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FUNCTIONAL IDENTITIES OF DEGREE 2 IN CSL ALGEBRAS

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Abstract. Let $\mathscr{L}$ be a commutative subspace lattice generated by finite many commuting independent nests on a complex separable Hilbert space $\mathbf{H}$ with $\dim \mathbf{H} \geq 3$, Alg$\mathscr{L}$ the CSL algebra associated with $\mathscr{L}$ and $\mathscr{M}$ be an algebra containing Alg$\mathscr{L}$. This article is aimed at describing the form of additive mapppings $F_1, F_2, G_1, G_2\colon \mathrm{Alg}\mathscr{L} \longrightarrow \mathscr{M}$ satisfying functional identity $F_1(X)Y + F_2(Y)X + XG_2(Y) + YG_1(X) = 0$ for all $X, Y \in \mathrm{Alg}\mathscr{L}$. As an application generalized inner biderivations and commuting additive mappings are determined.
Keywords: Functional identity, CSL algebra, generalized inner biderivation, commuting mapping.
MSC(2010): Primary: 16R60; Secondary: 47L35.

1. Introduction

A *functional identity* (FI) on an algebra $\mathscr{A}$ is, roughly speaking, an identity holding in $\mathscr{A}$ which involves some mappings on $\mathscr{A}$. The involved mappings can be looked on as "functions". If the identity, besides mappings (or functions), also includes some fixed elements of $\mathscr{A}$, then one has the notion of *generalized functional identity* (GFI). The usual objective in studying (generalized) functional identities is to "solve" these functions or, in case this is not possible, to determine the structure of the algebra admitting the given FI and to obtain information concerning the intrinsic structure of the algebra. That such identities extend the notions of polynomial identity (PI) and generalized polynomial identity (GPI) is clear, but it seems that, especially from the perspective of possible applications, the theory of FI's shows its strength in the non-PI case (a similar remark applies to the GFI theory versus GPI's). In certain classes of algebras, FIs have only trivial solutions, that is, solutions which do not depend on some structural properties of the algebra but are merely consequences of algebra axioms and formal calculations. We call them the *standard solutions*.

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In case there exists a non-standard solution, it reflects structural properties of the algebra. The first functional identities were introduced in the early 90's by Brešar as an attempt to unify several results on centralizing mappings. Then the theory quickly developed through a decade until reaching an ultimate stage that covers and unifies a number of results previously obtained. It turned out to be the right tool in proving several conjectures formulated by Herstein in 1961 concerning the description of Lie-type mappings in associative algebras. For a clear and full account on the development of the theory of functional identities and their applications, we refer the reader to the technical literature [5].

Let $\mathscr{A}$ be a nonempty subset of a unital algebra $\mathscr{B}$ with center $\mathscr{Z}(\mathscr{B})$. Let $F_1, F_2, G_1, G_2\colon \mathscr{A} \longrightarrow \mathscr{B}$ be mappings satisfying the identity

$$F_1(X)Y + F_2(Y)X + XG_2(Y) + YG_1(X) = 0 \quad \text{for all} \quad X, Y \in \mathscr{A}. \tag{1.1}$$

Identities of this kind are called *functional identities of degree* 2, because they involve two variables X and Y. This is one of the most basic functional identities, which were widely studied in (semi-)prime algebras. An ordinary task in the theory of functional identities is to characterize set-theoretic mappings satisfying certain identities. Therefore, the above mappings F_1, F_2, G_1, G_2 are reasonably considered as unknowns. It is not difficult to check that mappings of the form

$$\begin{aligned} F_1(X) &= XQ_1 + \Gamma_1(X), \\ F_2(X) &= XQ_2 + \Gamma_2(X), \\ G_1(X) &= -Q_2X - \Gamma_1(X), \\ G_2(X) &= -Q_1X - \Gamma_2(X), \end{aligned} \tag{1.2}$$

where $Q_1, Q_2 \in \mathscr{B}, \Gamma_1, \Gamma_2\colon \mathscr{A} \longrightarrow \mathscr{Z}(\mathscr{B})$ are exactly a solution of (1.1). According to [5], the solution of the form (1.2) is called a *standard solution* of (1.1). It was Brešar who initiated the study of functional identity of type (1.1) in the setting of prime algebras [4]. Let $\mathscr{I}$ be an ideal of a prime algebra $\mathscr{A}$ with extended centroid $\mathscr{C}$. If the mappings $F_1, F_2, G_1, G_2\colon \mathscr{I} \longrightarrow \mathscr{A}$ are additive modulo $\mathscr{C}$, then (1.1) has only standard solution (1.2). Using this result he characterized the form of generalized inner biderivations of prime algebras [4, Theorem 4.7]. Zhang et al. [15] investigated the functional identity of type (1.1) in nest algebras and discovered that the functional identity (1.1) has only the standard solution in a certain class of nest algebras. It should be pointed out that the general theory of functional identities, which was elaborated in [5], can not be applicable to the background of triangular algebras and operator algebras, since these algebras may not be d-free. It's not so bad, and it's not the end. Beidar, Brešar and Chebotar's joint work [3] puts the matter in a new light. They considered certain functional identities on upper triangular matrix algebras. In [7], Cheung independently considered the question of when all commuting mappings of a triangular algebra take the so-called "proper" form. Later, several problems on certain types of mappings on triangular rings and algebras have been studied, where some special examples of functional identities appear. In a recent article [9], Eremita studied the functional identity of type (1.1) in triangular algebras. He was in an effort to describe the form of additive mappings $F_1, F_2, G_1, G_2 \colon \mathscr{R} \longrightarrow \mathscr{R}$ satisfying (1.1) when a triangular ring $\mathscr{R}$ satisfies some additional conditions, see [9, Theorem 2.2]. And then, Eremita continued to generalize this result in a surprising manner [10]. He used the notion of the maximal left ring of quotients and described the form of mappings $F_1, F_2, G_1, G_2 \colon \mathscr{R} \longrightarrow \mathscr{R}$ satisfying (1.1) for a much wider class of triangular rings.

Motivated by the afore-mentioned results, we are concerned with CSL algebras in this article. As a matter of fact, CSL algebras are exactly the reflexive algebras with commutative invariant projection lattice, which were introduced by Arveson [2] and have been extensively studied since then(see [1,6,11,13,14], etc). Most of existing work are contributed to linear mappings of CSL algebras, such as derivations, higher derivations, Jordan derivations, Lie derivations, isomorphisms, Jordan isomorphisms et al. We will study the functional identity of type (1.1) in a CSL algebra $\mathrm{Alg}\mathscr{L}$. The main purpose of this paper is to determine the form of mappings $F_1, F_2, G_1, G_2 \colon \mathrm{Alg}\mathscr{L} \longrightarrow \mathscr{M}$, where $\mathscr{M}$ is an algebra containing $\mathrm{Alg}\mathscr{L}$.

We now fix some notation and terminology. Let $\mathbf{H}$ be a complex separable Hilbert space, and $\mathbf{B}(\mathbf{H})$ the algebra of all bounded linear operators on $\mathbf{H}$. The terms *operator* and *projection* on $\mathbf{H}$ will mean "bounded linear mapping of $\mathbf{H}$ into itself" and "self-adjoint idempotent operator on $\mathbf{H}$", respectively. A *subspace lattice* $\mathscr{L}$ of $\mathbf{H}$ is a family of orthogonal projections on $\mathbf{B}(\mathbf{H})$ which contains the zero operator $\{0\}$ and the identity operator I, and is closed under the usual lattice operations $\vee$ and $\wedge$. A *nest* is a totally ordered subspace lattice; a *commutative subspace lattice* (*CSL* in brief) is a subspace lattice in which all projections commute pairwise. Given a subspace lattice $\mathscr{L}$ of $\mathbf{H}$, the corresponding *subspce lattice algebra* $\mathrm{Alg}\,\mathscr{L}$ is defined to be the collection of operators in $\mathbf{B}(\mathbf{H})$ which leave invariant each element in $\mathscr{L}$, that is,

$$\mathrm{Alg}\,\mathscr{L} = \{A \in \mathbf{B}(\mathbf{H}) \colon PAP = AP \ \text{ for all } P \in \mathscr{L}\}.$$

$\mathrm{Alg}\,\mathscr{L}$ is called a *CSL algebra* if the subspace lattice $\mathscr{L}$ is a CSL; $\mathrm{Alg}\,\mathscr{L}$ is called a *nest algebra* if the subspace lattice $\mathscr{L}$ is a nest [8]. Dually, for a subalgebra $\mathscr{A}$ of $\mathbf{B}(\mathbf{H})$, we define the invariant subspace lattice $\mathrm{Lat}\,\mathscr{A}$ to be the collection of orthogonal projections which are left invariant by each operator in $\mathscr{A}$. An algebra $\mathscr{A}$ is called *reflexive* if $\mathrm{Alg}\,\mathrm{Lat}\,\mathscr{A} = \mathscr{A}$; a lattice is said to be *reflexive* if $\mathrm{Lat}\,\mathrm{Alg}\,\mathscr{L} = \mathscr{L}$. Each CSL is reflexive [2].

