mathcal{P}_N^{SE}\mathcal{V}_N\| \to 0, \qquad \|\mathcal{V}_{\theta N} - \mathcal{P}_N^{SE}\mathcal{V}_{\theta N}\| \to 0, \qquad as \quad N\to\infty.$$

From [1, Theorem 4.1.2] there exists the inverse of $(\mathcal{I}-\mathcal{V}_N-\mathcal{V}_{\theta N})$ for all sufficiently large N, so we can pick n such that

$$\epsilon_n = \sup_{n\geqslant N}\|\mathcal{V}_N+\mathcal{V}_{\theta N}-\mathcal{P}_N^{SE}(\mathcal{V}_N+\mathcal{V}_{\theta N})\| \leqslant \frac{1}{\|(\mathcal{I}-\mathcal{V}_N-\mathcal{V}_{\theta N})^{-1}\|}.$$

Then the inverse of $[\mathcal{I}+(\mathcal{I}-\mathcal{V}_N-\mathcal{V}_{\theta N})^{-1}(\mathcal{V}_N+\mathcal{V}_{\theta N}-\mathcal{P}_N^{SE}(\mathcal{V}_N+\mathcal{V}_{\theta N}))]$ exists and uniformly bounded by the geometric series theorem

$$\|\big[\mathcal{I}+(\mathcal{I}-\mathcal{V}_N-\mathcal{V}_{\theta N})^{-1}(\mathcal{V}_N+\mathcal{V}_{\theta N}-\mathcal{P}_N^{SE}(\mathcal{V}_N+\mathcal{V}_{\theta N}))\big]^{-1}\| \leqslant \frac{1}{1-\epsilon_n\|(\mathcal{I}-\mathcal{V}_N-\mathcal{V}_{\theta N})^{-1}\|}.$$

On the other hand,

$$\begin{aligned} &\mathcal{I}-\mathcal{P}_N^{SE}(\mathcal{V}_N+\mathcal{V}_{\theta N}) = (\mathcal{I}-\mathcal{V}_N-\mathcal{V}_{\theta N}) + (\mathcal{V}_N+\mathcal{V}_{\theta N}-\mathcal{P}_N^{SE}(\mathcal{V}_N+\mathcal{V}_{\theta N})) \\ &= (\mathcal{I}-\mathcal{V}_N-\mathcal{V}_{\theta N})\Big[\mathcal{I}+(\mathcal{I}-\mathcal{V}_N-\mathcal{V}_{\theta N})^{-1}(\mathcal{V}_N+\mathcal{V}_{\theta N}-\mathcal{P}_N^{SE}(\mathcal{V}_N+\mathcal{V}_{\theta N}))\Big]. \end{aligned}$$

Using above equation, $(\mathcal{I}-\mathcal{P}_N^{SE}(\mathcal{V}_N+\mathcal{V}_{\theta N}))^{-1}$ exists, and

$$\begin{aligned} &(\mathcal{I}-\mathcal{P}_N^{SE}(\mathcal{V}_N+\mathcal{V}_{\theta N}))^{-1} \\ &= \Big[\mathcal{I}+(\mathcal{I}-\mathcal{V}_N-\mathcal{V}_{\theta N})^{-1}(\mathcal{V}_N+\mathcal{V}_{\theta N}-\mathcal{P}_N^{SE}(\mathcal{V}_N+\mathcal{V}_{\theta N}))\Big]^{-1}(\mathcal{I}-\mathcal{V}_N-\mathcal{V}_{\theta N})^{-1} \\ &\|(\mathcal{I}-\mathcal{P}_N^{SE}(\mathcal{V}_N+\mathcal{V}_{\theta N}))^{-1}\| \leqslant \frac{\|(\mathcal{I}-\mathcal{V}_N-\mathcal{V}_{\theta N})^{-1}\|}{1-\epsilon_n\|(\mathcal{I}-\mathcal{V}_N-\mathcal{V}_{\theta N})^{-1}\|}. \end{aligned}$$

This shows (5.1). Also it is true for $\mathcal{P}_N^{DE}$. □

Lemma 5.2 ([16]). *Let $h > 0$. Then it holds that*

$$\sup_{x\in\mathbb{R}} \sum_{j=-N}^{N} |S(j,h)(x)| \leqslant \frac{2}{\pi}(3 + \ln N).$$

Based on this lemma, it has been concluded $\|\mathcal{P}_N^{SE}\| \leqslant C_{SE}\ln(N)$ and $\|\mathcal{P}_N^{SE}\| \leqslant C_{SE}\ln(N)$ where C_{SE} and C_{DE} are constants independent of N. In the following theorem, we will find an upper bound for the error.

