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STABILITY OF F-BIHARMONIC MAPS

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Abstract. The paper studies some properties of F-biharmonic maps between Riemannian manifolds. By considering the first variation formula of the F-bienergy functional, F-biharmonicity of conformal maps are investigated. Moreover, the second variation formula for F-biharmonic maps is obtained. As an application, instability and nonexistence theorems for F-biharmonic maps are given.

Keywords: Biharmonic maps, F-biharmonic maps, conformal maps, stability.

MSC(2010): Primary: 53C43; Secondary: 58E20.

1. Introduction

A map $\phi : (M, g) \longrightarrow (N, h)$ between Riemannian manifolds is called biharmonic if ϕ is a critical point of the bienergy functional E_2 defined by

$$E_2(\phi) := \frac{1}{2} \int_M \mid \tau(\phi) \mid^2 d\upsilon_g, \tag{1.1}$$

where $\mid \tau(\phi) \mid$ denotes the Hilbert-Schmidt norm of the tension field of ϕ, and $d\upsilon_g$ is the volume element of (M, g). In 1964, biharmonic maps between Riemannian manifolds were first studied by Eells and Sampson [9]. In [17], the first and second variation formulas for biharmonic maps in spheres were obtained. Furthermore, a new method to construct biharmonic submanifolds of n-dimensional sphere was given, [6]. Jiang [15] obtained the first and second variation formulas of the bienergy functional. Recently this topic has been extensively studied (for instance, see [8,12–14]). Biharmonic maps play an important role in many fields of physics and mathematics where they may serve as the model of elasticity and fluid dynamics.

Han and Feng [13] extended the notions of biharmonic, p-biharmonic and exponentially biharmonic maps to F-biharmonic maps, and studied the harmonicity

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of F-biharmonic maps under the curvature conditions on the target manifold. They considered a C^3- strictly increasing function $F:[0,\infty)\longrightarrow[0,\infty)$ and defined the F-bienergy functional as follows:

$$E_{F,2}(\phi)=\int_M F(\frac{\mid\tau(\phi)\mid^2}{2})dv_g. \tag{1.2}$$

A map ϕ is called *F-biharmonic* if ϕ is a critical point of the F-bienergy functional. F-biharmonic maps can be categorized as biharmonic, p-biharmonic or exponentially biharmonic when $F(t)$ is equal to t, $(2t)^{\frac{p}{2}}/p$ ($p\geq 4$) or e^t respectively. In terms of the Euler-Lagrange equation, ϕ is F-biharmonic if ϕ satisfies the following equation

$$\begin{aligned}\tau_{F,2}(\phi):&=-\Delta^\phi(F'(\frac{\mid\tau(\phi)\mid^2}{2})\tau(\phi))-trace_g\, R^N(d\phi,F'(\frac{\mid\tau(\phi)\mid^2}{2})\tau(\phi))d\phi\\ &=0,\end{aligned}\tag{1.3}$$

where Δ^ϕ denotes the rough Laplacian on sections of the pull-back bundle $\phi^{-1}TN$. $\tau_{F,2}(\phi)$ is called F-bitension field of ϕ, [13].

In [14], Y. Han and W. Zhang investigated harmonicity of p-biharmonic maps, as the cases of F-biharmonic maps. Moreover, in [12] Y.B. Han introduced the notion of p-biharmonic submanifold and proved that p-biharmonic submanifold (M,g) in a Riemannian manifold (N,h) with non-positive sectional curvature which satisfies certain conditions must be minimal.

The present article is organized as follows.

In section 2, the concepts of harmonic, biharmonic and conformal maps are reviewed and some essential formulas which are necessary for the paper are given. In section 3, we recall the definition of F-biharmonic maps and analyse the conditions for conformal maps between the same dimensional manifolds to be F-biharmonic maps. In the last section, the second variation formula for F-biharmonic maps is obtained. Then, the stability of F-biharmonic maps from a compact Riemannian manifold into an arbitrary Riemannian manifold with constant positive sectional curvature is studied.

2. **Preliminaries**

In this section, we recall some basic concepts for computing the first and second variation formulas of the F-bienergy functional. Let $\phi:(M,g)\longrightarrow(N,h)$ be a smooth map between Riemannian manifolds. Throughout this paper we consider (M,g) as a compact Riemannian manifold, ∇ and ∇^N denote the Levi-Civita connections on M and N respectively, and ∇^ϕ is the induced connection on the pullback bundle $\phi^{-1}TN$ which is defined by $\nabla^\phi_X W=\nabla^N_{d\phi(X)}W$, where X is a vector field on M and W is a section of $\phi^{-1}TN$. The energy of ϕ

is defined by

$$E(\phi) = \frac{1}{2}\int_M \mid d\phi \mid^2 dv_g, \tag{2.1}$$

Where $\mid d\phi \mid$ denotes the Hilbert-Schmidt norm of the differential $d\phi \in \Gamma(T^*M \otimes \phi^{-1}TN)$. ϕ is called *harmonic* if