Let $\mathscr{L}$ be a nest and $P \in \mathscr{L}$, we let $P_+ = \inf\{Q \in \mathscr{L} \colon Q > P\}$ and $P_- = \sup\{Q \in \mathscr{L} \colon Q < P\}$. If $P, Q \in \mathscr{L}$ with $Q < P$, then the projection $E = P - Q$ is called an *interval* from $\mathscr{L}$. A collection of nests $\{\mathscr{L}_1, \mathscr{L}_2, \dots, \mathscr{L}_n\}$ is said to be *independent* if the product $\prod_i^n E_i \neq 0$ whenever E_i is an interval from

$\mathscr{L}_i$. We say that $\mathscr{L}$ is an independent finite-width CSL if $\mathscr{L}$ is generated by finitely many commuting independent nests. By [12, Lemma 1.1], we know that the commutant of $\operatorname{Alg}\mathscr{L}$ is the von Neumann algebra generated by reducing projections of $\operatorname{Alg}\mathscr{L}$ in $\mathscr{L}$. It follows that the commutant of $\operatorname{Alg}\mathscr{L}$ is $\mathbb{C}I$ if $\mathscr{L}$ is an independent finite-width CSL.

2. **Functional identities of degree 2 in CSL algebras**

Let $\mathscr{L}$ be a commutative subspace lattice generated by independent nests. We assume that $\mathscr{L}$ is nontrivial in the present study. Let $P \in \mathscr{L}$ be a nontrivial projection. Denote $\mathscr{A}_{11} = P(\operatorname{Alg}\mathscr{L})P$, $\mathscr{A}_{12} = P(\operatorname{Alg}\mathscr{L})P^{\perp}$ and $\mathscr{A}_{22} = P^{\perp}(\operatorname{Alg}\mathscr{L})P^{\perp}$. Then $\operatorname{Alg}\mathscr{L} = \mathscr{A}_{11} + \mathscr{A}_{12} + \mathscr{A}_{22}$(see [14]).

The following lemmas will be used in the sequel.

Lemma 2.1 ([14, Lemma 2.3])**.** *The commutant of $\mathscr{A}_{11}$ in $\mathbf{B}(P\mathbf{H})$ and the commutant of $\mathscr{A}_{22}$ in $\mathbf{B}(P^{\perp}\mathbf{H})$ are $\mathbb{C}P$ and $\mathbb{C}P^{\perp}$, respectively.*

Lemma 2.2 ([14, Lemma 2.4])**.** *Let $X \in \mathbf{B}(\mathbf{H})$. We have (a) If $XA_{12} = 0$ for all $A_{12} \in \mathscr{A}_{12}$, then $XP = 0$; (b) If $A_{12}X = 0$ for all $A_{12} \in \mathscr{A}_{12}$, then $P^{\perp}X = 0$.*

Lemma 2.3. *Let $\mathscr{L}$ be an independent finite-width* CSL *on a complex separable Hilbert space* $\mathbf{H}$ *with* $\dim \mathbf{H} \geq 3$. *Suppose that $f, g : P(\operatorname{Alg}\mathscr{L})P^{\perp} \to \mathbb{C}$ are arbitrary maps such that*

$$f(PXP^{\perp})PYP^{\perp} + g(PYP^{\perp})PXP^{\perp} = 0 \tag{2.1}$$

for all $X, Y \in \operatorname{Alg}\mathscr{L}$. Then $f = g = 0$.

Proof. By (2.1), we see that

$$f(PXP^{\perp})PZPYP^{\perp} + g(PYP^{\perp})PZPXP^{\perp} = 0$$

and

$$f(PXP^{\perp})PZPYP^{\perp} + g(PZPYP^{\perp})PXP^{\perp} = 0$$

for all $X, Y, Z \in \operatorname{Alg}\mathscr{L}$. Comparing the above two relations, we obtain

$$g(PYP^{\perp})PZPXP^{\perp} - g(PZPYP^{\perp})PXP^{\perp} = 0$$

for all $X, Y, Z \in \operatorname{Alg}\mathscr{L}$. Hence by Lemma 2.2, one sees that

$$g(PYP^{\perp})PZP - g(PZPYP^{\perp})P = 0 \tag{2.2}$$

for all $Y, Z \in \operatorname{Alg}\mathscr{L}$. Likewise, by (2.1), we can get

$$f(PXP^{\perp})PYP^{\perp}ZP^{\perp} + g(PYP^{\perp})PXP^{\perp}ZP^{\perp} = 0$$

and

$$f(PXP^{\perp})PYP^{\perp}ZP^{\perp} + g(PYP^{\perp}ZP^{\perp})PXP^{\perp} = 0$$

for all $X, Y, Z \in \operatorname{Alg}\mathscr{L}$. It follows that

$$g(PYP^{\perp})PXP^{\perp}ZP^{\perp} - g(PYP^{\perp}ZP^{\perp})PXP^{\perp} = 0$$

for all $X, Y, Z \in \operatorname{Alg}\mathscr{L}$. Applying Lemma 2.2 again, we obtain

$$g(PYP^{\perp})P^{\perp}ZP^{\perp} - g(PYP^{\perp}ZP^{\perp})P^{\perp} = 0 \tag{2.3}$$

for all $Y, Z \in \operatorname{Alg}\mathscr{L}$.

Since $\dim \mathbf{H} \geq 3$, we have $\dim P^{\perp}\mathbf{H} \geq 2$ or $\dim P\mathbf{H} \geq 2$. Suppose that $\dim P^{\perp}\mathbf{H} \geq 2$. If $g(PYP^{\perp}) \neq 0$ for some $Y \in \operatorname{Alg}\mathscr{L}$, then by equation (2.3), $P^{\perp}ZP^{\perp} \in \mathbb{C}P^{\perp}$ for all $Z \in \operatorname{Alg}\mathscr{L}$, and so the commutant of $\mathscr{A}_{22}$ in $\mathbf{B}(P^{\perp}\mathbf{H})$ is $\mathbf{B}(P^{\perp}\mathbf{H}) \neq \mathbb{C}P^{\perp}$, which contradicts the result of Lemma 2.1. Hence $g(PYP^{\perp}) = 0$ for all $Y \in \operatorname{Alg}\mathscr{L}$. Suppose that $\dim P\mathbf{H} \geq 2$. If $g(PYP^{\perp}) \neq 0$ for some $Y \in \operatorname{Alg}\mathscr{L}$, then by equation (2.2), $PZP \in \mathbb{C}P$ for all $Z \in \operatorname{Alg}\mathscr{L}$, and so the commutant of $\mathscr{A}_{11}$ in $\mathbf{B}(P\mathbf{H})$ is $\mathbf{B}(P\mathbf{H}) \neq \mathbb{C}P$, which is also a contradiction. So $g(PYP^{\perp}) = 0$ for all $Y \in \operatorname{Alg}\mathscr{L}$. From (2.1) we know that $f = 0$ as well. ☐

Let $\mathscr{A}$ be an arbitrary subalgebra of $\mathscr{M}$. A map $F : \mathscr{A} \to \mathscr{M}$ is said to be additive modulo $\mathbb{C}I$, if $F(X+Y) - F(X) - F(Y) \in \mathbb{C}I$ for all $X, Y \in \mathscr{A}$. For each map $F : \mathscr{A} \to \mathscr{M}$ we define a map $\Delta_{n,F} : \mathscr{A}^n \to \mathscr{M}$ by $\Delta_{n,F}(X_1, \ldots, X_n) := F(X_1 + \cdots + X_n) - F(X_1) - \cdots - F(X_n)$. Obviously, $\Delta_{n,F} \subseteq \mathbb{C}I$, if F is additive modulo $\mathbb{C}I$. In order to prove our main theorem we need the following result.

Lemma 2.4. *Let $\mathscr{L}$ be an independent finite-width* CSL *on a complex separable Hilbert space $\mathbf{H}$ with $\dim \mathbf{H} \geq 3$, and $\mathscr{M}$ an algebra containing $\operatorname{Alg}\mathscr{L}$. Suppose that maps $F_1, F_2, G_1, G_2 : \operatorname{Alg}\mathscr{L} \to \mathscr{M}$ are additive modulo $\mathbb{C}I$ such that*

$$F_1(X)Y + F_2(Y)X + XG_2(Y) + YG_1(X) = 0$$

for all $X, Y \in \operatorname{Alg}\mathscr{L}$. If we have

$$\begin{aligned} P^{\perp}F_1(PXP^{\perp})P^{\perp} - PG_1(PXP^{\perp})P \in \mathbb{C}I, \\ P^{\perp}F_2(PXP^{\perp})P^{\perp} - PG_2(PXP^{\perp})P \in \mathbb{C}I, \end{aligned} \tag{2.4}$$

then there exist $U_1, U_2, Q_1, Q_2, R_1, R_2 \in \mathscr{M}$ and maps $\Gamma_1, \Gamma_2 : \operatorname{Alg}\mathscr{L} \to \mathbb{C}I$ such that $U_1 + U_2 = R_1 + R_2 \in \mathbb{C}I$, $U_i[X,Y] = [X,Y]R_i, i = 1, 2,$ and

$$\begin{aligned} F_1(X) &= XQ_1 - U_1X + \Gamma_1(X), \\ F_2(X) &= XQ_2 - U_2X + \Gamma_2(X), \\ G_1(X) &= XR_2 - Q_2X - \Gamma_1(X), \\ G_2(X) &= XR_1 - Q_1X - \Gamma_2(X) \end{aligned} \tag{2.5}$$

for all $X, Y \in \operatorname{Alg}\mathscr{L}$.

