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FIVE-VALUE RICH LINES, BOREL DIRECTIONS AND UNIQUENESS OF MEROMORPHIC FUNCTIONS

J.R. LONG

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Abstract. For a meromorphic function f in the complex plane, we shall introduce the notion of five-value rich line of f, and study the uniqueness of meromorphic functions of finite order in an angular domain by involving the five-value rich line and Borel directions. Finally, the relationship between a five-value rich line and a Borel direction is discussed, that is, every Borel direction of f is its five-value rich line, and the inverse statement holds when f is of infinite order.
Keywords: Borel direction, five-value rich line, meromorphic function, sharing value, uniqueness.
MSC(2010): Primary: 30D30; Secondary: 30D35.

1. Introduction and main results

As usual, the abbreviations IM and CM refer to sharing values ignoring multiplicities and counting multiplicities in a domain $D \subseteq \mathbb{C}$, respectively, where $\mathbb{C}$ denotes the complex plane. In addition, $\rho(f)$ denotes the order of growth of a meromorphic function f in $\mathbb{C}$. For further notation and basic results in Nevanlinna theory, we refer to [8, 20].

In [14], Nevanlinna proved the remarkable five-value theorem and four-value theorem by using his value distribution theory. We recall the former result as follows.

Theorem 1.1 ([14]). *Let f and g be two non-constant meromorphic functions in $\mathbb{C}$, and let $a_i \in \bar{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, $i = 1, 2, 3, 4, 5$, be five distinct values. If f and g share the values a_i, $i = 1, 2, 3, 4, 5$, IM in $\mathbb{C}$, then $f = g$.*

The five-value and four-value theorems gave birth to a new research field nowadays known as the uniqueness theory [22]. In [25, 26] Zheng obtained analogues of these two fundamental theorems valid for meromorphic functions

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in angular domains. Further development on such results can be found in [1, 10–13, 18, 24].

In this paper we consider uniqueness theory in an angular domain by using the concept of proximate order and type. The following result is our starting point.

Theorem 1.2 ([9]). *Let $B(r)$ be a positive and continuous function in $[0,\infty)$ which satisfies $\limsup_{r\to\infty}\frac{\log B(r)}{\log r}=\infty$. Then there exists a continuously differentiable function $\rho(r)$, which satisfies the following conditions.*

- (i) *$\rho(r)$ is continuous and nondecreasing for $r\geq r_0$ and tends to ∞ as $r\to\infty$, where $r_0>0$ is a constant;*
- (ii) *The function $U(r)=r^{\rho(r)}$, $r\geq r_0$, satisfies the condition*

$$\lim_{r\to\infty}\frac{\log U(R)}{\log U(r)}=1,\quad R=r+\frac{r}{\log U(r)};$$

- (iii)

$$\limsup_{r\to\infty}\frac{\log B(r)}{\log U(r)}=1.$$

Theorem 1.2 is due to Hiong [9]. A simple proof of the existence of $\rho(r)$ was given by Chuang [2].

Definition 1.3. We call $\rho(r)$ and $U(r)$ discussed in Theorem 1.2 as the proximate order and type function of $B(r)$, respectively. For a meromorphic function f of infinite order, we define its proximate order and type function as the proximate order and type function of $T(r,f)$. If $\rho(r)$ is a proximate order of a meromorphic function f of infinite order in $\mathbb{C}$, we denote by $M(\rho(r))$ the set of all meromorphic functions g in $\mathbb{C}$ such that

$$\limsup_{r\to\infty}\frac{\log T(r,g)}{\rho(r)\log r}\leq 1.$$

In order to recall the definition of the Borel direction, we also need the following notation. For $0\leq\theta<2\pi$ and $\delta\in(0,\pi)$, define

$$\Delta(\theta,\delta)=\{z:|\arg z-\theta|\leq\delta\}.$$

Next we define a Borel direction of order $\rho(r)$. The concept is originally due to Hiong [9], and it can also be found in [3, p. 140] or [4].