$$\frac{d}{dt} \mid_{t=0} E(\phi_t) = 0, \tag{2.2}$$

for any smooth variation $\phi_t : M \longrightarrow N$ with $\phi_0 = \phi$. The corresponding Euler-Lagrange equation of the energy functional E is given by

$$\tau(\phi) := \nabla^{\phi}_{e_i} d\phi(e_i) - d\phi(\nabla_{e_i} e_i) = 0, \tag{2.3}$$

where $\{e_i\}_{i=1}^m$ is a local orthonormal frame field on M (here henceforward we sum over repeated indices). $\tau(\phi)$ is called the *tension field* of ϕ. In terms of the Euler-Lagrange equation, a map ϕ is called harmonic if $\tau(\phi) = 0$, [9]. Furthermore, the *stress energy* tensor of ϕ is defined by

$$S_\phi = \mid d\phi \mid^2 g - \phi^* h.$$

ϕ is said to satisfy the *conservation law* if

$$0 = div S_\phi(X) = -h(\tau(\phi), d\phi(X)), \tag{2.4}$$

for any vector field X on M, [8]. Harmonic maps have been studied by many authors (see [1,2,11,16,18]).
Biharmonic maps $\phi : (M, g) \longrightarrow (N, h)$ are critical points of the bienergy functional

$$E_2(\phi) = \frac{1}{2}\int_M \mid \tau(\phi) \mid^2 dv_g. \tag{2.5}$$

The Euler- Lagrange equation associated to E_2 is, (see [15]):

$$\tau_2(\phi) := -J^\phi(\tau(\phi)) = 0, \tag{2.6}$$

where J^ϕ denotes the Jacobi operator of ϕ on sections of the pull-back bundle $\phi^{-1}TN$, defined by

$$J^\phi(V) := -\Delta^\phi V - trace_g R^N(d\phi, V)d\phi, \qquad V \in \phi^{-1}TN, \tag{2.7}$$

here $R^N(X,Y) = [\nabla^N X, \nabla^N Y] - \nabla^N_{[X,Y]}$ is the curvature operator on N, and Δ^ϕ is the rough Laplacian on sections of $\phi^{-1}TN$ which is defined with respect to a local orthonormal frame field $\{e_i\}$ on M as follows:

$$-\Delta^\phi V := \nabla^\phi_{e_i} \nabla^\phi_{e_i} V - \nabla^\phi_{\nabla_{e_i} e_i} V, \tag{2.8}$$

Remark 2.1. It has been proven that biharmonic maps between Riemannian manifolds always exist while harmonic maps do not exist neccesarily as demonstrated by the nonexistence of harmonic maps from T^2 to S^2, [10].

Theorem 2.2 (The second variation formula of bienergy functional, see [15])**.** *Let $\phi : (M, g) \longrightarrow (N, h)$ be a biharmonic map, and $\{\phi_t\}$ be an arbitrary smooth variation of ϕ such that $\phi_0 = \phi$. Then the second variation formula of $\frac{1}{2}E_2(\phi_t)$ is given by*

$$
\begin{aligned}
&\frac{1}{2}\frac{d^2}{dt^2}\ |_{t=0}\ E_2(\phi_t) \\
&= \int_M |\ -\Delta^\phi V - R^N(d\phi(e_i), V)d\phi(e_i)\ |^2 \\
&-h\Big(V, (\nabla^\phi_{e_i} R^N)(d\phi(e_i), \tau(\phi))V + (\nabla^\phi_{\tau(\phi)} R^N)(d\phi(e_i), V)d\phi(e_i) \\
&+R^N(\tau(\phi), V)\tau(\phi) + 2R^N(d\phi(e_i), V)\nabla^\phi_{e_i}\tau(\phi) \\
&+2R^N(d\phi(e_i), \tau(\phi))\nabla^\phi_{e_i} V\Big) dv_g,
\end{aligned}
\tag{2.9}
$$

where $V = \frac{d\phi_t}{dt}\ |_{t=0}$.

ϕ is said to be *stable biharmonic* if $\frac{d^2}{dt^2}\ |_{t=0}\ E_2(\phi_t) \geq 0$ for every smooth variation $\{\phi_t\}$ of ϕ, [15]. In particular, ϕ is called *strongly stable biharmonic* if $k_\phi(V)$ is non-negative for every vector field V along ϕ, where k_ϕ is defined by

$$
\begin{aligned}
k_\phi(V) :=& |\ -\Delta^\phi V - R^N(d\phi(e_i), V)d\phi(e_i)\ |^2 \\
&- h\Big(V, (\nabla^\phi_{e_i} R^N)(d\phi(e_i), \tau(\phi))V + (\nabla^\phi_{\tau(\phi)} R^N)(d\phi(e_i), V)d\phi(e_i) \\
&+ R^N(\tau(\phi), V)\tau(\phi) + 2R^N(d\phi(e_i), V)\nabla^\phi_{e_i}\tau(\phi) \\
&+ 2R^N(d\phi(e_i), \tau(\phi))\nabla^\phi_{e_i} V\Big).
\end{aligned}
\tag{2.10}
$$