Proof. Taking $X = I$ or $Y = I$ into (1.1) yields

$$\begin{aligned} F_1(X) + G_1(X) = -F_2(I)X - XG_2(I), \\ F_2(X) + G_2(X) = -F_1(I)X - XG_1(I) \end{aligned} \tag{2.6}$$

for all $X \in \operatorname{Alg}\mathscr{L}$.

Setting $Y = P$ in (1.1) and multiplying by $P^\perp$ from the left hand side, we obtain

$$P^\perp F_1(X)P = -P^\perp F_2(P)X - P^\perp XG_2(P) \tag{2.7}$$

for all $X \in \operatorname{Alg}\mathscr{L}$. Analogously, we get

$$P^\perp F_2(X)P = -P^\perp F_1(P)X - P^\perp XG_1(P), \tag{2.8}$$

$$P^\perp G_1(X)P = -F_2(P^\perp)XP - XG_2(P^\perp)P, \tag{2.9}$$

$$P^\perp G_2(X)P = -F_1(P^\perp)XP - XG_1(P^\perp)P, \tag{2.10}$$

$$PF_1(X)P^\perp = -PF_2(P^\perp)X - PXG_2(P^\perp), \tag{2.11}$$

$$PF_2(X)P^\perp = -PF_1(P^\perp)X - PXG_1(P^\perp), \tag{2.12}$$

$$PG_1(X)P^\perp = -F_2(P)XP^\perp - XG_2(P)P^\perp, \tag{2.13}$$

$$PG_2(X)P^\perp = -F_1(P)XP^\perp - XG_1(P)P^\perp \tag{2.14}$$

for all $X \in \operatorname{Alg}\mathscr{L}$.

Next we will consider the following different cases of (1.1) for the proof of the lemma.

Case 1. Replacing X by PXP and Y by $P^\perp YP^\perp$ in (1.1) gives

$$F_1(PXP)P^\perp YP^\perp + F_2(P^\perp YP^\perp)PXP + PXPG_2(P^\perp YP^\perp) + P^\perp YP^\perp G_1(PXP) = 0 \tag{2.15}$$

for all $X, Y \in \operatorname{Alg}\mathscr{L}$. Multiplying the above relation by $P^\perp$ from the left hand side, we obtain

$$P^\perp F_1(PXP)P^\perp YP^\perp + P^\perp F_2(P^\perp YP^\perp)PXP + P^\perp YP^\perp G_1(PXP) = 0,$$

and consequently

$$P^\perp F_1(PXP)P^\perp YP^\perp + P^\perp YP^\perp G_1(PXP)P^\perp = 0 \tag{2.16}$$

for all $X, Y \in \operatorname{Alg}\mathscr{L}$.

Similarly, multiplying (2.15) by P from the left hand side gives

$$PF_1(PXP)P^\perp YP^\perp + PF_2(P^\perp YP^\perp)PXP + PXPG_2(P^\perp YP^\perp) = 0,$$

from which we can get

$$PF_2(P^\perp YP^\perp)PXP + PXPG_2(P^\perp YP^\perp)P = 0 \tag{2.17}$$

for all $X, Y \in \operatorname{Alg}\mathscr{L}$.

Case 2. Replacing X by $P^\perp XP^\perp$ and Y by PYP, we obtain

$$F_1(P^\perp XP^\perp)PYP + F_2(PYP)P^\perp XP^\perp + P^\perp XP^\perp G_2(PYP) + PYPG_1(P^\perp XP^\perp) = 0 \tag{2.18}$$

for all $X, Y \in \operatorname{Alg}\mathscr{L}$. Multiplying the above relation by $P^\perp$ from the left hand side gives

$$P^\perp F_1(P^\perp XP^\perp)PYP + P^\perp F_2(PYP)P^\perp XP^\perp + P^\perp XP^\perp G_2(PYP) = 0,$$

and then we have

$$P^\perp F_2(PYP)P^\perp XP^\perp + P^\perp XP^\perp G_2(PYP)P^\perp = 0 \tag{2.19}$$

for all $X, Y \in \operatorname{Alg}\mathscr{L}$.

Similarly, multiplying (2.18) by P from the left hand side gives

$$PF_1(P^\perp XP^\perp)PYP + PF_2(PYP)P^\perp XP^\perp + PYPG_1(P^\perp XP^\perp) = 0,$$

and hence

$$PF_1(P^\perp XP^\perp)PYP + PYPG_1(P^\perp XP^\perp)P = 0 \tag{2.20}$$

for all $X, Y \in \operatorname{Alg}\mathscr{L}$.

Remark 2.5. Now in view of (2.16), (2.17), (2.19), (2.20) and Lemma 2.1, we arrive at

$$\begin{aligned} P^\perp F_1(PXP)P^\perp &= -P^\perp G_1(PXP)P^\perp \in \mathbb{C}P^\perp \\ PF_2(P^\perp XP^\perp)P &= -PG_2(P^\perp XP^\perp)P \in \mathbb{C}P \\ P^\perp F_2(PXP)P^\perp &= -P^\perp G_2(PXP)P^\perp \in \mathbb{C}P^\perp \\ PF_1(P^\perp XP^\perp)P &= -PG_1(P^\perp XP^\perp)P \in \mathbb{C}P \end{aligned} \tag{2.21}$$

for all $X \in \operatorname{Alg}\mathscr{L}$. We might as well use $\mathbb{C}_{P^\perp G_1(PXP)P^\perp}P^\perp$ to denote $P^\perp G_1(PXP)P^\perp$, and so on.

Case 3. Replacing X by PXP and Y by $PYP^\perp$ in (1.1) yields

$$F_1(PXP)PYP^\perp + F_2(PYP^\perp)PXP + PXPG_2(PYP^\perp) + PYP^\perp G_1(PXP) = 0,$$

which further yields

$$P^\perp F_1(PXP)PYP^\perp = P^\perp F_2(PYP^\perp)PXP = 0, \tag{2.22}$$

$$PF_2(PYP^\perp)PXP + PXPG_2(PYP^\perp)P + PYP^\perp G_1(PXP)P = 0 \tag{2.23}$$

and

$$PF_1(PXP)PYP^\perp + PXPG_2(PYP^\perp)P^\perp + PYP^\perp G_1(PXP)P^\perp = 0 \tag{2.24}$$

for all $X, Y \in \operatorname{Alg}\mathscr{L}$.

Furthermore, putting (2.14) into the last relation and applying (2.21), we can compute that

$$\begin{aligned}0 =&PF_1(PXP)PYP^{\perp} + PXP(-F_1(P)PYP^{\perp} - PYP^{\perp}G_1(P)P^{\perp})\\&+ PYP^{\perp}G_1(PXP)P^{\perp}\\=&PF_1(PXP)PYP^{\perp} - PXPF_1(P)PYP^{\perp} - PXPYP^{\perp}G_1(P)P^{\perp}\\&+ PYP^{\perp}G_1(PXP)P^{\perp}\\=&(PF_1(PXP)P - PXPF_1(P)P - \mathbb{C}_{P^{\perp}G_1(P)P^{\perp}}PXP\\&+ \mathbb{C}_{P^{\perp}G_1(PXP)P^{\perp}}P)PYP^{\perp}\end{aligned}$$

for all $X, Y \in \operatorname{Alg}\mathscr{L}$. By Lemma 2.2 we immediately get

$$PF_1(PXP)P = PXPF_1(P)P + \mathbb{C}_{P^{\perp}G_1(P)P^{\perp}}PXP - \mathbb{C}_{P^{\perp}G_1(PXP)P^{\perp}}P \tag{2.25}$$

for all $X \in \operatorname{Alg}\mathscr{L}$. Analogously, we obtain

$$PF_2(PXP)P = PXPF_2(P)P + \mathbb{C}_{P^{\perp}G_2(P)P^{\perp}}PXP - \mathbb{C}_{P^{\perp}G_2(PXP)P^{\perp}}P \tag{2.26}$$

for all $X \in \operatorname{Alg}\mathscr{L}$.