Definition 1.4. Suppose that $\rho(r)$ is a proximate order of a meromorphic function f of infinite order in $\mathbb{C}$. A ray $\arg z=\theta\in[0,2\pi)$ from the origin is called a Borel direction of order $\rho(r)$ of f, if for any $\varepsilon>0$,

$$\limsup_{r\to\infty}\frac{\log n(r,\Delta(\theta,\varepsilon),\frac{1}{f-a})}{\rho(r)\log r}=1,\tag{1.1}$$

for all $a \in \bar{\mathbb{C}}$, with at most two exceptions, where $n(r, \Delta(\theta, \varepsilon), \frac{1}{f-a})$ denotes the number of zeros of $f - a$ in $\Delta(\theta, \varepsilon) \cap \{z : |z| \leq r\}$, each zero of $f - a$ being counted according to its multiplicity; the zeros are replaced with poles of f if $a = \infty$.

It is well known that every meromorphic function of infinite order must have at least one Borel direction of order $\rho(r)$. The proof can be found in [3, pp. 140-145]. Next we also need the definition of Borel direction of order $\rho \in (0, \infty)$, which can be found in [20, pp. 83-84].

Definition 1.5. Let f be a meromorphic function of finite positive order ρ in $\mathbb{C}$. A ray $\arg z = \theta \in [0, 2\pi)$ from the origin is called a Borel direction of order ρ of f, if for any $\varepsilon > 0$,

$$\limsup_{r\to\infty} \frac{\log n(r, \Delta(\theta, \varepsilon), \frac{1}{f-a})}{\log r} = \rho, \tag{1.2}$$

for all $a \in \bar{\mathbb{C}}$, with at most two exceptions.

According to the fundamental result [20, Theorem 3.8] due to Valiron, a meromorphic function of finite positive order ρ must have at least one Borel direction of order ρ. In the theory of angular value distribution of meromorphic functions, Borel directions play a fundamental role [5, 15–17, 19, 21, 23]. In [13] the authors take advantage of the proximate order, and prove the following uniqueness result in an angular domain.

Theorem 1.6 ([13])**.** *Let $\rho(r)$ be a proximate order of a meromorphic function f of infinite order in $\mathbb{C}$, and let $g \in M(\rho(r))$. Suppose that $\arg z = \theta \in [0, 2\pi)$ is a Borel direction of order $\rho(r)$ of f. If for any $\varepsilon > 0$, the functions f and g share five distinct values $a_i \in \bar{\mathbb{C}}$, $i = 1, 2, 3, 4, 5$, IM in $\Delta(\theta, \varepsilon)$, then $f = g$.*

By Theorem 1.6, we can pose the question: What can we say if f and g are two non-constant meromorphic functions of finite order? To approach this question, we need a new technique. To this end, we introduce the definition of five-value rich line.

Definition 1.7. Let f be a non-constant meromorphic function in $\mathbb{C}$, and let $a_i \in \bar{\mathbb{C}}$, $i = 1, 2, 3, 4, 5$, be any five distinct values. A ray $\arg z = \theta \in [0, 2\pi)$ from the origin is called a five-value rich line of f, if for any $\varepsilon > 0$, either

$$\limsup_{r\to\infty} \frac{\log \sum_{i=1}^{5} n(r, \Delta(\theta, \varepsilon), \dfrac{1}{f - a_i})}{\log r} = \rho(f) \tag{1.3}$$

when f is of finite positive order $\rho(f)$, or

$$\limsup_{r\to\infty} \frac{\log \sum_{i=1}^{5} n(r, \Delta(\theta,\varepsilon), \frac{1}{f-a_i})}{\rho(r)\log r} = 1 \tag{1.4}$$

when f is of infinite order, where $\rho(r)$ is a proximate order of f.

For stating our results, we also need the following definitions.

Definition 1.8. Let f be a non-constant meromorphic function in $\mathbb{C}$, and let X be a set of all five-value rich lines of f. For any $\varepsilon > 0$, define $X_\varepsilon = \cup\Delta(\theta,\varepsilon)$, where $\arg z = \theta \in X$.

Definition 1.9. Let f and g be two non-constant meromorphic functions in $\mathbb{C}$, and let X be a set of all five-value rich lines of f. We say f and g share $a \in \bar{\mathbb{C}}$ IM in X, if for any $\varepsilon > 0$, f and g share a IM in X_ε.