3. **The first variation formula**

A smooth map $\phi : (M, g) \longrightarrow (N, h)$ is called *F-biharmonic* if ϕ is a critical point of the F-bienergy functional defined by (1.2), i.e.,

$$
\frac{d}{dt}\ |_{t=0}\ E_{F,2}(\phi_t) = 0, \tag{3.1}
$$

for any smooth variation $\phi_t : M \longrightarrow N\ \ (-\epsilon < t < \epsilon)$ with $\phi_0 = \phi$.

Theorem 3.1 (The first variation formula, see [13])**.** *Let $\phi : M \longrightarrow N$ be a smooth map, and $\phi_t : M \longrightarrow N\ \ (-\epsilon < t < \epsilon)$ be a smooth variation such that $\phi_0 = \phi$. Then*

$$
\frac{d}{dt}\ |_{t=0}\ E_{F,2}(\phi_t) = \int_M h(\tau_{F,2}, V)dv_g, \tag{3.2}
$$

where $V = \frac{d\phi_t}{dt}\ |_{t=0}$ and $\tau_{F,2}$ is defined by (1.3).

Using the above theorem, ϕ is F-biharmonic if and only if $\tau_{F,2}(\phi) \equiv 0$. By (1.3), every harmonic map is F-biharmonic . Thus, we focus on non-trivial F-biharmonic maps which are non-harmonic F-biharmonic maps.
A smooth map $\phi : (M, g) \longrightarrow (N, h)$ is said to be *conformal* if there exists a positve real function λ on M such that $\phi^* h = \lambda^2 g$, λ is called the dilation of ϕ. In particular, ϕ is called *homothetic* if λ is constant.

Remark 3.2. The tension field of a conformal map $\phi : (M^n, g) \to (N^n, h)$ with dilation λ, is given by, [4]:

$$\tau(\phi) = (2 - n)\, d\phi(grad\ \ln \lambda). \tag{3.3}$$

In order to investigate the F-biharmonicity of conformal maps, we need the following lemma

Lemma 3.3. *Let $\phi : (M, g) \longrightarrow (N, h)$ be a smooth map. Then ϕ is F-biharmonic if and only if*

$$\begin{aligned} 0 &= 4F'(\frac{\mid \tau(\phi) \mid^2}{2})\tau_2(\phi) + (F'''(\frac{\mid \tau(\phi) \mid^2}{2}) \mid grad\ (\mid \tau(\phi) \mid^2) \mid^2 \\ &- 2F''(\frac{\mid \tau(\phi) \mid^2}{2})\Delta \mid \tau(\phi) \mid^2)\tau(\phi) + 4F''(\frac{\mid \tau(\phi) \mid^2}{2})\nabla^{\phi}_{grad\ (\mid\tau(\phi)\mid^2)}\tau(\phi), \end{aligned} \tag{3.4}$$

where $\mid \nabla^{\phi}\tau(\phi) \mid^2 := h(\nabla^{\phi}_{e_i}\tau(\phi), \nabla^{\phi}_{e_i}\tau(\phi))$ for any local orthonormal frame field $\{e_i\}$ on M.

Proof. Let $\{e_i\}$ be a local orthonormal frame field on M. Using (1.3), we have:

$$\begin{aligned} \tau_{F,2}(\phi) &= -\Delta^{\phi}(F'(\frac{\mid \tau(\phi) \mid^2}{2})\tau(\phi)) - R^N(d\phi(e_i), F'(\frac{\mid \tau(\phi) \mid^2}{2})\tau(\phi))d\phi(e_i) \\ &= \Big(F''(\frac{\mid \tau(\phi) \mid^2}{2})h(-\Delta^{\phi}\tau(\phi), \tau(\phi)) + F''(\frac{\mid \tau(\phi) \mid^2}{2}) \mid \nabla^{\phi}\tau(\phi) \mid^2 \\ &+ \frac{1}{4}F'''(\frac{\mid \tau(\phi) \mid^2}{2}) \mid grad\ (\mid \tau(\phi) \mid^2) \mid^2 \Big)\tau(\phi) \\ &+ F''(\frac{\mid \tau(\phi) \mid^2}{2})\nabla^{\phi}_{grad\ (\mid\tau(\phi)\mid^2)}\tau(\phi) + F'(\frac{\mid \tau(\phi) \mid^2}{2})\tau_2(\phi). \end{aligned} \tag{3.5}$$

On the other hand, with straightforward calculation, we have

$$h(-\Delta^{\phi}\tau(\phi), \tau(\phi)) = -\frac{1}{2}\Delta(\mid \tau(\phi) \mid^2) - \mid \nabla^{\phi}\tau(\phi) \mid^2 . \tag{3.6}$$

By substituting (3.6) in (3.5), we obtain formula (3.4), and this completes the proof. ∎

In the following example we find a non-trivial F-biharmonic map by using Lemma 3.3.