Case 4. Replacing X by $PXP^{\perp}$ and Y by PYP in (1.1) gives

$$F_1(PXP^{\perp})PYP + F_2(PYP)PXP^{\perp} + PXP^{\perp}G_2(PYP) + PYPG_1(PXP^{\perp}) = 0$$

for all $X, Y \in \operatorname{Alg}\mathscr{L}$. It follows that

$$P^{\perp}F_2(PYP)PXP^{\perp} = 0, \; P^{\perp}F_1(PXP^{\perp})PYP = 0 \tag{2.27}$$

and

$$PF_1(PXP^{\perp})PYP + PXP^{\perp}G_2(PYP)P + PYPG_1(PXP^{\perp})P = 0 \tag{2.28}$$

for all $X, Y \in \operatorname{Alg}\mathscr{L}$.

Case 5. Replacing X by $PXP^{\perp}$ and Y by $P^{\perp}YP^{\perp}$ in (1.1), we obtain

$$\begin{aligned}0 =&F_1(PXP^{\perp})P^{\perp}YP^{\perp} + F_2(P^{\perp}YP^{\perp})PXP^{\perp}\\&+ PXP^{\perp}G_2(P^{\perp}YP^{\perp}) + P^{\perp}YP^{\perp}G_1(PXP^{\perp})\end{aligned}$$

for all $X, Y \in \operatorname{Alg}\mathscr{L}$. Thus we have

$$P^{\perp}YP^{\perp}G_1(PXP^{\perp})P = 0, \; PXP^{\perp}G_2(P^{\perp}YP^{\perp})P = 0 \tag{2.29}$$

and

$$P^{\perp}F_1(PXP^{\perp})P^{\perp}YP^{\perp} + P^{\perp}F_2(P^{\perp}YP^{\perp})PXP^{\perp} + P^{\perp}YP^{\perp}G_1(PXP^{\perp})P^{\perp} = 0$$

for all $X, Y \in \operatorname{Alg}\mathscr{L}$.

Case 6. Replacing X by $P^{\perp}XP^{\perp}$ and Y by $PYP^{\perp}$ in (1.1) leads to

$$\begin{aligned}0 =&F_1(P^{\perp}XP^{\perp})PYP^{\perp} + F_2(PYP^{\perp})P^{\perp}XP^{\perp}\\&+ P^{\perp}XP^{\perp}G_2(PYP^{\perp}) + PYP^{\perp}G_1(P^{\perp}XP^{\perp}),\end{aligned}$$

which implies that

$$PYP^{\perp}G_1(P^{\perp}XP^{\perp})P = 0,\ P^{\perp}XP^{\perp}G_2(PYP^{\perp})P = 0, \tag{2.30}$$

$$P^{\perp}F_1(P^{\perp}XP^{\perp})PYP^{\perp}+P^{\perp}F_2(PYP^{\perp})P^{\perp}XP^{\perp}+P^{\perp}XP^{\perp}G_2(PYP^{\perp})P^{\perp} = 0 \tag{2.31}$$

and

$$PF_1(P^{\perp}XP^{\perp})PYP^{\perp} + PF_2(PYP^{\perp})P^{\perp}XP^{\perp} + PYP^{\perp}G_1(P^{\perp}XP^{\perp})P^{\perp} = 0 \tag{2.32}$$

for all $X, Y \in \operatorname{Alg}\mathscr{L}$.

Taking $X = PYP^{\perp}$ into (2.12), we can further get

$$PF_2(PYP^{\perp})P^{\perp} = -PF_1(P^{\perp})PYP^{\perp} - PYP^{\perp}G_1(P^{\perp}) \tag{2.33}$$

for all $Y \in \operatorname{Alg}\mathscr{L}$. Combining (2.32) with (2.33) gives

$$\begin{aligned} 0 =& PF_1(P^{\perp}XP^{\perp})PYP^{\perp} + (-PF_1(P^{\perp})PYP^{\perp} - PYP^{\perp}G_1(P^{\perp}))P^{\perp}XP^{\perp} \\ &+ PYP^{\perp}G_1(P^{\perp}XP^{\perp})P^{\perp} \\ =& PYP^{\perp}(\mathbb{C}_{PF_1(P^{\perp}XP^{\perp})P}P^{\perp} - \mathbb{C}_{PF_1(P^{\perp})P}P^{\perp}XP^{\perp} - P^{\perp}G_1(P^{\perp})P^{\perp}XP^{\perp} \\ &+ P^{\perp}G_1(P^{\perp}XP^{\perp})P^{\perp}) \end{aligned}$$

for all $X, Y \in \operatorname{Alg}\mathscr{L}$. It follows from Lemma 2.2 that

$$\begin{aligned} P^{\perp}G_1(P^{\perp}XP^{\perp})P^{\perp} =& -\mathbb{C}_{PF_1(P^{\perp}XP^{\perp})P}P^{\perp} + \mathbb{C}_{PF_1(P^{\perp})P}P^{\perp}XP^{\perp} \\ &+ P^{\perp}G_1(P^{\perp})P^{\perp}XP^{\perp} \end{aligned} \tag{2.34}$$

for all $X \in \operatorname{Alg}\mathscr{L}$.

In addition, taking into account (2.22), (2.27), (2.29) and (2.30) and applying Lemma 2.2 again yields

$$\begin{aligned} &P^{\perp}F_1(PXP)P = 0, \quad P^{\perp}F_2(PXP)P = 0, \\ &P^{\perp}G_2(P^{\perp}XP^{\perp})P = 0, \quad P^{\perp}G_1(P^{\perp}XP^{\perp})P = 0 \end{aligned} \tag{2.35}$$

for all $X \in \operatorname{Alg}\mathscr{L}$.

Case 7. Replacing X by $P^{\perp}XP^{\perp}$ and Y by $P^{\perp}YP^{\perp}$ in (1.1) gives

$$\begin{aligned} 0 =& F_1(P^{\perp}XP^{\perp})P^{\perp}YP^{\perp} + F_2(P^{\perp}YP^{\perp})P^{\perp}XP^{\perp} \\ &+ P^{\perp}XP^{\perp}G_2(P^{\perp}YP^{\perp}) + P^{\perp}YP^{\perp}G_1(P^{\perp}XP^{\perp}) \end{aligned}$$

for all $X, Y \in \operatorname{Alg}\mathscr{L}$. Taking $Y = P^{\perp}$ into the above equality yields

$$F_1(P^{\perp}XP^{\perp})P^{\perp} + F_2(P^{\perp})P^{\perp}XP^{\perp} + P^{\perp}XP^{\perp}G_2(P^{\perp}) + P^{\perp}G_1(P^{\perp}XP^{\perp}) = 0, \tag{2.36}$$

which leads to

$$\begin{aligned} 0 =& P^{\perp}F_1(P^{\perp}XP^{\perp})P^{\perp} + P^{\perp}F_2(P^{\perp})P^{\perp}XP^{\perp} \\ &+ P^{\perp}XP^{\perp}G_2(P^{\perp})P^{\perp} + P^{\perp}G_1(P^{\perp}XP^{\perp})P^{\perp} \end{aligned}$$

for all $X \in \operatorname{Alg}\mathscr{L}$. Using (2.34), we have

$$\begin{aligned} P^{\perp}F_1(P^{\perp}XP^{\perp})P^{\perp} =& - P^{\perp}F_2(P^{\perp})P^{\perp}XP^{\perp} - P^{\perp}XP^{\perp}G_2(P^{\perp})P^{\perp} \\ &+ \mathbb{C}_{PF_1(P^{\perp}XP^{\perp})P}P^{\perp} - \mathbb{C}_{PF_1(P^{\perp})P}P^{\perp}XP^{\perp} \\ &- P^{\perp}G_1(P^{\perp})P^{\perp}XP^{\perp} \end{aligned} \tag{2.37}$$

for all $X \in \operatorname{Alg}\mathscr{L}$.