Now we state the following result which gives a partial affirmative answer to uniqueness question of a meromorphic function of finite order mentioned above.

Theorem 1.10. *Let f be a meromorphic function of finite positive order in $\mathbb{C}$, and let $a_i \in \bar{\mathbb{C}}$, $i = 1, 2, 3, 4, 5$, be five distinct values. Let X be a set of all five-value rich lines of f. If f and g share the values a_i, $i = 1, 2, 3, 4, 5$, IM in X for a meromorphic function g in $\mathbb{C}$ satisfying the growth condition $\log T(r, g) = O(\log T(r, f))$ as $r \to \infty$, possibly outside a set E of finite linear measure, then $f = g$.*

In Theorem 1.10 the set X can be replaced with the set of all Borel directions of f, and the result stays the same by Theorem 1.10, see Theorem 1.11 below.

Theorem 1.11. *Let f be a meromorphic function of finite positive order in $\mathbb{C}$, and let $a_i \in \bar{\mathbb{C}}$, $i = 1, 2, 3, 4, 5$, be five distinct values. Let X be a set of all Borel directions of f. If f and g share the values a_i, $i = 1, 2, 3, 4, 5$, IM in X for a meromorphic function g in $\mathbb{C}$ satisfying the growth condition $\log T(r, g) = O(\log T(r, f))$ as $r \to \infty$, possibly outside a set E of finite linear measure, then $f = g$.*

The set X of Borel directions of f in Theorem 1.11 can be quite arbitrary. Indeed, we recall the following result due to Yang and Zhang [23], which can also be found in [20, Theorem 5.1].

Theorem 1.12 ([23])**.** *If ρ is a positive number and F is a non-empty closed set of real numbers (mod 2π), then there exists a meromorphic function f of order ρ such that all its Borel directions constitute exactly $\{\arg z = \theta : \theta \in F\}$.*

For an entire function of a positive finite order, Drasin and Weitsman [5] obtained a complete result on the distribution of its Borel directions, see also [20, p. 148]. According to Theorem 1.12, there exists a meromorphic function f such that $F = [0, 2\pi]$, that is, every ray $\arg z = \theta \in [0, 2\pi]$ is a Borel direction of f. Hence, if $X = \{\arg z = \theta : \theta \in F\}$ and $F = [0, 2\pi]$ in Theorem 1.11, where F defined as in Theorem 1.12, then Theorem 1.11 implies Theorem 1.1 for the case of meromorphic functions of finite order.

Finally, we discuss the relationship between a five-value rich line and a Borel direction. By Definitions 1.4, 1.5, 1.7, we know that, for a non-constant meromorphic function f in $\mathbb{C}$, every Borel direction of f is its five-value rich line. It is natural to ask the question: For a non-constant meromorphic function f, is every five-value rich line of f equal to its Borel direction? We proceed to answer the question by using Nevanlinna theory in an angular domain [7, Chapters 1 and 3] if f is a meromorphic function of infinite order in $\mathbb{C}$.

Theorem 1.13. *Let $\rho(r)$ be a proximate order of a meromorphic function f of infinite order in $\mathbb{C}$. Then every five-value rich line of f is its Borel direction of order $\rho(r)$.*

Remark 1.14. It is an open problem whether the five-value rich line of a meromorphic function of finite order is its Borel direction.

This paper is organized as follows. In Section 2, we recall some properties of the sectorial Nevanlinna characteristic and state some auxiliary results which are needed in proving our results. The proofs of Theorems 1.10 and 1.13 are given in Section 3.

2. Auxiliary results

Let f be a non-constant meromorphic function in $\mathbb{C}$, and let $\Delta(\theta, \delta)$ be an angular domain for any $\delta \in (0, \pi)$. We recall the following notations which can be found in [7, p. 25]:

$$A(r, \Delta(\theta,\delta), f) = \frac{\omega}{\pi}\int_1^r \left(\frac{1}{t^\omega} - \frac{t^\omega}{r^{2\omega}}\right)\{\log^+|f(te^{i(\theta-\delta)})| + \log^+|f(te^{i(\theta+\delta)})|\}\frac{dt}{t},$$