Theorem 5.3. *Let $\mathcal{U}_N^{SE}(x)$ be the approximate solution of integral equation (3.1). Then there exists a constant c_6 independent of N such that*

$$\sup_{x\in(0,T)} |u(x) - \mathcal{U}_N^{SE}(x)| \leqslant c_6\sqrt{N}\ln N e^{-\sqrt{\pi d\alpha N}}. \tag{5.2}$$

Proof. The estimation (5.2) is obtained as follows:

$$\begin{aligned} u - \mathcal{U}_N^{SE} &= f - \mathcal{P}_N^{SE} f + \mathcal{V}u - \mathcal{P}_N^{SE}\mathcal{V}_N\mathcal{U}_N^{SE} + \mathcal{V}_\theta u - \mathcal{P}_N^{SE}\mathcal{V}_{\theta N}\mathcal{U}_N^{SE} \\ &= (f - \mathcal{P}_N^{SE} f) + (\mathcal{V}u - \mathcal{P}_N^{SE}\mathcal{V}u) + (\mathcal{P}_N^{SE}\mathcal{V}u - \mathcal{P}_N^{SE}\mathcal{V}_N u) \\ &\quad + (\mathcal{P}_N^{SE}\mathcal{V}_N u - \mathcal{P}_N^{SE}\mathcal{V}_N\mathcal{U}_N^{SE}) + (\mathcal{V}_\theta u - \mathcal{P}_N^{SE}\mathcal{V}_\theta u) \\ &\quad + (\mathcal{P}_N^{SE}\mathcal{V}_\theta u - \mathcal{P}_N^{SE}\mathcal{V}_{\theta N} u) + (\mathcal{P}_N^{SE}\mathcal{V}_{\theta N} u - \mathcal{P}_N^{SE}\mathcal{V}_{\theta N}\mathcal{U}_N^{SE}) \\ &= (f - \mathcal{P}_N^{SE} f) + (\mathcal{V}u - \mathcal{P}_N^{SE}\mathcal{V}u) + \mathcal{P}_N^{SE}(\mathcal{V}u - \mathcal{V}_N u) \\ &\quad + \mathcal{P}_N^{SE}(\mathcal{V}_N u - \mathcal{V}_N\mathcal{U}_N^{SE}) + (\mathcal{V}_\theta u - \mathcal{P}_N^{SE}\mathcal{V}_\theta u) \\ &\quad + \mathcal{P}_N^{SE}(\mathcal{V}_\theta u - \mathcal{V}_{\theta N} u) + \mathcal{P}_N^{SE}(\mathcal{V}_{\theta N} u - \mathcal{V}_{\theta N}\mathcal{U}_N^{SE}), \end{aligned}$$

then we can write

$$\begin{aligned} &(\mathcal{I} - \mathcal{P}_N^{SE}\mathcal{V}_N - \mathcal{P}_N^{SE}\mathcal{V}_{\theta N})(u - \mathcal{U}_N^{SE}) \\ &\qquad = (f - \mathcal{P}_N^{SE} f) + (\mathcal{V}u - \mathcal{P}_N^{SE}\mathcal{V}u) + \mathcal{P}_N^{SE}(\mathcal{V}u - \mathcal{V}_N u) \\ &\qquad\quad + (\mathcal{V}_\theta u - \mathcal{P}_N^{SE}\mathcal{V}_\theta u) + \mathcal{P}_N^{SE}(\mathcal{V}_\theta u - \mathcal{V}_{\theta N} u). \end{aligned}$$

By using Lemma 5.1 we have

$$\begin{aligned} (u - \mathcal{U}_N^{SE}) &= (\mathcal{I} - \mathcal{P}_N^{SE}\mathcal{V}_N - \mathcal{P}_N^{SE}\mathcal{V}_{\theta N})^{-1}\Big[(f - \mathcal{P}_N^{SE} f) + (\mathcal{V}u - \mathcal{P}_N^{SE}\mathcal{V}u) \\ &\quad + \mathcal{P}_N^{SE}(\mathcal{V}u - \mathcal{V}_N u) + (\mathcal{V}_\theta u - \mathcal{P}_N^{SE}\mathcal{V}_\theta u) + \mathcal{P}_N^{SE}(\mathcal{V}_\theta u - \mathcal{V}_{\theta N} u)\Big]. \end{aligned}$$