Now we can describe F_1. In view of (2.7), (2.11), (2.21), (2.25) and (2.37) we routinely compute that

$$\begin{aligned} F_1(X) =& F_1(PXP) + F_1(PXP^{\perp}) + F_1(P^{\perp}XP^{\perp}) + \Delta_{3,F_1}(PXP, PXP^{\perp}, P^{\perp}XP^{\perp}) \\ =& PF_1(PXP)P + PF_1(PXP^{\perp})P + PF_1(P^{\perp}XP^{\perp})P \\ &+ P^{\perp}F_1(PXP)P^{\perp} + P^{\perp}F_1(PXP^{\perp})P^{\perp} + P^{\perp}F_1(P^{\perp}XP^{\perp})P^{\perp} \\ &+ PF_1(X)P^{\perp} + P^{\perp}F_1(X)P + \Delta_{3,F_1}(PXP, PXP^{\perp}, P^{\perp}XP^{\perp}) \\ =& PXPF_1(P)P + \mathbb{C}_{P^{\perp}G_1(P)P^{\perp}}PXP - \mathbb{C}_{P^{\perp}G_1(PXP)P^{\perp}}P \\ &+ PF_1(PXP^{\perp})P + PF_1(P^{\perp}XP^{\perp})P - P^{\perp}G_1(PXP)P^{\perp} \\ &+ P^{\perp}F_1(PXP^{\perp})P^{\perp} - P^{\perp}F_2(P^{\perp})P^{\perp}XP^{\perp} - P^{\perp}XP^{\perp}G_2(P^{\perp})P^{\perp} \\ &+ \mathbb{C}_{PF_1(P^{\perp}XP^{\perp})P}P^{\perp} - \mathbb{C}_{PF_1(P^{\perp})P}P^{\perp}XP^{\perp} - P^{\perp}G_1(P^{\perp})P^{\perp}XP^{\perp} \\ &- PF_2(P^{\perp})X - PXG_2(P^{\perp}) - P^{\perp}F_2(P)X \\ &- P^{\perp}XG_2(P) + \Delta_{3,F_1}(PXP, PXP^{\perp}, P^{\perp}XP^{\perp}) \end{aligned}$$

for all $X \in \operatorname{Alg}\mathscr{L}$. Let us define a commutant map Γ_1 by

$$\begin{aligned} \Gamma_1(X) =& - \mathbb{C}_{P^{\perp}G_1(PXP)P^{\perp}}P + \mathbb{C}_{PF_1(P^{\perp}XP^{\perp})P}P - \mathbb{C}_{P^{\perp}G_1(PXP)P^{\perp}}P^{\perp} \\ &+ \mathbb{C}_{PF_1(P^{\perp}XP^{\perp})P}P^{\perp} + \Delta_{3,F_1}(PXP, PXP^{\perp}, P^{\perp}XP^{\perp}) \\ =& - \mathbb{C}_{P^{\perp}G_1(PXP)P^{\perp}}I + \mathbb{C}_{PF_1(P^{\perp}XP^{\perp})P}I \\ &+ \Delta_{3,F_1}(PXP, PXP^{\perp}, P^{\perp}XP^{\perp}) \end{aligned}$$

for all $X \in \operatorname{Alg}\mathscr{L}$. It follows from (2.28) and (2.35) that

$$\begin{aligned} F_1(X) =& PXPF_1(P)P + \mathbb{C}_{P^{\perp}G_1(P)P^{\perp}}PXP - PF_2(P^{\perp})XP^{\perp} \\ &- PXG_2(P^{\perp})P^{\perp} - P^{\perp}F_2(P^{\perp})P^{\perp}XP^{\perp} - P^{\perp}XP^{\perp}G_2(P^{\perp})P^{\perp} \\ &- \mathbb{C}_{PF_1(P^{\perp})P}P^{\perp}XP^{\perp} - P^{\perp}G_1(P^{\perp})P^{\perp}XP^{\perp} - PXP^{\perp}G_2(P)P \\ &- PG_1(PXP^{\perp})P + P^{\perp}F_1(PXP^{\perp})P^{\perp} - P^{\perp}F_2(P)XP \\ &- P^{\perp}XG_2(P)P + \Gamma_1(X) \\ =& PXPF_1(P)P + \mathbb{C}_{P^{\perp}G_1(P)P^{\perp}}PXP - PF_2(P^{\perp})XP^{\perp} \\ &- PXG_2(P^{\perp})P^{\perp} - P^{\perp}F_2(P^{\perp})P^{\perp}XP^{\perp} - P^{\perp}XP^{\perp}G_2(P^{\perp})P^{\perp} \\ &- \mathbb{C}_{PF_1(P^{\perp})P}P^{\perp}XP^{\perp} - P^{\perp}G_1(P^{\perp})P^{\perp}XP^{\perp} - XP^{\perp}G_2(P)P \\ &- PG_1(PXP^{\perp})P + P^{\perp}F_1(PXP^{\perp})P^{\perp} + \Gamma_1(X) \end{aligned}$$

for all $X \in \operatorname{Alg}\mathscr{L}$. Note that $PF_2(P^\perp)PXP^\perp = PF_2(P^\perp)PX - XPF_2(P^\perp)P$, so we can rewrite the last relation as

$$\begin{aligned}
F_1(X) =&PXPF_1(P)P + \mathbb{C}_{P^\perp G_1(P)P^\perp}PXP - PF_2(P^\perp)PX + XPF_2(P^\perp)P\\
&- PF_2(P^\perp)P^\perp XP^\perp - PXG_2(P^\perp)P^\perp - P^\perp F_2(P^\perp)P^\perp XP^\perp\\
&- P^\perp XP^\perp G_2(P^\perp)P^\perp - \mathbb{C}_{PF_1(P^\perp)P}P^\perp XP^\perp - P^\perp G_1(P^\perp)P^\perp XP^\perp\\
&- XP^\perp G_2(P)P - PG_1(PXP^\perp)P + P^\perp F_1(PXP^\perp)P^\perp + \Gamma_1(X)\\
=&PXPF_1(P)P + \mathbb{C}_{P^\perp G_1(P)P^\perp}PXP + XPF_2(P^\perp)P - PXG_2(P^\perp)P^\perp\\
&- P^\perp XP^\perp G_2(P^\perp)P^\perp - P^\perp F_2(P^\perp)P^\perp XP^\perp - \mathbb{C}_{PF_1(P^\perp)P}P^\perp XP^\perp\\
&- P^\perp G_1(P^\perp)P^\perp XP^\perp - PF_2(P^\perp)PX - PF_2(P^\perp)P^\perp XP^\perp\\
&- XP^\perp G_2(P)P - PG_1(PXP^\perp)P + P^\perp F_1(PXP^\perp)P^\perp + \Gamma_1(X)
\end{aligned}$$

for all $X \in \operatorname{Alg}\mathscr{L}$. Furthermore,

$$\begin{aligned}
F_1(X) =&PXPF_1(P)P + \mathbb{C}_{P^\perp G_1(P)P^\perp}PXP + XPF_2(P^\perp)P - XG_2(P^\perp)P^\perp\\
&- XP^\perp G_2(P)P - P^\perp F_2(P^\perp)P^\perp XP^\perp - \mathbb{C}_{PF_1(P^\perp)P}P^\perp XP^\perp\\
&- P^\perp G_1(P^\perp)P^\perp XP^\perp - PF_2(P^\perp)P^\perp XP^\perp - PF_2(P^\perp)PX\\
&- PG_1(PXP^\perp)P + P^\perp F_1(PXP^\perp)P^\perp + \Gamma_1(X)\\
=&X(PF_1(P)P + \mathbb{C}_{P^\perp G_1(P)P^\perp}P + \mathbb{C}_{PF_2(P^\perp)P}P - G_2(P^\perp)P^\perp - P^\perp G_2(P)P)\\
&- (F_2(P^\perp)P^\perp + \mathbb{C}_{PF_1(P^\perp)P}P^\perp + P^\perp G_1(P^\perp)P^\perp + \mathbb{C}_{PF_2(P^\perp)P}P)X + \Gamma_1(X)
\end{aligned}$$

for all $X \in \operatorname{Alg}\mathscr{L}$. In view of (2.21) and (2.35), we further get

$$\begin{aligned}
F_1(X) =&X(PF_1(P)P + \mathbb{C}_{P^\perp G_1(P)P^\perp}P - G_2(P^\perp) - P^\perp G_2(P)P)\\
&- (F_2(P^\perp)P^\perp + \mathbb{C}_{PF_1(P^\perp)P}P^\perp + P^\perp G_1(P^\perp)P^\perp + \mathbb{C}_{PF_2(P^\perp)P}P)X + \Gamma_1(X)
\end{aligned}\tag{2.38}$$

for all $X \in \operatorname{Alg}\mathscr{L}$. By the symmetry of (1.1) we immediately get

$$\begin{aligned}
F_2(X) =&X(PF_2(P)P + \mathbb{C}_{P^\perp G_2(P)P^\perp}P - G_1(P^\perp) - P^\perp G_1(P)P)\\
&- (F_1(P^\perp)P^\perp + \mathbb{C}_{PF_2(P^\perp)P}P^\perp + P^\perp G_2(P^\perp)P^\perp + \mathbb{C}_{PF_1(P^\perp)P}P)X + \Gamma_2(X)
\end{aligned}\tag{2.39}$$

for all $X \in \operatorname{Alg}\mathscr{L}$. Now using (2.6) we can also describe G_1 and G_2, namely,

$$G_1(X) = -F_1(X) - F_2(I)X - XG_2(I)\tag{2.40}$$

and

$$G_2(X) = -F_2(X) - F_1(I)X - XG_1(I)\tag{2.41}$$

for all $X \in \operatorname{Alg}\mathscr{L}$.