$$B(r, \Delta(\theta,\delta), f) = \frac{2\omega}{\pi r^\omega}\int_{\theta-\delta}^{\theta+\delta} \log^+|f(te^{i\varphi})|\sin\omega(\varphi-\theta+\delta)d\varphi,$$

$$C(r, \Delta(\theta,\delta), f) = 2\omega\int_1^r c(t, \Delta(\theta,\delta), f)\left(\frac{1}{t^\omega} + \frac{t^\omega}{r^{2\omega}}\right)\frac{dt}{t},$$

where $c(r, \Delta(\theta,\delta), f) = \sum_{1<|b_n|<r, |\theta_n-\theta|\le\delta} \sin\omega(\theta_n - \theta + \delta)$, $\omega = \pi/2\delta$ and $b_n = |b_n|e^{i\theta_n}$ are the poles of f in $\Delta(\theta, \delta)$ with respect to multiplicities. The function $C(r, \Delta(\theta, \delta), f)$ is called the sectorial counting function of the poles of f in $\Delta(\theta, \delta)$. In the corresponding counting function $\bar{C}(r, \Delta(\theta, \delta), f)$ these

multiplicities are ignored. For $a \in \mathbb{C}$, the definitions of $A(r, \Delta(\theta, \delta), \frac{1}{f-a})$, $B(r, \Delta(\theta, \delta), \frac{1}{f-a})$ and $C(r, \Delta(\theta, \delta), \frac{1}{f-a})$ are immediate. Finally, the sectorial Nevanlinna characteristic function is given by

$$S\left(r, \Delta(\theta, \delta), f\right) = A\left(r, \Delta(\theta, \delta), f\right) + B\left(r, \Delta(\theta, \delta), f\right) + C\left(r, \Delta(\theta, \delta), f\right).$$

We state sectorial analogues of Nevanlinna's first and second fundamental theorems as follows.

Lemma 2.1 ([7]). *Let f be a meromorphic function in $\mathbb{C}$, and let $\Delta(\theta, \delta)$ be an angular domain. Then, for any $a \in \mathbb{C}$,*

$$S(r, \Delta(\theta, \delta), \frac{1}{f-a}) = S(r, \Delta(\theta, \delta), f) + O(1).$$

Moreover, let $a_1, a_2, \ldots, a_q \in \bar{\mathbb{C}}$, $q \geq 3$, be distinct numbers. Then

$$(q-2)S(r, \Delta(\theta, \delta), f) \leq \sum_{j=1}^{q} \bar{C}(r, \Delta(\theta, \delta), \frac{1}{f-a_j}) + R(r, \Delta(\theta, \delta), f),$$

where

$$\begin{aligned} R(r, \Delta(\theta, \delta), f) &= A(r, \Delta(\theta, \delta), \frac{f'}{f}) + B(r, \Delta(\theta, \delta), \frac{f'}{f}) \\ &+ \sum_{j=1}^{q} \{A(r, \Delta(\theta, \delta), \frac{f'}{f-a_j}) + B(r, \Delta(\theta, \delta), \frac{f'}{f-a_j})\} + O(1). \end{aligned} \tag{2.1}$$

Nevanlinna conjectured [7, p. 104] that

$$R'(r, \Delta(\theta, \delta), f) = o(S(r, \Delta(\theta, \delta), f)) \tag{2.2}$$

as $r \to \infty$ outside an exceptional set of finite linear measure, where

$$R'(r, \Delta(\theta, \delta), f) = A(r, \Delta(\theta, \delta), \frac{f'}{f}) + B(r, \Delta(\theta, \delta), \frac{f'}{f}),$$

and he proved that $R'(r, \Delta(\theta, \delta), f) = O(1)$, when f is a meromorphic function of finite order in $\mathbb{C}$. In 1974 Goldberg [6] constructed a counterexample to show that (2.2) is not valid. However, for any meromorphic function f in $\mathbb{C}$, the following formula is true [7, Chapter 3]:

$$R'(r, \Delta(\theta, \delta), f) = \begin{cases} O(1), & \rho(f) < \infty, \\ O(\log rT(r, f))(r \to \infty, r \notin E), & \rho(f) = \infty, \end{cases}$$

where E is a set of finite linear measure.