This leads to

$$\begin{aligned} \|u - \mathcal{U}_N^{SE}\| &\leqslant \|(\mathcal{I} - \mathcal{P}_N^{SE}\mathcal{V}_N - \mathcal{P}_N^{SE}\mathcal{V}_{\theta N})^{-1}\|\Big[\|f - \mathcal{P}_N^{SE} f\| + \|\mathcal{V}u - \mathcal{P}_N^{SE}\mathcal{V}u\| \\ &\quad + \|\mathcal{P}_N^{SE}\|\|\mathcal{V}u - \mathcal{V}_N u\| + \|\mathcal{V}_\theta u - \mathcal{P}_N^{SE}\mathcal{V}_\theta u\| + \|\mathcal{P}_N^{SE}\|\|\mathcal{V}_\theta u - \mathcal{V}_{\theta N} u\|\Big]. \end{aligned}$$

We can apply Theorem 2.2 and get

$$\|f - \mathcal{P}_N^{SE} f\| \leqslant C_1\sqrt{N}e^{-\sqrt{\pi d\alpha N}},$$
$$\|\mathcal{V}u - \mathcal{P}_N^{SE}\mathcal{V}u\| \leqslant C_2\sqrt{N}e^{-\sqrt{\pi d\alpha N}},$$
$$\|\mathcal{V}_\theta u - \mathcal{P}_N^{SE}\mathcal{V}_\theta u\| \leqslant C_3\sqrt{N}e^{-\sqrt{\pi d\alpha N}}.$$

By using Theorem 2.3, the following result is concluded

$$\|\mathcal{V}u - \mathcal{V}_N u\| \leqslant C_4 e^{-\sqrt{\pi d\alpha N}},$$
$$\|\mathcal{V}_\theta u - \mathcal{V}_{\theta N} u\| \leqslant C_5 e^{-\sqrt{\pi d\alpha N}},$$

and finally $\|\mathcal{P}_N^{SE}\|$ is estimated by conclusion of Lemma 3.1. So

$$\|u - \mathcal{U}_N^{SE}\| \leqslant C\|(\mathcal{I} - \mathcal{P}_N^{SE}\mathcal{V}_N - \mathcal{P}_N^{SE}\mathcal{V}_{\theta N})^{-1}\|\sqrt{N}e^{-\sqrt{\pi d\alpha N}}.$$

□

It is clear that the arguments employed in the above proof can be used to DE-sinc method.

Theorem 5.4. *Let $\mathcal{U}_N^{DE}(x)$ be the approximate solution of integral equation (3.1). Then there exists a constant c_7 independent of N such that*

$$\sup_{x\in(0,T)} |u(x) - \mathcal{U}_N^{DE}(x)| \leqslant c_7 e^{-\pi dN/\ln(2dN/\alpha)}.$$

In the following we try to discuss the conditions under which Newtons method is convergent for the nonlinear equation. For this reason we will state and prove the following theorem.

Theorem 5.5. *Assume that $\boldsymbol{U}_N^{SE}$ is the exact solution of the nonlinear system (4.3), and hypotheses of Theorem 5.3 are satisfied. Also, suppose that $\frac{\partial \mathcal{K}_1}{\partial u}$ and $\frac{\partial \mathcal{K}_2}{\partial u}$ are Lipschitz with respect to u. Then there exist $\delta > 0$ and $\overline{h} > 0$ such that if $\| \boldsymbol{U}_{N(0)}^{SE} - \boldsymbol{U}_N^{SE}\| \leqslant \delta$, the Newton's sequence $\{\boldsymbol{U}_{N(m)}^{SE}\}$ for any $h \in (0, \overline{h})$ is well-defined and convergence to $\boldsymbol{U}_N^{SE}$. Furthermore, for some constant l with $l\delta < 1$, we have the error bounds*

$$\| \boldsymbol{U}_{N(m)}^{SE} - \boldsymbol{U}_N^{SE}\| \leqslant \frac{(l\delta)^{2^m}}{l}.$$

Proof. We must solve the nonlinear system

$$\mathbf{U}_N^{SE} - \mathcal{A}^{SE}(\mathbf{U}_N^{SE}) = 0.$$

The Newton method reads as follow. Choose an initial guess $\mathbf{U}_{N(0)}^{SE}$; for $m = 0, 1, \ldots$, compute

$$\text{(5.3)} \quad \mathbf{U}_{N(m+1)}^{SE} = \mathbf{U}_{N(m)}^{SE} - [\mathcal{I} - (\mathcal{A}^{SE}(\mathbf{U}_{N(m)}^{SE}))']^{-1}[\mathbf{U}_{N(m)}^{SE} - \mathcal{A}^{SE}(\mathbf{U}_{N(m)}^{SE})].$$

