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SOME RESULTS IN GENERALIZED ŠERSTNEV SPACES

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ABSTRACT. In this paper, we show that D-compactness in Generalized Šerstnev spaces implies D-boundedness and as in the classical case, a D-bounded and closed subset of a characteristic Generalized Šerstnev is not D-compact in general. Finally, in the finite dimensional Generalized Šerstnev spaces a subset is D-compact if and only if it is D-bounded and closed.

1. Preliminaries

Probabilistic normed spaces (PN spaces henceforth) were introduced by Šerstnev in 1963 [9]. In the sequel, we adopt the new definition of Generalized Šerstnev PN spaces given in the paper by

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Lafuerza-Guillén and Rodríguez [7]. The notations and concepts used are those of [2, 3, 4, 5, 7] and [10].

In the sequel, the space of probability distribution functions is denoted by $\Delta^+ = \{F : \mathbf{R} \cup \{-\infty, +\infty\} \longrightarrow [0,1] : F$ is left-continuous and non-decreasing on $\mathbf{R}$, $F(0) = 0$ and $F(+\infty) = 1\}$ and $D^+ \subseteq \Delta^+$ is defined as follows $D^+ = \{F \in \Delta^+ : l^- F(+\infty) = 1\}$. Here $l^- f(x)$ denotes the left limit of the function f at the point x, $l^- f(x) = \lim_{t \to x^-} f(t)$. The space Δ^+ is partially ordered by the usual point-wise ordering of functions i.e., $F \leq G$ if and only if $F(x) \leq G(x)$ for all x in $\mathbf{R}$. The maximal element for Δ^+ in this order is ε_0 , a distribution defined by

$$\varepsilon_0 = \begin{cases} 0, & \text{if } x \leq 0, \\ 1, & \text{if } x > 0. \end{cases}$$

A triangle function is a binary operation on Δ^+, namely a function $\tau : \Delta^+ \times \Delta^+ \longrightarrow \Delta^+$ that is associative, commutative, non-decreasing and which has ε_0 as unit. Typical continuous triangle functions are $\tau_T(F,G)(x) = \sup_{s+t=x} T(F(s), G(t))$ and $\tau_{T^*}(F,G) = \inf_{s+t=x} T^*(F(s), G(t))$. Here, T is a continuous t-norm, i.e. a continuous binary operation on $[0,1]$ that is commutative, associative, non-decreasing in each variable and has 1 as identity, and T^* is a continuous t-conorm, namely a continuous binary operation on $[0,1]$ which is related to the continuous t-norm T through $T^*(x,y) = 1 - T(1-x, 1-y)$.

Definition 1.1. A *probabilistic normed* (briefly, PN) space is a quadruple (V, ν, τ, τ^*), where V is a real vector space, τ and τ^* are continuous triangle functions, and ν is a mapping from V into Δ^+ such that, for all p, q in V, the following conditions hold:

(N1) $\nu_p = \varepsilon_0$ if and only if $p = \theta$, where θ is the null vector in V;

(N2) $\nu_{-p} = \nu_p$ for each $p \in V$;

(N3) $\nu_{p+q} \geq \tau(\nu_p, \nu_q)$ for all $p, q \in V$;

(N4) $\nu_p \leq \tau^*(\nu_{\alpha p}, \nu_{(1-\alpha)p})$ for all α in $[0, 1]$.

If the inequality (N4) is replaced by the equality $\nu_p = \tau_M(\nu_{\alpha p}, \nu_{(1-\alpha)p})$, then the PN space is called *Šerstnev* space and, as a consequence, a condition stronger than (N2) holds, namely, $\nu_{\lambda p}(x) = \nu_p(\frac{x}{|\lambda|})$ for all $p \in V$, $\lambda \neq 0$ and $x \in \mathbf{R}$.

Following [1, 7], for $0 < b \leq +\infty$, let M_b be the set of m-*transforms* consisting of all continuous and strictly increasing functions from $[0, b]$ onto $[0, +\infty]$. More generally, let $\widetilde{M}$ be the set of non-decreasing left-continuous functions $\phi : [0, +\infty] \longrightarrow [0, +\infty]$, $\phi(0) = 0$, $\phi(+\infty) = +\infty$ and $\phi(x) > 0$ for $x > 0$. Then $M_b \subseteq \widetilde{M}$ once m is extended to $[0, +\infty]$ by $m(x) = +\infty$ for all $x \geq b$. Note that a function $\phi \in \widetilde{M}$ is bijective if and only if $\phi \in M_{+\infty}$. Sometimes, the probabilistic norms ν and ν' of two given PN spaces satisfy $\nu' = \nu\phi$ for some $\phi \in M_{+\infty}$, not necessarily bijective. Let $\hat{\phi}$ be the (unique) quasi-inverse of ϕ which is left-continuous. Recall from [10] page 49, that $\hat{\phi}$ is defined by $\hat{\phi}(0) = 0$, $\hat{\phi}(+\infty) = +\infty$ and $\hat{\phi}(t) = \sup\{u : \phi(u) < t\}$ for all $0 < t < +\infty$. It follows that $\hat{\phi}(\phi(x)) \leq x$ and $\phi(\hat{\phi}(y)) \leq y$ for all x and y.