Example 3.4. Let $(\mathbb{S}^m, g)$ be the m-dimensional Euclidean sphere and define a Riemannian metric h on $N = \mathbb{R}^n - \{0\} \times \mathbb{S}^m$ by

$$h = g_{_{Eucl}} + f^2 g,$$

where $g_{_{Eucl}}$ is the Euclidean metric on $\mathbb{R}^n - \{0\}$, and $f \in C^\infty(\mathbb{R}^n - \{0\})$ is defined by $f(x) = \sqrt{\mid x \mid}$ for $x \in \mathbb{R}^n - \{0\}$. In this case

$$\mid grad\ f^2 \mid = 1. \tag{3.7}$$

For any arbitrary point $x_0 \in \mathbb{R}^n - \{0\}$, the tension and bitension field of the inclusion map

$$\begin{aligned} \phi : \mathbb{S}^m &\longrightarrow N \\ y &\longrightarrow (x_0, y) \end{aligned}$$

are given by

$$\tau(\phi) = -\frac{m}{2}(grad\ f^2, 0) \circ \phi, \tag{3.8}$$

$$\tau_2(\phi) = \frac{m^2}{8}(grad \mid grad\ f^2 \mid^2, 0) \circ \phi = 0, \tag{3.9}$$

for more details (see [5, heorem 3.1]). By setting $F(t) = e^t$, and substituting (3.7)-(3.9) in (3.4), we conclude that i_{x_0} is an F-biharmonic map.

In the following theorem, the F-biharmonicity of conformal maps between manifolds of the same dimension is studied.

Theorem 3.5. *Let $\phi : (M^n, g) \longrightarrow (N^n, h)$ be a conformal map with dilation λ. Then ϕ is F-biharmonic if and only if*

$$\begin{aligned} 0 &= 4F'(\frac{\mid \tau(\phi) \mid^2}{2})\tau_2(\phi) \\ &+ (n-2)^5 \lambda^4 F'''(\frac{\mid \tau(\phi) \mid^2}{2}) \mid grad\ (\mid grad\ \ln\lambda \mid^2) \\ &+ 2 \mid grad\ \ln\lambda \mid^2 grad\ \ln\lambda \mid^2 grad\ \ln\lambda \\ &- 2(n-2)^3\lambda^2 F''(\frac{\mid \tau(\phi) \mid^2}{2})\Bigg\{\Bigg(\Delta(\mid grad\ \ln\lambda \mid^2) \\ &+ 2 \mid grad\ \ln\lambda \mid^2 \Delta \ln\lambda + 4 grad\ \ln\lambda(\mid grad\ \ln\lambda \mid^2)\Bigg) grad\ \ln\lambda \\ &- 2\nabla_{grad\ (\mid grad\ \ln\lambda\mid^2)} grad\ \ln\lambda \\ &- 4 \mid grad\ \ln\lambda \mid^2 grad\ (\mid grad\ \ln\lambda \mid^2)\Bigg\}. \end{aligned} \tag{3.10}$$

Proof. Let x_0 be an arbitrary fixed point on M, and $\{e_i\}$ be an orthonormal frame field in a neighbourhood of x_0, such that $d\phi(e_i) = \lambda(f_i \circ \phi)$ for $i \in$

$\{1,\ldots,n\}$. Here $\{f_i\}$ is the orthonormal frame field, defined in a neighbourhood of $\phi(x_0)$ such that $\nabla_{f_i} f_j = 0$ at $\phi(x_0)$. By the construction of $\{e_i\}$, we have

$$\nabla_{e_i} e_j = -e_j(ln\lambda)e_i + \delta_{ij} grad \ \ \ln\lambda, \tag{3.11}$$

at the point x_0, (see [3], pp.407). By (3.3), the following equalities hold:

(i) $grad\ (\mid \tau(\phi) \mid^2) = (n-2)^2\lambda^2\Big(grad\ (\mid grad\ \ln\lambda \mid^2)$
$+2 \mid grad\ \ln\lambda \mid^2) grad\ \ln\lambda \Big).$

(ii) $\Delta \mid \tau(\phi) \mid^2 = (n-2)^2\lambda^2(\Delta(\mid grad\ \ln\lambda \mid^2) + 2 \mid grad\ \ln\lambda \mid^2 \Delta \ln\lambda$
$+4 grad\ \ln\lambda(\mid grad\ \ln\lambda \mid^2) + 4 \mid grad\ \ln\lambda \mid^4).$