We know that

$$
\begin{aligned}
[(\mathcal{A}^{SE}(\mathbf{U}_N^{SE}))']_{k,l} &= h\psi'(lh)\Big[J(l,h)(kh)\frac{\partial \mathcal{K}_1}{\partial u}(t_k^{SE},t_l^{SE},u_k^{SE}) \\
&\quad +J(l,h)(\phi_k^{SE})\frac{\partial \mathcal{K}_2}{\partial u}(t_k^{SE},t_l^{SE},u_k^{SE})\Big],\ k,l=-N,\dots,N.
\end{aligned}
$$

Using Lemma 3.1 and differentiability of $\mathcal{K}_1$ and $\mathcal{K}_2$, there exists $c>0$ so that $|[(\mathcal{A}^{SE}(\mathbf{U}_N^{SE}))']_{k,l}| < ch$ and then $\|(\mathcal{A}^{SE}(\mathbf{U}_N^{SE}))'\| < 1$ whenever h is sufficiently small. In other words, there is a $\overline{h}>0$ so that for any $h<\overline{h}$ the matrix $(I-(\mathcal{A}^{SE}(\mathbf{U}_N^{SE}))')$ has a uniformaly bounded inverse.

The conclusion is straightforwardly achievable by applying [2, Theorem 5.4.1] and the above discussion. □

In the following theorem, we summarize the conclusions of theorems proved in this section.

Theorem 5.6. *Assume that u is an isolated solution of (4.1). Furthermore, $\mathcal{U}_N^{SE}$ and $\mathcal{U}_{N,(m)}^{SE}$ are the solutions of (4.3) and (5.3), respectively. Suppose that hypotheses of Theorems 5.3 and 5.5 are satisfied. Then there exists a positive constant $C(m)$ independent of N and dependant on m such that*

$$\|u-\mathcal{U}_{N,(m)}^{SE}\| \leqslant C(m)\sqrt{N}\ln N e^{-\sqrt{\pi d\alpha N}}.$$

Proof. The conclusion is obtained by using the triangular inequality and conclusions of Theorems 5.3 and 5.5. □

Theorem 5.7. *Assume that u is an isolated solution of (4.1). Furthermore, $\mathcal{U}_N^{DE}$ and $\mathcal{U}_{N,(m)}^{DE}$ are the solutions of (4.6) and (5.3), respectively. Suppose that hypotheses of Theorems 5.4 and 5.5 are satisfied. Then there exists a positive constant $C(m)$ independent of N and dependant on m such that*

$$\|u-\mathcal{U}_{N,(m)}^{DE}\| \leqslant C(m)e^{-\pi dN/\ln(2dN/\alpha)}.$$

Proof. The proof of this theorem goes almost in the same way as in the SE case. □

6. Illustrative examples

In this section, we illustrate the theoretical results of the previous sections by the following three exampls with $\tau=1$. The numerical experiments are implemented in MATLAB.

It is assumed that $\alpha=1$. The d values are $\frac{\pi}{2}$ and $\frac{\pi}{4}$ for the SE-sinc and DE-sinc methods, respectively. The errors of the two methods for different N are reported. These tables show that increasing N the error significantly is reduced. As expected, the tables show that the convergence rate of the DE-sinc method is faster than the SE-sinc scheme.

Example 6.1. Consider the delay integral equation (see [3, Example 4.1])

$$y(t) = g(t) + \int_0^t K_1(t,s)y(s)ds + \int_0^{t-\tau} K_2(t,s)y(s)ds, \quad t \in (0,5],$$
$$\varphi(t) = \sin(t) + 1, \quad t \in [-1,0],$$

with $K_1(t,s) = e^{s-t}$, $K_2(t,s) = t\sin(s)$ and g is chosen so that its exact solution is $y(t) = \sin(t) + 1$. Table 1 shows the absolute error of sinc method and the numerical results of [3] for this example.

Table 1. Values of $\|E\|_\infty$ for Example 6.1.

N	30	50	60	75
SE	1.16E-4	1.85E-6	1.38E-7	1.84E-9
DE	5.68E-7	5.19E-12	1.74E-13	1.54E-13
Result in [3]	1E-3	1.41E-5	1.65E-6	1.39E-7

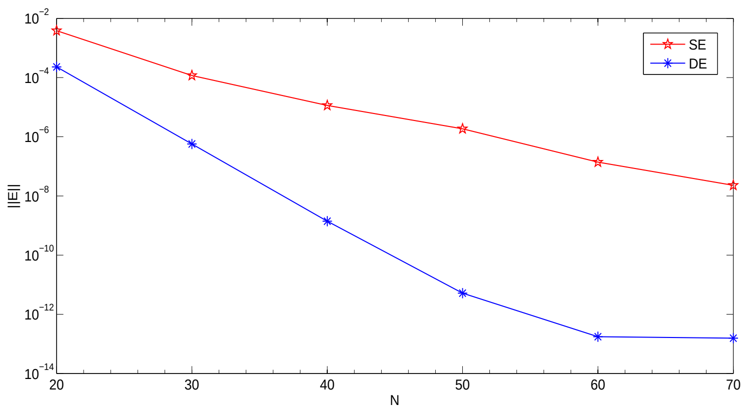

Figure 1. Results for Example 6.1.