Let us define

$$\begin{aligned}
&Q_1 = PF_1(P)P + \mathbb{C}_{P^\perp G_1(P)P^\perp} P - G_2(P^\perp) - P^\perp G_2(P)P,\\
&Q_2 = PF_2(P)P + \mathbb{C}_{P^\perp G_2(P)P^\perp} P - G_1(P^\perp) - P^\perp G_1(P)P,\\
&\lambda = \mathbb{C}_{PF_2(P^\perp)P} I + \mathbb{C}_{P^\perp G_1(P)P^\perp} I,\\
&\mu = \mathbb{C}_{PF_1(P^\perp)P} I + \mathbb{C}_{P^\perp G_2(P)P^\perp} I.
\end{aligned}$$

Equalities (2.10), (2.14), (2.21) and (2.35) jointly lead to

$$\begin{aligned}
\lambda - Q_1 &= \mathbb{C}_{PF_2(P^\perp)P} I + \mathbb{C}_{P^\perp G_1(P)P^\perp} P^\perp - PF_1(P)P + G_2(P^\perp) + P^\perp G_2(P)P\\
&= \mathbb{C}_{PF_2(P^\perp)P} P^\perp + \mathbb{C}_{P^\perp G_1(P)P^\perp} P^\perp - PF_1(P)P\\
&\quad + PG_2(P^\perp)P^\perp + P^\perp G_2(P^\perp)P^\perp + P^\perp G_2(P^\perp)P + P^\perp G_2(P)P\\
&= \mathbb{C}_{PF_2(P^\perp)P} P^\perp + \mathbb{C}_{P^\perp G_1(P)P^\perp} P^\perp - PF_1(P)P\\
&\quad - F_1(P)P^\perp - P^\perp G_1(P)P^\perp + P^\perp G_2(P^\perp)P^\perp + P^\perp G_2(P^\perp)P + P^\perp G_2(P)P\\
&= \mathbb{C}_{PF_2(P^\perp)P} P^\perp - F_1(P) + P^\perp G_2(P^\perp)P^\perp - F_1(P^\perp)P - PG_1(P^\perp)P.
\end{aligned}$$

Similarly, using (2.9), (2.13), (2.21) and (2.35), we obtian

$$\begin{aligned}
\mu - Q_2 &= \mathbb{C}_{PF_1(P^\perp)P} I + \mathbb{C}_{P^\perp G_2(P)P^\perp} P^\perp - PF_2(P)P + G_1(P^\perp) + P^\perp G_1(P)P\\
&= \mathbb{C}_{PF_1(P^\perp)P} P^\perp + \mathbb{C}_{P^\perp G_2(P)P^\perp} P^\perp - PF_2(P)P\\
&\quad + PG_1(P^\perp)P^\perp + P^\perp G_1(P^\perp)P^\perp + P^\perp G_1(P^\perp)P + P^\perp G_1(P)P\\
&= \mathbb{C}_{PF_1(P^\perp)P} P^\perp + \mathbb{C}_{P^\perp G_2(P)P^\perp} P^\perp - PF_2(P)P - F_2(P)P^\perp - P^\perp G_2(P)P^\perp\\
&\quad - P^\perp G_2(P^\perp)P - F_2(P^\perp)P - PG_2(P^\perp)P + P^\perp G_1(P^\perp)P^\perp\\
&= \mathbb{C}_{PF_1(P^\perp)P} P^\perp - F_2(P) + P^\perp G_1(P^\perp)P^\perp - F_2(P^\perp)P - PG_2(P^\perp)P.
\end{aligned}$$

A simple computation shows that

$$\begin{aligned}
&F_2(P^\perp)P^\perp + \mathbb{C}_{PF_1(P^\perp)P} P^\perp + P^\perp G_1(P^\perp)P^\perp + \mathbb{C}_{PF_2(P^\perp)P} P\\
&= \mu - Q_2 + F_2(P) + F_2(P^\perp),\\
&F_1(P^\perp)P^\perp + \mathbb{C}_{PF_2(P^\perp)P} P^\perp + P^\perp G_2(P^\perp)P^\perp + \mathbb{C}_{PF_1(P^\perp)P} P\\
&= \lambda - Q_1 + F_1(P) + F_1(P^\perp),\\
&PF_1(P)P + \mathbb{C}_{P^\perp G_1(P)P^\perp} P - G_2(P^\perp) - P^\perp G_2(P)P + G_2(I)\\
&= Q_1 + G_2(I),\\
&PF_2(P)P + \mathbb{C}_{P^\perp G_2(P)P^\perp} P - G_1(P^\perp) - P^\perp G_1(P)P + G_1(I)\\
&= Q_2 + G_1(I).
\end{aligned}$$

By the previous facts we can rewrite (2.38), (2.39), (2.40) and (2.41) as

$$
\begin{aligned}
F_1(X) =& XQ_1 - (\mu - Q_2 + F_2(P) + F_2(P^\perp))X + \Gamma_1(X),\\
F_2(X) =& XQ_2 - (\lambda - Q_1 + F_1(P) + F_1(P^\perp))X + \Gamma_2(X),\\
G_1(X) =& - X(Q_1 + G_2(I)) + (\mu - Q_2 - \Delta_{2,F_2}(P, P^\perp))X - \Gamma_1(X)\\
=& - X(Q_1 + G_2(I) - \mu + \Delta_{2,F_2}(P, P^\perp)) - Q_2X - \Gamma_1(X),\\
G_2(X) =& - X(Q_2 + G_1(I)) + (\lambda - Q_1 - \Delta_{2,F_1}(P, P^\perp))X - \Gamma_2(X)\\
=& - X(Q_2 + G_1(I) - \lambda + \Delta_{2,F_1}(P, P^\perp)) - Q_1X - \Gamma_2(X)
\end{aligned}
\tag{2.42}
$$

for all $X \in \operatorname{Alg}\mathscr{L}$. Set

$$
\begin{aligned}
U_1 :=& \mu - Q_2 + F_2(P) + F_2(P^\perp),\\
U_2 :=& \lambda - Q_1 + F_1(P) + F_1(P^\perp),\\
R_2 :=& - Q_1 - G_2(I) + \mu - \Delta_{2,F_2}(P, P^\perp),\\
R_1 :=& - Q_2 - G_1(I) + \lambda - \Delta_{2,F_1}(P, P^\perp).
\end{aligned}
$$

This gives (2.5) directly.

Now (1.1) yields

$$
0 = -U_1XY - U_2YX + XYR_1 + YXR_2 \tag{2.43}
$$

for all $X, Y \in \operatorname{Alg}\mathscr{L}$. Taking $Y = I$ into (2.43), we get

$$
(-U_1 - U_2)X + X(R_1 + R_2) = 0
$$

for all $X \in \operatorname{Alg}\mathscr{L}$. Since the commutant of $\operatorname{Alg}\mathscr{L}$ is $\mathbb{C}I$, we see $U_1 + U_2 = R_1 + R_2 \in \mathbb{C}I$. This fact together with equality (2.43) implies that

$$
U_i[X, Y] = [X, Y]R_i (i = 1, 2)
$$

for all $X, Y \in \operatorname{Alg}\mathscr{L}$. We complete the proof of the lemma. □

We are now ready to prove our main results.

Theorem 2.6. *Let $\mathscr{L}$ be an independent finite-width* CSL *on a complex separable Hilbert space* $\mathbf{H}$ *with* $\dim \mathbf{H} \geq 3$, *and $\mathscr{M}$ an algebra containing* $\operatorname{Alg}\mathscr{L}$ *satisfying $\mathscr{A}_{12}P^\perp\mathscr{M}P \subseteq \mathbb{C}P$ and $P^\perp\mathscr{M}P\mathscr{A}_{12} \subseteq \mathbb{C}P^\perp$. Suppose that maps F_1, F_2, G_1, G_2:* $\operatorname{Alg}\mathscr{L} \to \mathscr{M}$ *are additive modulo $\mathbb{C}I$ such that*

$$
F_1(X)Y + F_2(Y)X + XG_2(Y) + YG_1(X) = 0
$$

for all $X, Y \in$ $\operatorname{Alg}\mathscr{L}$. *Then there exist $U_1, U_2, Q_1, Q_2, R_1, R_2 \in \mathscr{M}$ and maps Γ_1, Γ_2 :* $\operatorname{Alg}\mathscr{L}$ *$\to \mathbb{C}I$ such that $U_1 + U_2 = R_1 + R_2 \in \mathbb{C}I$, $U_i[X, Y] =$*

$[X,Y]R_i (i=1,2)$ *and*

$$\begin{aligned} F_1(X) &= XQ_1 - U_1X + \Gamma_1(X), \\ F_2(X) &= XQ_2 - U_2X + \Gamma_2(X), \\ G_1(X) &= XR_2 - Q_2X - \Gamma_1(X), \\ G_2(X) &= XR_1 - Q_1X - \Gamma_2(X) \end{aligned}$$

for all $X, Y \in \operatorname{Alg}\mathscr{L}$.