Definition 1.2. A quadruple (V, ν, τ, τ^*) satisfying the ϕ-Šerstnev condition

$$\nu_{\lambda p}(x) = \nu_p(\hat{\phi}(\frac{\phi(x)}{|\lambda|})),$$

for all $x \in \mathbf{R}^+$, $p \in V$ and $\lambda \in \mathbf{R} - \{0\}$ is called a ϕ-Šerstnev PN space (*generalized Šerstnev space*).

Lemma 1.3. [2] *If* $|\alpha| \leq |\beta|$, *then* $\nu_{\beta p} \leq \nu_{\alpha p}$ *for every* p *in* V.

Definition 1.4. Let (V, ν, τ, τ^*) be a PN space. For each p in V and $\lambda > 0$, the strong $\lambda -$ *neighborhood* of p is the set

$$N_p(\lambda) \;=\; \{q \in V : \nu_{p-q}(\lambda) > 1 - \lambda\},$$

and the strong neighborhood system for V is the union $\bigcup_{p \in V} \mathcal{N}_p$ where $\mathcal{N}_p = \{N_p(\lambda) : \lambda > 0\}$. The strong neighborhood system for V determines a Hausdorff topology for V.

Definition 1.5. Let (V, ν, τ, τ^*) be a PN space, a sequence $\{p_n\}_n$ in V is said to be *strongly convergent* to p in V if for each $\lambda > 0$, there exists a positive integer N such that $p_n \in N_p(\lambda)$, for $n \geq N$. Also the sequence $\{p_n\}_n$ in V is called *strongly Cauchy* sequence if for every $\lambda > 0$ there is a positive integer N such that $\nu_{p_n - p_m}(\lambda) > 1 - \lambda$, whenever $m, n > N$. A PN space (V, ν, τ, τ^*) is said to be *strongly complete* in the strong topology if every strongly Cauchy sequence in V is strongly convergent to a point in V.

Definition 1.6. Let (V, ν, τ, τ^*) be a PN space and A be the nonempty subset of V. The *probabilistic radius* of A is the function R_A defined on $\mathbf{R}^+$ by

$$R_A(x) = \begin{cases} l^- \inf\{\nu_p(x) : p \in A\}, & \text{if } x \in [0, +\infty), \\ 1, & \text{if } \quad x = +\infty. \end{cases}$$

A nonempty set A in a PN space (V, ν, τ, τ^*) is said to be

(a) *certainly bounded*, if $R_A(x_0) = 1$ for some $x_0 \in (0, +\infty)$;

(b) *perhaps bounded*, if one has $R_A(x) < 1$ for every $x \in (0, +\infty)$ and $l^- R_A(+\infty) = 1$;

(c) *perhaps unbounded*, if $R_A(x_0) > 0$ for some $x_0 \in (0, +\infty)$ and $l^- R_A(+\infty) \in (0, 1)$;

(d) *certainly unbounded*, if $l^- R_A(+\infty) = 0$ i.e., if $R_A = \varepsilon_\infty$.
Moreover, A is said to be *D-bounded* if either (a) or (b) holds, i.e. $R_A \in D^+$. If $R_A \in \Delta^+ \setminus D^+$, A is called *D-unbounded*.

Theorem 1.7. [5] *A subset A in the PN space (V, ν, τ, τ^*) is D-bounded if and only if there exists a d.f. $G \in D^+$ such that $\nu_p \geq G$ for every $p \in A$.*

Definition 1.8. A subset A of *Topological Vector Space* (briefly, TVS) V is said to be *topologically bounded* if for every sequence $\{\alpha_n\}$ of real numbers that converges to zero as $n \longrightarrow +\infty$ and for every $\{p_n\}$ of elements of A, one has $\alpha_n p_n \longrightarrow \theta$, in the strong topology.

Theorem 1.9. [3] *Suppose (V, ν, τ, τ^*) is a PN space, endowed with the strong topology induced by the probabilistic norm ν. Then it is a TVS if and only if for every $p \in V$ the map from $\mathbf{R}$ into V defined by*

$$\alpha \longmapsto \alpha p$$

is continuous. The PN space (V, ν, τ, τ^*) is called characteristic whenever $\nu(V) \subseteq D^+$.