The uniqueness of meromorphic functions in an angular domain is studied by using the sectorial Nevanlinna characteristic, and the sectorial Nevanlinna characteristic function plays critical role in proving the uniqueness theorem, see, for example, [1, 24–26]. As for reasoning that (2.2) is not true, we need carefully analyze the error term $R(r, \Delta(\theta, \delta), f)$ in proving the uniqueness results in an angular domain. For the case of meromorphic functions of infinite order, many five-value theorems in an angular domain have been proved by applying directly the sectorial Nevanlinna characteristic, see, for example, [12,13,24,26]. However, in the finite order case, the uniqueness of meromorphic functions in an angular domain has been studied under some additional conditions between the magnitude of the angular domain and the growth of the sectorial Nevanlinna characteristic function, see, for example, [1, 12, 24, 25]. In particular, if f is a meromorphic function of order $\rho \in (0, \infty)$, and the magnitude of the angular domain is too small, then the sectorial Nevanlinna characteristic function is bounded. Therefore, it is of no use to study the value distribution of meromorphic functions in these angular domains. Typically the angular domain $\Delta(\theta, \delta) = \{z : \theta - \delta \leq \arg z \leq \theta + \delta\}$ relates to the order of growth ρ in terms of

$$2\delta > \pi/\rho, \tag{2.3}$$

or in terms of

$$\lim_{r\to\infty} \frac{S(r, \Delta(\theta, \delta), f)}{\log(rT(r, f))} = \infty. \tag{2.4}$$

In fact, for any $a \in \bar{\mathbb{C}}$, from the definition of sectorial counting function and the facts that $c(r, \Delta(\theta, \delta), f = a) \leq n(r, \Delta(\theta, \delta), f = a)$ and $1/t^{\omega} + t^{\omega}/r^{2\omega} \leq 2/t^{\omega}$, we get

$$\begin{aligned} C(r, \Delta(\theta, \delta), \frac{1}{f-a}) &\leq 4\omega \int_1^r n(t, \Delta(\theta, \delta), \frac{1}{f-a}) \frac{dt}{t^{\omega+1}} \\ &\leq 4\omega \int_1^r \frac{1}{t^{\omega}} d(\int_1^t \frac{n(s, \Delta(\theta, \delta), \frac{1}{f-a})}{s} ds) \\ &\leq 4\omega \frac{N(r, \Delta(\theta, \delta), \frac{1}{f-a})}{r^{\omega}} + 4\omega^2 \int_1^r \frac{N(t, \Delta(\theta, \delta), \frac{1}{f-a})}{t^{\omega+1}} dt. \end{aligned} \tag{2.5}$$

If $2\delta \leq \pi/\rho$, i.e., $\omega \geq \rho$, then it follows from this and (2.5) that

$$C(r, \Delta(\theta, \delta), \frac{1}{f-a}) \leq r^{\rho-\omega} + O(1) = O(1), \text{ as } r \to \infty.$$

Hence, we get $S(r, \Delta(\theta, \delta), f) = O(1)$ by Lemma 2.1. There is an example to show that the sectorial Nevanlinna characteristic function is bounded if $2\delta > \pi/\rho$ is not true, see [1, p. 83] for more details.

Lemma 2.2 ([7]). *Let f be a meromorphic function of order ρ in $\mathbb{C}$, let $\Delta(\theta, \delta)$ be an angular domain, and denote $\omega = \pi/2\delta$. Then*

$$A(r, \Delta(\theta, \delta), \frac{f'}{f}) \leq K\{(\frac{R}{r})^{\omega} \int_r^R \frac{\log^+ T(t, f)}{t^{\omega+1}} dt + \log^+ \frac{r}{R-r} + \log \frac{R}{r} + 1\},$$

$$B(r, \Delta(\theta, \delta), \frac{f'}{f}) \leq \frac{4\omega}{r^{\omega}} m(r, \frac{f'}{f}),$$

where $1 < r < R < \infty$, and K is a nonzero constant.

The next result follows from Lemma 2.2 and the lemma on the logarithmic derivative.