Proof. First, using the conditions $\mathscr{A}_{12}P^{\perp}\mathscr{M}P \subseteq \mathbb{C}P$ and $P^{\perp}\mathscr{M}P\mathscr{A}_{12} \subseteq \mathbb{C}P^{\perp}$, we know that

$$P^{\perp}F_2(P^{\perp}YP^{\perp})PXP^{\perp} \in \mathbb{C}P^{\perp}, \quad PXP^{\perp}F_2(P^{\perp}YP^{\perp})P \in \mathbb{C}P$$

for all $X, Y \in \operatorname{Alg}\mathscr{L}$. This shows there exist two functionals $\varphi, \psi : \mathscr{A}_{12} \to \mathbb{C}I$ such that

$$P^{\perp}F_2(P^{\perp}YP^{\perp})PXP^{\perp} = \varphi(PXP^{\perp})P^{\perp}, \quad PXP^{\perp}F_2(P^{\perp}YP^{\perp})P = \psi(PXP^{\perp})P$$

for all $X \in \operatorname{Alg}\mathscr{L}$. Consequently, we have

$$\varphi(PXP^{\perp})P^{\perp}ZP^{\perp} = P^{\perp}F_2(P^{\perp}YP^{\perp})PXP^{\perp}ZP^{\perp} = \varphi(PXP^{\perp}ZP^{\perp})P^{\perp} \tag{2.44}$$

for all $Z \in \operatorname{Alg}\mathscr{L}$, and

$$\psi(PXP^{\perp})PZP = PZPXP^{\perp}F_2(P^{\perp}YP^{\perp})P = \psi(PZPXP^{\perp})P \tag{2.45}$$

for all $Z \in \operatorname{Alg}\mathscr{L}$.

Since $\dim \mathbf{H} \geq 3$, we have $\dim P^{\perp}\mathbf{H} \geq 2$ or $\dim P\mathbf{H} \geq 2$. Suppose that $\dim P^{\perp}\mathbf{H} \geq 2$. If $\varphi(PXP^{\perp}) \neq 0$ for some $X \in \operatorname{Alg}\mathscr{L}$, then by equation (2.44), $P^{\perp}ZP^{\perp} \in \mathbb{C}P^{\perp}$ for all $Z \in \operatorname{Alg}\mathscr{L}$, and so the commutant of $\mathscr{A}_{22}$ in $\mathbf{B}(P^{\perp}\mathbf{H})$ is $\mathbf{B}(P^{\perp}\mathbf{H}) \neq \mathbb{C}P^{\perp}$, which contradicts the result of Lemma 2.1. Hence $\varphi(PXP^{\perp})$
$= 0$ for all $X \in \operatorname{Alg}\mathscr{L}$. Suppose that $\dim P\mathbf{H} \geq 2$. If $\psi(PXP^{\perp}) \neq 0$ for some $X \in \operatorname{Alg}\mathscr{L}$, then by equation (2.45), $PZP \in \mathbb{C}P$ for all $Z \in \operatorname{Alg}\mathscr{L}$, and so the commutant of $\mathscr{A}_{11}$ in $\mathbf{B}(P\mathbf{H})$ is $\mathbf{B}(P\mathbf{H}) \neq \mathbb{C}P$, which is also a contradiction. So $\psi(PXP^{\perp}) = 0$ for all $X \in \operatorname{Alg}\mathscr{L}$. Now by Lemma 2.2, $P^{\perp}F_2(P^{\perp}YP^{\perp})P = 0$ for all $Y \in \operatorname{Alg}\mathscr{L}$, and we have from (2.29) that

$$P^{\perp}F_1(PXP^{\perp})P^{\perp}YP^{\perp} + P^{\perp}YP^{\perp}G_1(PXP^{\perp})P^{\perp} = 0$$

for all $X, Y \in \operatorname{Alg}\mathscr{L}$. In particular, $P^{\perp}F_1(PXP^{\perp})P^{\perp} + P^{\perp}G_1(PXP^{\perp})P^{\perp} = 0$ and hence

$$P^{\perp}F_1(PXP^{\perp})P^{\perp} = -P^{\perp}G_1(PXP^{\perp})P^{\perp} \in \mathbb{C}P^{\perp} \tag{2.46}$$

for all $X \in \operatorname{Alg}\mathscr{L}$. In an analogous manner, using (2.23), (2.28) and (2.31), we can get

$$\begin{aligned} PF_2(PXP^{\perp})P &= -PG_2(PXP^{\perp})P \in \mathbb{C}P, \\ PF_1(PXP^{\perp})P &= -PG_1(PXP^{\perp})P \in \mathbb{C}P, \\ P^{\perp}F_2(PXP^{\perp})P^{\perp} &= -P^{\perp}G_2(PXP^{\perp})P^{\perp} \in \mathbb{C}P^{\perp} \end{aligned} \tag{2.47}$$

for all $X \in \operatorname{Alg}\mathscr{L}$.

Substituting X by $PXP^{\perp}$ and Y by $PYP^{\perp}$ in (1.1), we have

$$\begin{aligned} 0 =& F_1(PXP^{\perp})PYP^{\perp} + F_2(PYP^{\perp})PXP^{\perp} \\ &+ PXP^{\perp}G_2(PYP^{\perp}) + PYP^{\perp}G_1(PXP^{\perp}) \end{aligned}$$

for all $X, Y \in \operatorname{Alg}\mathscr{L}$. It follows that

$$\begin{aligned} 0 =& PF_1(PXP^{\perp})PYP^{\perp} + PF_2(PYP^{\perp})PXP^{\perp} \\ &+ PXP^{\perp}G_2(PYP^{\perp})P^{\perp} + PYP^{\perp}G_1(PXP^{\perp})P^{\perp} \end{aligned}$$

for all $X, Y \in \operatorname{Alg}\mathscr{L}$. This together with (2.47) implies that

$$\begin{aligned} 0 =& (\mathbb{C}_{PF_1(PXP^{\perp})P} - \mathbb{C}_{P^{\perp}F_1(PXP^{\perp})P^{\perp}})PYP^{\perp} \\ &+ (\mathbb{C}_{PF_2(PYP^{\perp})P} - \mathbb{C}_{P^{\perp}F_2(PYP^{\perp})P^{\perp}})PXP^{\perp} \end{aligned}$$

for all $X, Y \in \operatorname{Alg}\mathscr{L}$. Applying Lemma 2.3, we obtain

$$\mathbb{C}_{PF_1(PXP^{\perp})P} = \mathbb{C}_{P^{\perp}F_1(PXP^{\perp})P^{\perp}}, \quad \mathbb{C}_{PF_2(PYP^{\perp})P} = \mathbb{C}_{P^{\perp}F_2(PYP^{\perp})P^{\perp}}$$

for all $X, Y \in \operatorname{Alg}\mathscr{L}$. Now taking into account (2.46) and (2.47), we obtain

$$\begin{aligned} P^{\perp}F_1(PXP^{\perp})P^{\perp} - PG_1(PXP^{\perp})P &= P^{\perp}F_1(PXP^{\perp})P^{\perp} + PF_1(PXP^{\perp})P \in \mathbb{C}I, \\ P^{\perp}F_2(PXP^{\perp})P^{\perp} - PG_2(PXP^{\perp})P &= P^{\perp}F_2(PXP^{\perp})P^{\perp} + PF_2(PXP^{\perp})P \in \mathbb{C}I \end{aligned}$$

for all $X \in \operatorname{Alg}\mathscr{L}$. The desired result follows from Lemma 2.4. $\square$

From the above theorem, we have the following corollary directly.

Corollary 2.7. *Let $\mathscr{L}$ be an independent finite-width* CSL *on a complex separable Hilbert space* $\mathbf{H}$ *with* $\dim \mathbf{H} \geq 3$*, and $\mathscr{M}$ an algebra containing* $\operatorname{Alg}\mathscr{L}$ *satisfying $P^{\perp}\mathscr{M}P = \{0\}$. Suppose that maps $F_1, F_2, G_1, G_2 : \operatorname{Alg}\mathscr{L} \to \mathscr{M}$ are additive modulo $\mathbb{C}I$ such that*

$$F_1(X)Y + F_2(Y)X + XG_2(Y) + YG_1(X) = 0$$

for all $X, Y \in \operatorname{Alg}\mathscr{L}$. Then there exist $U_1, U_2, Q_1, Q_2, R_1, R_2 \in \mathscr{M}$ and maps $\Gamma_1, \Gamma_2 : \operatorname{Alg}\mathscr{L} \to \mathbb{C}I$ such that $U_1 + U_2 = R_1 + R_2 \in \mathbb{C}I$, $U_i[X, Y] =$

$[X, Y]R_i (i = 1, 2)$ *and*

$$\begin{aligned} F_1(X) &= XQ_1 - U_1X + \Gamma_1(X), \\ F_2(X) &= XQ_2 - U_2X + \Gamma_2(X), \\ G_1(X) &= XR_2 - Q_2X - \Gamma_1(X), \\ G_2(X) &= XR_1 - Q_1X - \Gamma_2(X) \end{aligned}$$

for all $X, Y \in \operatorname{Alg}\mathscr{L}$.

The next result shows that functional identity (1.1) of degree 2 in CSL algebras has only the standard solution.