Theorem 1.10. [7] *Let $\phi \in \widetilde{M}$ such that $\lim_{x \longrightarrow \infty} \hat{\phi}(x) = \infty$. Then a ϕ-Šerstnev PN space (V, ν, τ, τ^*) is a TVS if and only if it is characteristic.*

2. **Results**

In this section we study D-bounded and D-compact sets in ϕ-Šerstnev PN spaces.

Proposition 2.1. *Let $\phi \in \widetilde{M}$ such that $\lim_{x\longrightarrow\infty} \hat{\phi}(x) = \infty$. Let (V, ν, τ, τ^*) be a characteristic ϕ-Šerstnev PN space. Then for a subset A of V the following are equivalent.*

(a) *For every $n \in \mathbf{N}$, there is a $k \in \mathbf{N}$ such that $A \subset kN_\theta(1/n)$.*

(b) *A is D-bounded.*

(c) *A is topologically bounded .*

Proof. From [7], (a) and (b) are equivalent. For $(b) \Longrightarrow (c)$, let A be any D-bounded subset of V, $\{p_n\}$ be any sequence in A and $\{\alpha_n\}$ any sequence of real numbers that converges to zero. Without loss of generality we may assume that $\alpha_n \neq 0$ for every $n \in \mathbf{N}$. Then for every $x > 0$ and $n \in \mathbf{N}$

$$\nu_{\alpha_n p_n}(x) = \nu_{p_n}(\hat{\phi}(\frac{\phi(x)}{\alpha_n})) \geq R_A(\hat{\phi}(\frac{\phi(x)}{\alpha_n})) \longrightarrow 1$$

as $n \longrightarrow \infty$. Thus $\alpha_n p_n \longrightarrow \theta$.

$(c) \Longrightarrow (b)$. Let A be a subset of V which is not D-bounded. Then

$$R_A(x) \longrightarrow \gamma < 1,$$

as $x \longrightarrow \infty$. By definition of R_A, for every $n \in \mathbf{N}$ there is $p_n \in A$ such that

$$\nu_{p_n}(\hat{\phi}(n\phi(n)) < \frac{1+\gamma}{2} < 1.$$

Choosing $\alpha_n = 1/n$,

$$\nu_{\alpha_n p_n}(1) \leq \nu_{\alpha_n p_n}(n) = \nu_{p_n}(\hat{\phi}(n\phi(n)) < \frac{1+\gamma}{2} < 1,$$

which shows that $\{\nu_{\alpha_n p_n}\}$ does not tend to ε_0, even it has a weak limit, i.e., $\{\alpha_n p_n\}$ does not tend to θ in the strong topology, so A is not topologically bounded. $\square$

Lemma 2.2. *Let* $\phi \in \widetilde{M}$ *such that* $\lim_{x\longrightarrow\infty} \hat{\phi}(x) = \infty$ *and let* (V, ν, τ, τ^*) *be a characteristic* ϕ*-Šerstnev PN space. If* $p_m \longrightarrow p_0$ *in* V *and* $A = \{p_m : m \in \mathbf{N}\}$*, then* A *is a D-bounded subset of* V*.*

Proof. Let $p_m \longrightarrow p_0$ and $\alpha_m \longrightarrow 0$. Then there exists $m_0 \in \mathbf{N}$ such that for every $m \geq m_0$, $0 < \alpha_m < 1$; so that

$$\begin{aligned} \nu_{\alpha_m p_m} &\geq \tau(\nu_{\alpha_m(p_m - p_0)}, \nu_{\alpha_m p_0}) \\ &\longrightarrow \tau(\varepsilon_0, \varepsilon_0) \\ &= \varepsilon_0, \end{aligned}$$

as m tends to infinity. $\square$

Example 2.3. The quadruple $(\mathbf{R}, \nu, \tau_\pi, \tau_\pi^*)$, where $\nu : \mathbf{R} \longrightarrow \mathbf{\Delta}^+$ is defined by

$$\nu_p(x) = \begin{cases} 0 & \text{if } x = 0, \\ exp(-|p|^{1/2}) & \text{if } 0 < x < +\infty, \\ 1 & \text{if } x = +\infty, \end{cases}$$

and $\nu_0 = \varepsilon_0$ is a PN space but is not Šerstnev space (see [2]). The sequence $\{\frac{1}{m}\}$ is convergent but $A = \{\frac{1}{m} : m \in \mathbf{N}\}$ is not D-bounded set. The only D-bounded set in this space is $\{0\}$.

Definition 2.4. The characteristic ϕ-Šerstnev PN space (V, ν, τ, τ^*) is said to be *distributionally compact* (simply *D-compact*) if every sequence $\{p_m\}_m$ in V has a convergent subsequence $\{p_{m_k}\}$. A subset A of a characteristic ϕ-Šerstnev PN space (V, ν, τ, τ^*) is said to be *D-compact* if every sequence $\{p_m\}$ in A has a subsequence $\{p_{m_k}\}$ convergent to an element $p \in A$.