Lemma 2.3. *Let f be a meromorphic function in $\mathbb{C}$, and let $\Delta(\theta,\delta)$ be an angular domain. Then*

$$R(r,\Delta(\theta,\delta),f)=\begin{cases} O(1), & \rho<\infty, \\ O(\log U(r)), & \rho=\infty, \end{cases}$$

where $R(r,\Delta(\theta,\delta),f)$ is defined as in (2.1), $U(r)=r^{\rho(r)}$ and $\rho(r)$ is a proximate order of the meromorphic function f of infinite order.

3. Proofs of Theorems 1.10 and 1.13

Proof of Theorem 1.13. Let $\rho(r)$ be a proximate order of f, and let $a_i\in\bar{\mathbb{C}}$, $i=1,2,3,4,5$, be any five distinct values. Suppose $\arg z=\theta_0\in[0,2\pi)$ is any five-value rich line of f. For any given $\delta\in(0,\pi)$, it follows from Definition 1.7 that there exists at least one value a_j, such that

$$\limsup_{r\to\infty}\frac{\log n(r,\Delta(\theta_0,\delta),\frac{1}{f-a_j})}{\rho(r)\log r}=1,$$

where $\Delta(\theta_0,\delta)=\{z:|\arg z-\theta_0|\le\delta\}$. Without loss of generality, we may assume that a_1 is such a value, that is,

$$\limsup_{r\to\infty}\frac{\log n(r,\Delta(\theta_0,\delta),\frac{1}{f-a_1})}{\rho(r)\log r}=1.$$

Therefore, there exist a sequence r_n with $r_n\to\infty$ as $n\to\infty$, such that

$$\lim_{n\to\infty}\frac{\log n(r_n,\Delta(\theta_0,\delta),\frac{1}{f-a_1})}{\rho(r_n)\log r_n}=1. \tag{3.1}$$

For any given $\varepsilon\in(0,\delta)$, by the definition of sectorial counting function, we have

$$\begin{aligned} c(r_n,\Delta(\theta_0,\delta),\frac{1}{f-a_1}) &\ge \sum_{1<|b_m|<r_n,|\theta_m-\theta_0|\le\delta-\varepsilon}\sin\omega(\theta_m-\theta_0+\delta) \\ &\ge \sin(\omega\varepsilon)[n(r_n,\Delta(\theta_0,\delta-\varepsilon),\frac{1}{f-a_1})-n(1,\Delta(\theta_0,\delta-\varepsilon),\frac{1}{f-a_1})], \end{aligned}$$

where $\omega=\pi/2\delta$ and $|b_m|e^{i\theta_m}$ are the zeros of $f-a_1$ in the angular domain $\Delta(\theta_0,\delta)$, each zero being counted according to its multiplicity. Since $1/t^\omega+t^\omega/r_n^{2\omega}\ge 2/r_n^\omega$ for $1<t<r_n$, we obtain

$$C(r_n,\Delta(\theta_0,\delta),\frac{1}{f-a_1})\ge\frac{4\omega\sin(\omega\varepsilon)}{r_n^\omega}N(r_n,\Delta(\theta_0,\delta-\varepsilon),\frac{1}{f-a_1})+O(1), \tag{3.2}$$

where

$$N(r,\Delta(\theta_0,\delta),\frac{1}{f-a_1})=\int_0^r\frac{n(t,\Delta(\theta_0,\delta),\frac{1}{f-a_1})-n(0,\Delta(\theta_0,\delta),\frac{1}{f-a_1})}{t}dt+n(0,\Delta(\theta_0,\delta),\frac{1}{f-a_1})\log r.$$

Similarly as in [20, pp. 27-28], it follows that

$$\limsup_{r\to\infty}\frac{\log n(r,\Delta(\theta_0,\delta),\frac{1}{f-a_1})}{\rho(r)\log r}=\limsup_{r\to\infty}\frac{\log N(r,\Delta(\theta_0,\delta),\frac{1}{f-a_1})}{\rho(r)\log r}. \tag{3.3}$$

We deduce from (3.1), (3.2), (3.3), and the arbitrariness of ε that

$$\lim_{n\to\infty}\frac{\log C(r_n,\Delta(\theta_0,\delta),\frac{1}{f-a_1})}{\rho(r_n)\log r_n}\geq 1.$$