Proposition 2.8. *Let* $\mathscr{L}$ *be an independent finite-width* CSL *on a complex separable Hilbert space* $\mathbf{H}$ *with* $\dim \mathbf{H} \geq 3$, *and* $\mathscr{M}$ *an algebra containing* $\operatorname{Alg}\mathscr{L}$ *satisfying* $\mathscr{A}_{12}P^{\perp}\mathscr{M}P \subseteq \mathbb{C}P$ *and* $P^{\perp}\mathscr{M}P\mathscr{A}_{12} \subseteq \mathbb{C}P^{\perp}$. *Then functional identity* (1.1) *has only the standard solution if and only if*

$$U[X, Y] = [X, Y]R \text{ for all } X, Y \in \operatorname{Alg}\mathscr{L} \Longrightarrow U = R \in \mathbb{C}I.$$

Proof. Suppose that (1.1) has only the standard solution. Further, assume that $U[X, Y] = [X, Y]R$ for all $X, Y \in \operatorname{Alg}\mathscr{L}$. We might as well assume that $F_1(X) = -F_2(X) = UX$ and $G_1(X) = -G_2(X) = XR$ for all $X, Y \in \operatorname{Alg}\mathscr{L}$. Then we must have

$$UX = XQ + \Gamma_1(X), \quad XR = QX + \Gamma_2(X),$$

where $X \in \operatorname{Alg}\mathscr{L}$, $Q \in \mathscr{M}$ and Γ_1, Γ_2 are mappings from $\operatorname{Alg}\mathscr{L}$ to $\mathbb{C}I$. Obviously, the above two identities imply $PUP^{\perp} = 0$, $P^{\perp}UP = 0$ and $U = Q = R$. Now we have

$$UPXP^{\perp} - PXP^{\perp}U = PUPXP^{\perp} - PXP^{\perp}UP^{\perp} = 0 \tag{2.48}$$

for all $X \in \operatorname{Alg}\mathscr{L}$. This implies that for any $A \in \mathscr{A}_{11}$ we can get

$$\begin{aligned} &(PUPA - APUP)PXP^{\perp} = PUP(APXP^{\perp}) - A(PUPXP^{\perp}) \\ &= (APXP^{\perp})P^{\perp}UP^{\perp} - A(PXP^{\perp}UP^{\perp}) = 0. \end{aligned}$$

So by Lemma 2.2 we can see that $(PUPA - APUP) = 0$ for all $A \in \mathscr{A}_{11}$. This fact and Lemma 2.1 imply that $PUP \in \mathbb{C}P$. Using (2.48) again we have $U \in \mathbb{C}I$. The converse follows immediately from Theorem 2.6. $\square$

3. Applications

In this section, as an application, we will consider generalized inner biderivations and commuting additive mappings of certain CSL algebras.

3.1. **Commuting maps.** Let $\mathscr{R}$ be a commutative ring with identity, $\mathcal{A}$ be a unital algebra over $\mathscr{R}$ and $\mathcal{Z}(\mathcal{A})$ be the center of $\mathcal{A}$. An $\mathscr{R}$-linear mapping f of $\mathcal{A}$ is called commuting if $[f(x), x] = 0$ for all $x \in \mathcal{A}$. A commuting mapping f of $\mathcal{A}$ is called proper if it is of the form

$$f(x) = xc + \alpha(x), \forall x \in \mathcal{A},$$

where $c \in \mathcal{Z}(\mathcal{A})$ and α is an $\mathscr{R}$-linear mapping from $\mathcal{A}$ into $\mathcal{Z}(\mathcal{A})$. As a simple application of our results, we give the following result.

Corollary 3.1. *Let $\mathscr{L}$ be an independent finite-width* CSL *on a complex separable Hilbert space* $\mathbf{H}$ *with* $\dim \mathbf{H} \geq 3$*, and $\mathscr{M}$ an algebra containing* $\operatorname{Alg}\mathscr{L}$ *with $\mathscr{A}_{12}P^{\perp}\mathscr{M}P \subseteq \mathbb{C}P$ and $P^{\perp}\mathscr{M}P\mathscr{A}_{12} \subseteq \mathbb{C}P^{\perp}$. Suppose that the map $F : \operatorname{Alg}\mathscr{L} \to \mathscr{M}$ is additive modulo $\mathbb{C}I$ satisfying $[F(X), X] = 0$ for all $X \in \operatorname{Alg}\mathscr{L}$. Then there exist $\lambda \in \mathbb{C}I$ and a map $\Gamma : \operatorname{Alg}\mathscr{L} \to \mathbb{C}I$ such that*

$$F(x) = \lambda X + \Gamma(X)$$

for all $X \in \operatorname{Alg}\mathscr{L}$.

Proof. Firstly, the linearization of the identity $F(X)X - XF(X) = 0$ gives

$$F(X)Y + F(Y)X - XF(Y) - YF(X) = 0$$

for all $X, Y \in \operatorname{Alg}\mathscr{L}$. Thus by Theorem 2.6 we obtain

$$\begin{aligned} F(X) &= XQ_1 - U_1X + \Gamma_1(X) = XQ_2 - U_2X + \Gamma_2(X), \\ -F(X) &= XR_2 - Q_2X - \Gamma_1(X) = XR_1 - Q_1X - \Gamma_2(X) \end{aligned} \tag{3.1}$$

for some $U_1, U_2, Q_1, Q_2, R_1, R_2 \in \mathscr{M}$ and maps $\Gamma_1, \Gamma_2 : \operatorname{Alg}\mathscr{L} \to \mathbb{C}I$ such that $U_1 + U_2 = R_1 + R_2 \in \mathbb{C}I$. From (3.1) we can see that

$$\begin{aligned} X(Q_1 - Q_2) - (U_1 - U_2)X \in \mathbb{C}I, \\ X(R_2 - R_1) - (Q_2 - Q_1)X \in \mathbb{C}I \end{aligned}$$

for all $X \in \operatorname{Alg}\mathscr{L}$. Repeating the same computational process in Proposition 2.8, we know that

$$Q_1 - Q_2 = U_1 - U_2 = R_1 - R_2 \in \mathbb{C}I,$$

which yields $U_1 = R_1 \in \mathbb{C}I$ and $U_2 = R_2 \in \mathbb{C}I$. Comparing the two relations in (3.1), we can also get $Q_1, Q_2 \in \mathbb{C}I$. Setting $\lambda := Q_1 - U_1 \in \mathbb{C}I$ and $\Gamma := \Gamma_1$, We conclude that $F(x) = \lambda X + \Gamma(X)$ for all $X \in \operatorname{Alg}\mathscr{L}$. □

3.2. **Generalized inner biderivation.** Let $\mathscr{R}$ be a commutative ring. A map $g : \mathscr{R} \to \mathscr{R}$ is called a generalized inner derivation, if $g(x) = ax + xb$ for some $a, b \in \mathscr{R}$. Further, a biadditive map $\Upsilon : \mathscr{R} \times \mathscr{R} \to \mathscr{R}$ is a generalized inner biderivation, if for each $y \in \mathscr{R}$ there exist unique elements $g_1(y), g_2(y), g_3(y), g_4(y) \in \mathscr{R}$ such that $\Upsilon(x, y) = g_1(y)x + xg_2(y)$ and $\Upsilon(y, x) = g_3(y)x + xg_4(y)$ for all $x \in \mathscr{R}$([4]). The main results in this paper also imply the following corollary.

Corollary 3.2. *Let $\mathscr{L}$ be an independent finite-width* CSL *on a complex separable Hilbert space* $\mathbf{H}$ *with* $\dim \mathbf{H} \geq 3$*, and* $\mathscr{M}$ *an algebra containing* $\operatorname{Alg}\mathscr{L}$ *with* $\mathscr{A}_{12}P^{\perp}\mathscr{M}P \subseteq \mathbb{C}P$ *and* $P^{\perp}\mathscr{M}P\mathscr{A}_{12} \subseteq \mathbb{C}P^{\perp}$*. Suppose that the map* $\Upsilon : \operatorname{Alg}\mathscr{L} \times \operatorname{Alg}\mathscr{L} \to \mathscr{M}$ *is a generalized inner biderivation. Then there exist* $U_1, U_2,\ Q_1, Q_2,\ R_1, R_2 \in \mathscr{M}$ *and maps* $\Gamma_1, \Gamma_2 : \operatorname{Alg}\mathscr{L} \to \mathbb{C}I$ *such that* $U_1 + U_2 = R_1 + R_2 \in \mathbb{C}I$*,* $U_i[X, Y] = [X, Y]R_i, i = 1, 2$*, and*

$$\Upsilon(X, Y) = (U_2Y - YQ_2)X + X(Q_1Y - YR_1) = (XQ_1 - U_1X)Y + Y(XR_2 - Q_2X)$$

for all $X, Y \in \operatorname{Alg}\mathscr{L}$*.*

Proof. The proof is similar to [9, Corollary 4.1]. □

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