(iii) $\nabla^\phi_{grad\ (\mid\tau(\phi)\mid^2)}\tau(\phi) = (2-n)^3\lambda^2\Big(d\phi(\nabla^\phi_{grad\ (\mid grad\ \ln\lambda\mid^2)} grad\ \ln\lambda)$
$+2 \mid grad\ \ln\lambda \mid^2 d\phi(grad\ (\mid grad\ \ln\lambda \mid^2))$
$+ 2 \mid grad\ \ln\lambda \mid^4 d\phi(grad\ \ln\lambda) \Big).$

By substituting the above identities in (3.4) and considering that all conformal maps between equidimensional manifolds are local diffeomorphism, formula (3.10) is obtained. The first two identities can be obtained directly by calculation, only the third identity needs to be proven. Using the first identity, the left hand side of (iii) can be rewritten as follows:

$$\begin{aligned}\nabla^\phi_{grad\ (\mid\tau(\phi)\mid^2)}\tau(\phi) = (2-n)^3\lambda^2(&\nabla^\phi_{grad\ (\mid grad\ \ln\lambda\mid^2)} d\phi(grad\ \ln\lambda)\\ &+ 2 \mid grad\ \ln\lambda \mid^2 \nabla^\phi_{grad\ \ln\lambda} d\phi(grad\ \ln\lambda)).\end{aligned} \tag{3.12}$$

Now we calculate each term of (3.12). We get

$$\begin{aligned}\nabla^\phi_{grad\ (\mid grad\ \ln\lambda\mid^2)} d\phi(grad\ \ln\lambda) &= e_i(\mid grad\ \ln\lambda \mid^2) e_i(e_j(\ln\lambda)) d\phi(e_j)\\ &+ grad\ \ln\lambda(\mid grad\ \ln\lambda \mid^2) d\phi(grad\ \ln\lambda).\end{aligned} \tag{3.13}$$

By (3.11), we have

$$\begin{aligned}\nabla_{grad\ (\mid grad\ \ln\lambda\mid^2)} grad\ \ln\lambda &= e_i(\mid grad\ \ln\lambda \mid^2) e_i(e_j(\ln\lambda)) e_j\\ &- \mid grad\ \ln\lambda \mid^2 grad\ (\mid grad\ \ln\lambda \mid^2)\\ &+ grad\ \ln\lambda(\mid grad\ \ln\lambda \mid^2) grad\ \ln\lambda.\end{aligned} \tag{3.14}$$

Following the relations (3.13) and (3.14) we get

$$\begin{aligned}\nabla^\phi_{grad\ (\mid grad\ \ln\lambda\mid^2)} d\phi(grad\ \ln\lambda) &= d\phi(\nabla_{grad\ (\mid grad\ \ln\lambda\mid^2)} grad\ \ln\lambda)\\ &+ \mid grad\ \ln\lambda \mid^2 d\phi(grad\ (\mid grad\ \ln\lambda \mid^2)).\end{aligned} \tag{3.15}$$

Similarly, we have

$$\nabla^{\phi}_{grad\ \ln\lambda} d\phi(grad\ \ln\lambda) = \mid grad\ \ln\lambda \mid^2 d\phi(grad\ \ln\lambda) + \frac{1}{2} d\phi(grad\ (\mid grad\ \ln\lambda \mid^2)). \tag{3.16}$$

Substituting (3.15) and (3.16) in (3.12), we obtain (iii). This completes the proof. □

Definition 3.6 ([5])**.** Let (M, g) be a Riemannian manifold and $\alpha \in C^{\infty}(M)$ be a real smooth function on M. α is said to be an affine function if $\alpha \circ \gamma : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ is an affine function for any geodesic γ on M.

Remark 3.7. It is well known that for an affine function α, $grad\ \alpha$ is a parallel vector field, hence $\mid grad\ \alpha \mid$ is constant, [19].

By the above argument and using (3.10), we have the following proposition

Proposition 3.8. *Let $\phi : (M^n, g) \longrightarrow (N^n, h)$ be a conformal map with dilation $\lambda(x) = e^{\sigma(x)}$, where σ is an affine function . Then ϕ is F-biharmonic if and only if ϕ is biharmonic.*

Based on the above proposition, every homothetic map between manifolds of the same dimension is F-biharmonic.

4. **The second variation formula**

In this section we derive the second variation formula of F-bienergy functional. Then instability and nonexistence theorems for F-biharmonic maps are given.