Example 6.2. Consider the following equation (see [3, Example 4.2])

$$y(t) = g(t) + \int_0^t K_1(t,s)y(s)ds + \int_0^{t-\tau} K_2(t,s)y(s)ds, \quad t \in (0,5],$$
$$\varphi(t) = \frac{1+t}{e^t}, \quad t \in [-1,0],$$

with $K_1(t,s) = \sin(s+t)$, $K_2(t,s) = \frac{s}{1+t}$ and g is chosen so that the exact solution $y(t) = \frac{1+t}{e^t}$. Table 2 shows the absolute error of DE-sinc method for $N = 50$ and the numerical results of [3] for this example.

TABLE 2. Comparison of results in [3] to sinc method

t	Result in [3]	SE	DE
1	3.5E-5	3.20E-9	1.13E-15
2	1.43E-5	4.35E-9	3.33E-16
3	2.83E-5	3.24E-9	0
4	1.18E-4	1.92E-9	1.15E-15
5	9.3E-5	1.42E-8	4.99E-16

Example 6.3. We consider the following nonvanishing delay Volterra integral equation

$$y(t) = g(t) + \int_0^{t-\tau} (s+t)[y(s)]^3 ds, \quad t \in (0,3],$$
$$\varphi(t) = t, \quad t \in [-1,0],$$

$g(t)$ chosen so that its exact solution is $y(t) = t^2 - t$. The absolute error of SE-sinc and DE-sinc methods for different values of N are reported in Table 3. In this example, Newton's method is iterated until the accuracy 10^{-8} is obtained.

TABLE 3. Values of $\|E\|_\infty$ for Example 6.3.

N	20	30	40	50	60
SE	3.6794E-3	8.3369E-4	3.3180E-5	4.7941E-7	1.5158E-7
DE	4.4596E-5	1.8916E-8	2.2818E-10	3.2987E-11	2.9460E-12

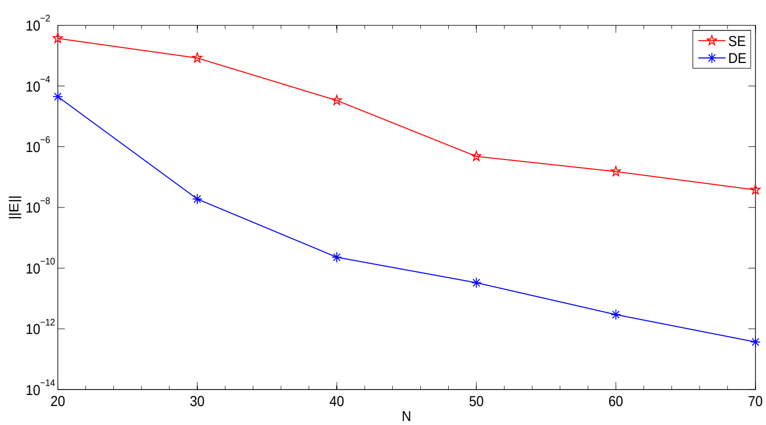

FIGURE 2. Results for Example 6.3.

7. Conclusion

We propose two numerical methods based on the sinc function, the SE-sinc and DE-sinc, in order to solve the linear and nonlinear delay integral equations (1.1) and (1.2), where θ is a general function. Our methods have been shown theoretically and numerically that they are extremely accurate and achieve exponential convergence with respect to N. These two methods have some strengths and weaknesses. In comparison with each other, as the theorems show, it is understood that the SE-sinc formulas are applicable to larger classes of functions than the DE-sinc formulas, whereas the DE-sinc formulas are more efficient for well-behaved functions.

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(Leila Shiri) Department of Mathematics, University of Mohagheghe Ardabili, 56199-11367, Ardabil, Iran.

E-mail address: `l.shiri@uma.ac.ir`

(Mohammad Zarebnia) Department of Mathematics, University of Mohagheghe Ardabili, 56199-11367, Ardabil, Iran.

E-mail address: `zarebnia@uma.ac.ir`