Proposition 2.5. *A D-compact subset of a characteristic ϕ-Šerstnev PN space (V, ν, τ, τ^*) is D-bounded and closed.*

Proof. Suppose that $A \subseteq V$ is D-compact. From Proposition 2.1 it is enough show that A is topologically bounded. Now assume that there is a sequence $\{p_m\} \subseteq A$ and a real sequence $\alpha_m \longrightarrow 0$ such that $\{\alpha_m p_m\}$ does not tend to the origin in V. Then there is an infinite set $J \subseteq \mathbf{N}$ such that the sequence $\{\alpha_m p_m\}_{m\in J}$ lies in the complement of a neighborhood of the origin. Now $\{p_m\}$ is a subset of D-compact set A, so it has a convergent subsequence $\{p_m\}_{m\in J'}$. From Proposition 2.1 and Lemma 2.2 $\{p_m\}_{m\in J'}$ is topologically bounded and so $\{\alpha_m p_m\}_{m\in J'}$ tends to origin which is a contradiction. The closedness of A is trivial . □

As in the classical case, a D-bounded and closed subset of a characteristic ϕ-Šerstnev is not D-compact in general, as one can see from the next example.

Example 2.6. We consider quadruple $(\mathbf{Q}, \nu, \tau_\pi, \tau_M)$, where $\pi(x, y) = xy$ for every $x, y \in [0, 1]$ and probabilistic norm $\nu_p(t) = \frac{t}{t+|p|}$. It is straightforward to check that $(\mathbf{Q}, \nu, \tau_\pi, \tau_M)$ is a characteristic ϕ-Šerstnev PN space. In this space, convergence of a sequence is equivalent to its convergence in $\mathbf{Q}$. We consider the subset $A = [a, b] \cap \mathbf{Q}$ with $a, b \in \mathbf{R} - \mathbf{Q}$. Since $R_A(t) = \frac{t}{t+\max\{|a|,|b|\}}$, the set A is D-bounded and since A is closed in $\mathbf{Q}$ classically, it is closed in $(\mathbf{Q}, \nu, \tau_\pi, \tau_M)$. We know that A is not classically compact in $\mathbf{Q}$, i.e., there exists a sequence in $\mathbf{Q}$ with no convergent subsequence in the classical sense and so in $(\mathbf{Q}, \nu, \tau_\pi, \tau_M)$. Hence, A is not D-compact.

We close the paper by quoting two recent results on D-compactness and strong completeness in ϕ-Šerstnev PN spaces.

Theorem 2.7. [8] *Consider a finite dimensional characteristic ϕ-Šerstnev PN space (V, ν, τ, τ^*) on real field $(\mathbf{R}, \nu', \tau', \tau'^*)$. Every subset A of V is D-compact if and only if A is D-bounded and closed.*

Theorem 2.8. [4] *Let (V, ν, τ, τ^*) be a characteristic ϕ-Šerstnev PN space and let S be a compact subset of V. Then S is strongly complete.*

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References

[1] C. Alsina and B. Schweizer, The countable product of probabilistic metric spaces, *Houston J. Math.* **9** (1983), 303–310.

[2] C. Alsina, B. Schweizer and A. Sklar, On the definition of a probabilistic normed space, *Aequationes Math.* **46** (1993), 91–98.

[3] C. Alsina, B. Schweizer and A. Sklar, Continuity properties of probabilistic norms, *J. Math. Anal. Appl.* **208** (1997), 446–452.

[4] S. B. Hosseini and R. Saadati, Completeness results in probabilistic metric spaces. I, *Commun. Appl. Anal.* **9** (2005), 549–554.

[5] B. Lafuerza-Guillén, A. Rodríguez-Lallena and C. Sempi, A study of boundedness in probabilistic normed spaces . *J. Math. Anal. Appl.* **232** (1999), 183–196.

[6] B. Lafuerza-Guillén, A. Rodríguez-Lallena and C. Sempi, Normability of probabilistic normed spaces, to appear in *J. Math. Anal. Appl.*

[7] B. Lafuerza-Guillén, J. L. Rodríguez , Boundedness in generalized Šerstnev spaces, http://front.math.ucdavis.edu/math.PR/0408207.

[8] R. Saadati and M. Amini, D-boundedness and D-compactness in finite dimensional probabilistic normed spaces, *Proc. Indian Acad. Sci. Math. Sci.* **115** (2005), 483–492.

[9] A. N. Šerstnev, On the notion of a random normed space, *Dokl. Akad. Nauk SSSR* **149** (2) (1963), 280–283 (English translation in *Soviet Math. Dokl.* **4** (1963), 388–390).

[10] B. Schweizer and A. Sklar, *Probabilistic Metric Spaces*, Elsevier North Holand, New York, 1983.

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