Theorem 4.1. *Let $\phi : (M, g) \longrightarrow (N, h)$ be an F-biharmonic map . Moreover, let $\{\phi_t\}$ be an arbitrary smooth variation of ϕ such that $\phi_0 = \phi$, and set $V = \frac{d\phi_t}{dt} \mid_{t=0}$. Then the second variation formula of $E_{F,2}(\phi_t)$ is given as follows:*

$$\begin{aligned}
&\frac{d^2}{dt^2} \mid_{t=0} E_{F,2}(\phi_t) \\
&= \int_M F'(\frac{\mid \tau(\phi) \mid^2}{2}) k_{\phi}(V) + h(V, T_{\phi,F}(V)\tau_2(\phi) \\
&-\Delta F'(\frac{\mid \tau(\phi) \mid^2}{2}) J^{\phi}(V) - \Delta T_{\phi,F}(V)\tau(\phi) \\
&+2\nabla^{\phi}_{grad\ (T_{\phi,F}(V))}\tau(\phi) + 2\nabla^{\phi}_{grad\ (F'(\frac{|\tau(\phi)|^2}{2}))} J^{\phi}(V) \\
&-2R^N(d\phi(grad\ F'(\frac{\mid \tau(\phi) \mid^2}{2})), V)\tau(\phi)) dv_g,
\end{aligned} \tag{4.1}$$

where $T_{\phi,F}(V) := \frac{d}{dt} \mid_{t=0} F'(\frac{|\tau(\phi_t)|^2}{2})$ and $k_{\phi}(V)$ is defined by (2.10).

Proof. Let $\Phi : M \times (-\epsilon, \epsilon) \longrightarrow N$ be a smooth map defined by $\Phi(t,x) = \phi_t(x)$, where $M \times (-\epsilon, \epsilon)$ is equipped with the product metric. Let $\bar{\nabla}, \nabla^{\Phi}$ and $\tilde{\nabla}$ denote the induced connections on $T(M \times (-\epsilon, \epsilon)), \Phi^{-1}TN$ and $T^*(M \times (-\epsilon, \epsilon)) \otimes \Phi^{-1}TN$ respectively. Considering canonical extension of $\frac{\partial}{\partial t}$ and X to $M \times (-\epsilon, \epsilon)$ and denoting the extensions by $\frac{\partial}{\partial t}$ and X again, we have

$$\bar{\nabla}_{\frac{\partial}{\partial t}} e_k - \bar{\nabla}_{e_k} \frac{\partial}{\partial t} = [e_k, \frac{\partial}{\partial t}] = 0. \tag{4.2}$$

Using (1.3) and (3.2), together with the assumption that ϕ is F-biharmonic, we have

$$\begin{aligned} \frac{d^2}{dt^2} E_{F,2}(\phi_t)\Big|_{t=0} = \int_M \Big\langle d\Phi(\frac{\partial}{\partial t}), \nabla^{\Phi}_{\frac{\partial}{\partial t}} \Big\{ &\nabla^{\Phi}_{e_k} \nabla^{\Phi}_{e_k} [F'(\frac{\mid \tau(\phi_t) \mid^2}{2})(\tilde{\nabla}_{e_i} d\Phi)(e_i)] \\ &- \nabla^{\Phi}_{\bar{\nabla}_{e_k} e_k} [F'(\frac{\mid \tau(\phi_t) \mid^2}{2})(\tilde{\nabla}_{e_i} d\Phi)(e_i)] \\ &- R^N(d\Phi(e_k), F'(\frac{\mid \tau(\phi_t) \mid^2}{2})(\tilde{\nabla}_{e_i} d\Phi)(e_i)) d\Phi(e_k) \Big\} \Big\rangle \Big|_{t=0} dv_g, \end{aligned} \tag{4.3}$$

where $\langle , \rangle$ is the inner product on $\Phi^{-1}TN$, with respect to g and h. Now, we have

$$\begin{aligned} &\frac{d^2}{dt^2} E_{F,2}(\phi_t)\Big|_{t=0} \\ &= \int_M \Big\langle \Big(e_k(e_k(F'(\frac{\mid \tau(\phi_t) \mid^2}{2}))) - \bar{\nabla}_{\bar{\nabla}_{e_k} e_k} F'(\frac{\mid \tau(\phi_t) \mid^2}{2}) \Big) \\ &\nabla^{\Phi}_{\frac{\partial}{\partial t}} [(\tilde{\nabla}_{e_i} d\Phi)(e_i)] + 2 e_k(F'(\frac{\mid \tau(\phi_t) \mid^2}{2})) \nabla^{\Phi}_{\frac{\partial}{\partial t}} \nabla^{\Phi}_{e_k} [(\tilde{\nabla}_{e_i} d\Phi)(e_i)] \\ &+ \Big(e_k(e_k(\frac{\partial F'(\frac{|\tau(\phi_t)|^2}{2})}{\partial t})) - \bar{\nabla}_{\bar{\nabla}_{e_k} e_k} (\frac{\partial F'(\frac{|\tau(\phi_t)|^2}{2})}{\partial t}) \Big) \\ &(\tilde{\nabla}_{e_i} d\Phi)(e_i) + 2 e_k(\frac{\partial F'(\frac{|\tau(\phi_t)|^2}{2})}{\partial t}) \nabla^{\Phi}_{e_k} [(\tilde{\nabla}_{e_i} d\Phi)(e_i)] \\ &+ \frac{\partial F'(\frac{|\tau(\phi_t)|^2}{2})}{\partial t} \Big(\nabla^{\Phi}_{e_k} \nabla^{\Phi}_{e_k} [(\tilde{\nabla}_{e_i} d\Phi)(e_i)] \\ &- \nabla^{\Phi}_{\bar{\nabla}_{e_k} e_k} [(\tilde{\nabla}_{e_i} d\Phi)(e_i)] - R^N(d\Phi(e_k), (\tilde{\nabla}_{e_i} d\Phi)(e_i)) d\Phi(e_k) \Big) \\ &+ F'(\frac{\mid \tau(\phi_t) \mid^2}{2}) \nabla^{\Phi}_{\frac{\partial}{\partial t}} \Big(\nabla^{\Phi}_{e_k} \nabla^{\Phi}_{e_k} [(\tilde{\nabla}_{e_i} d\Phi)(e_i)] - \nabla^{\Phi}_{\bar{\nabla}_{e_k} e_k} [(\tilde{\nabla}_{e_i} d\Phi)(e_i)] \\ &- R^N(d\Phi(e_k), (\tilde{\nabla}_{e_i} d\Phi)(e_i)) d\Phi(e_k) \Big), d\Phi(\frac{\partial}{\partial t}) \Big\rangle \Big|_{t=0} dv_g. \end{aligned} \tag{4.4}$$