Hence,

$$\limsup_{r\to\infty}\frac{\log C(r,\Delta(\theta_0,\delta),\frac{1}{f-a_1})}{\rho(r)\log r}\geq 1,$$

and Lemma 2.1 now yields

$$\limsup_{r\to\infty}\frac{\log S(r,\Delta(\theta_0,\delta),f)}{\rho(r)\log r}\geq 1. \tag{3.4}$$

Suppose on the contrary to the assertion that $\arg z=\theta_0$ is not a Borel direction of order $\rho(r)$ of f. Then there exist three values, say a_2, a_3, a_4, such that

$$\limsup_{r\to\infty}\frac{\log n(r,\Delta(\theta_0,\delta),\frac{1}{f-a_j})}{\rho(r)\log r}<1,\quad j=2,3,4. \tag{3.5}$$

By the definition of sectorial counting function and the inequalities

$$c(r,\Delta(\theta_0,\delta),\frac{1}{f-a_j})\leq n(r,\Delta(\theta_0,\delta),\frac{1}{f-a_j})$$

and $\int_1^r(1/t^\omega+t^\omega/r^{2\omega})dt/t\leq 1/\omega$, it follows that

$$C(r,\Delta(\theta_0,\delta),\frac{1}{f-a_j})\leq 2n(r,\Delta(\theta_0,\delta),\frac{1}{f-a_j}) \tag{3.6}$$

holds for $j=2,3,4$. Combining (3.5) and (3.6), we get

$$\limsup_{r\to\infty}\frac{\log C(r,\Delta(\theta_0,\delta),\frac{1}{f-a_j})}{\rho(r)\log r}<1,\quad j=2,3,4. \tag{3.7}$$

Applying Lemma 2.1, we have

$$S(r,\Delta(\theta_0,\delta),f)\leq\sum_{j=2}^{4}C(r,\Delta(\theta_0,\delta),\frac{1}{f-a_j})+R(r,\Delta(\theta_0,\delta),f). \tag{3.8}$$

Therefore, by (3.7), (3.8) and Lemma 2.3, we have

$$\limsup_{r\to\infty} \frac{\log S(r, \Delta(\theta_0, \delta), f)}{\rho(r)\log r} < 1.$$

This contradicts (3.4), and hence the proof is completed. □

Proof of Theorem 1.10. Suppose that X is a set of all five-value rich lines of f. For any given $\varepsilon > 0$, let X_ε be as in Definition 1.8, and set $Y = \mathbb{C} - X_\varepsilon$. Suppose on the contrary to the assertion that $f \not\equiv g$. By the usual Nevanlinna's second fundamental theorem, we obtain

$$\begin{aligned} 3T(r,f) &\le \sum_{i=1}^{5} \bar{N}(r, \frac{1}{f-a_i}) + S(r,f) \\ &= \sum_{i=1}^{5} \bar{N}(r, X_\varepsilon, \frac{1}{f-a_i}) + \sum_{i=1}^{5} \bar{N}(r, Y, \frac{1}{f-a_i}) + S(r,f) \\ &\le N(r, X_\varepsilon, \frac{1}{f-g}) + \sum_{i=1}^{5} N(r, Y, \frac{1}{f-a_i}) + S(r,f) \\ &\le T(r, \frac{1}{f-g}) + r^{\rho_1} + S(r,f), \end{aligned}$$

where $\rho_1 < \rho(f)$ is a constant. By the usual Nevanlinna's first fundamental theorem, we have

$$2T(r,f) \le T(r,g) + r^{\rho_1} + S(r,f). \tag{3.9}$$

By interchanging the roles of f and g, we have

$$2T(r,g) \le T(r,f) + \sum_{i=1}^{5} \bar{N}(r, Y, \frac{1}{g-a_i}) + S(r,g). \tag{3.10}$$

Combining (3.9) with (3.10) gives

$$T(r,f) + T(r,g) \le r^{\rho_1} + \sum_{i=1}^{5} \bar{N}(r, Y, \frac{1}{g-a_i}) + S(r,f) + S(r,g).$$

By $S(r,f) = O(\log r)$ and the growth of g with respect to that of f, this gives a contradiction. Hence $f = g$, and the proof is completed. □

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