By using ([15, Equation 23]), we have

$$\nabla^{\Phi}_{\frac{\partial}{\partial t}}[(\tilde{\nabla}_{e_i}d\Phi)(e_i)]\Big|_{t=0}=J^{\phi}(V) \tag{4.5}$$

We continue our calculation and get

$$\begin{aligned}
&e_k(F'(\frac{\mid\tau(\phi_t)\mid^2}{2}))\nabla^{\Phi}_{\frac{\partial}{\partial t}}\nabla^{\Phi}_{e_k}[(\tilde{\nabla}_{e_i}d\Phi)(e_i)]\Big|_{t=0}\\
=&e_k(F'(\frac{\mid\tau(\phi_t)\mid^2}{2}))\Big\{R^N(d\Phi(\frac{\partial}{\partial t}),d\Phi(e_k))[(\tilde{\nabla}_{e_i}d\Phi)(e_i)]\\
&+\nabla^{\Phi}_{e_k}\nabla^{\Phi}_{\frac{\partial}{\partial t}}[(\tilde{\nabla}_{e_i}d\Phi)(e_i)]+\nabla^{\Phi}_{[\frac{\partial}{\partial t},e_k]}[(\tilde{\nabla}_{e_i}d\Phi)(e_i)]\Big\}\Big|_{t=0}\\
=&R^N(V,d\phi(grad\ (F'(\frac{\mid\tau(\phi)\mid^2}{2}))))\tau(\phi)\\
&+\nabla^{\phi}_{grad(F'(\frac{|\tau(\phi)|^2}{2}))}J^{\phi}(V).
\end{aligned} \tag{4.6}$$

By direct calculation and using (2.10) and (4.5), the latter term of (4.4) can be obtained as follows

$$\begin{aligned}
&\int_M F'(\frac{\mid\tau(\phi_t)\mid^2}{2})\Big\langle d\phi(\frac{\partial}{\partial t}),\nabla^{\Phi}_{\frac{\partial}{\partial t}}\Big(\nabla^{\Phi}_{e_k}\nabla^{\Phi}_{e_k}[(\tilde{\nabla}_{e_i}d\Phi)(e_i)]\\
&-\nabla^{\Phi}_{\tilde{\nabla}_{e_k}e_k}[(\tilde{\nabla}_{e_i}d\Phi)(e_i)]\\
&-R^N(d\Phi(e_k),[(\tilde{\nabla}_{e_i}d\Phi)(e_i)])d\Phi(e_k)\Big)\Big\rangle\Big|_{t=0}d\upsilon_g\\
=&\int_M F'(\frac{\mid\tau(\phi_t)\mid^2}{2})k_{\phi}(V)d\upsilon_g.
\end{aligned} \tag{4.7}$$

Substituting formulas (4.5)-(4.7) into (4.4), we obtain formula (4.1). This completes the proof. □

Definition 4.2. Let $\phi:(M,g)\longrightarrow(N,h)$ be an F-biharmonic map, and $\{\phi_t\}$ be a smooth variation of ϕ such that $\phi_0=\phi$. We set

$$I^{\phi}_{F,2}(V,V):=\frac{d^2}{dt^2}\mid_{t=0}E_{F,2}(\phi_t), \tag{4.8}$$

where $V=\frac{d\phi_t}{dt}\mid_{t=0}$, ϕ is said to be stable F-biharmonic if $I^{\phi}_{F,2}(V,V)\geq 0$ for every vector field V along ϕ.

Remark 4.3. Let (M,g) be a Riemannian manifold and $Id:(M,g)\longrightarrow(M,g)$ be an identity map. Using (2.10) and (4.1) and considering that $\tau(Id)=0$, it can be concluded that Id is stable F-biharmonic. In fact Id is an absolute minimum of the F-bienergy functional.

Now, we use the methods of [13] and [15] to establish the following theorem, which extends a theorem in [15] for biharmonic maps.

Theorem 4.4. *Let (N, h) be a Riemannian manifold with a constant positive sectional curvature K, and $\phi : (M, g) \longrightarrow (N, h)$ be a non-trivial F-biharmonic map satisfies (2.4). Then ϕ is unstable F-biharmonic if ϕ is biharmonic.*

Proof. Using (2.10)and (4.1), together with the assumption that N is of constant sectional curvature, i.e., $\nabla^N R^N = 0$, we have

$$
\begin{aligned}
&\frac{d^2}{dt^2}\bigg|_{t=0} E_{F,2}(\phi_t) \\
&= \int_M F'(\frac{\mid \tau(\phi) \mid^2}{2})\bigg| - \Delta^\phi V - trace_g R^N(d\phi, V)d\phi\bigg|^2 dv_g \\
&- \int_M F'(\frac{\mid \tau(\phi) \mid^2}{2})\bigg(h(V, R^N(\tau(\phi), V)\tau(\phi)) \\
&+2\, trace_g\ R^N(d\phi, V)\nabla^\phi\tau(\phi) + 2\, trace_g\ R^N(d\phi, \tau(\phi))\nabla^\phi V\bigg)dv_g \\
&+ \int_M h\bigg(V, T_{\phi,F}(V)\tau_2(\phi) - \Delta F'(\frac{\mid \tau(\phi) \mid^2}{2})J^\phi(V) - \Delta T_{\phi,F}(V)\tau(\phi) \\
&+2\nabla^\phi_{grad\ \ (T_{\phi,F}(V))}\tau(\phi) + 2\nabla_{grad\ \ F'(\frac{|\tau(\phi)|^2}{2})}J^\phi(V) \\
&-2R^N(d\phi(grad\ \ F'(\frac{\mid \tau(\Phi) \mid^2}{2})), V)\tau(\phi)\bigg)dv_g.
\end{aligned}
\tag{4.9}
$$

On the other hand, $T_{\phi,F}(V) = \frac{d}{dt}|_{t=0}F'(\frac{|\tau(\phi_t)|^2}{2})$ can be obtained as follows

$$
T_{\phi,F}(V) = F''(\frac{\mid \tau(\phi) \mid^2}{2})h(J^\phi(V), \tau(\phi)). \tag{4.10}
$$

By setting $V = \tau(\phi)$ in (4.10) together with the assumption that ϕ is biharmonic, i.e., $\tau_2(\phi) = J^\phi(\tau(\phi)) = 0$, we obtain $T_{\phi,F}(\tau(\phi)) = 0$. Furthermore, by (2.4), we have

$$
h(d\phi(e_i), \nabla^\phi_{e_i}\tau(\phi)) = - \mid \tau(\phi) \mid^2 . \tag{4.11}
$$

Substituting $\tau_2(\phi) = 0$ and $T_{\phi,F}(\tau(\phi)) = 0$ in (4.9) and using (2.4) together with (4.11), the right hand side of (4.9) can be rewritten as follows:

$$\begin{aligned} I^{\phi}_{F,2}(\tau(\phi), \tau(\phi)) &= \int_M h(\tau(\phi), 4trace_g\ R^N(d\phi, \tau(\phi))\nabla^{\phi}\tau(\phi))dv_g \\ &= 4K \int_M \Big\{ h(d\phi(e_i), \nabla^{\phi}_{e_i}\tau(\phi))h(\tau(\phi), \tau(\phi)) \\ &\quad - h(d\phi(e_k), \tau(\phi))h(\tau(\phi), \nabla^{\phi}_{e_k}\tau(\phi)) \Big\} dv_g \\ &= -4K \int_M \mid \tau(\phi) \mid^4 dv_g \leq 0. \end{aligned} \tag{4.12}$$

It follows that $I^{\phi}_{F,2}(\tau(\phi), \tau(\phi)) = 0$ if and only if $\tau(\phi) = 0$. This is in contradiction with the fact that ϕ is a non-trivial F-biharmonic map. Hence $I^{\phi}_{F,2}(\tau(\phi), \tau(\phi)) < 0$, which completes the proof. $\square$

Along the lines of Theorem 4.1 and formula (4.10), we have the following theorem

Theorem 4.5. *An F-biharmonic map $\phi : (M, g) \longrightarrow (N, h)$ is stable if ϕ is strongly stable biharmonic and*

$$grad\ (F' \frac{\mid \tau(\phi) \mid^2}{2})) = grad\ (h(J^{\phi}(V), \tau(\phi))) = 0 \tag{4.13}$$

for any vector field V along ϕ.

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