

ON THE STABILITY OF THE ORTHOGONALLY QUARTIC FUNCTIONAL EQUATION

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ABSTRACT. We prove the Hyers–Ulam stability of the orthogonally quartic functional equation

$$f(2x+y) + f(2x-y) = 4f(x+y) + 4f(x-y) + 24f(x) - 6f(y)$$

for all x, y ($x \perp y$), where $\perp$ is the orthogonality in the sense of Rätz.

1. Introduction and preliminaries

Assume that X is a real inner product space and $f : X \rightarrow \mathbb{R}$ is a solution of the orthogonal Cauchy functional equation $f(x+y) = f(x) + f(y)$, $\langle x, y \rangle = 0$. By the Pythagorean theorem $f(x) = \|x\|^2$ is a solution of the conditional equation. Of course, this function does not satisfy the additivity equation everywhere. Thus orthogonal Cauchy equation is not equivalent to the classic Cauchy equation on the whole inner product space. This phenomenon may show the significance of study of orthogonal Cauchy equation.

G. Pinsker [18] characterized orthogonally additive functionals on an inner product space when the orthogonality is the ordinary one in such spaces. K. Sundaresan [26] generalized this result to

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arbitrary Banach spaces equipped with the Birkhoff–James orthogonality. The orthogonal Cauchy functional equation

$$f(x + y) = f(x) + f(y), \quad x \perp y,$$

in which $\perp$ is an abstract orthogonality relation, was first investigated by S. Gudder and D. Strawther [11]. They defined $\perp$ by a system consisting of five axioms and described the general semi-continuous real-valued solution of conditional Cauchy functional equation. In 1985, J. Rätz [23] introduced a new definition of orthogonality by using more restrictive axioms than of S. Gudder and D. Strawther. Moreover, he investigated the structure of orthogonally additive mappings. J. Rätz and Gy. Szabó [24] investigated the problem in a rather more general framework.

Let us recall the orthogonality in the sense of J. Rätz; cf. [23].

Suppose X is a real vector space with $\dim X \geq 2$ and $\perp$ is a binary relation on X with the following properties:

- (O1) totality of $\perp$ for zero: $x \perp 0, 0 \perp x$ for all $x \in X$;
- (O2) independence: if $x, y \in X - \{0\}, x \perp y$, then x, y are linearly independent;
- (O3) homogeneity: if $x, y \in X, x \perp y$, then $\alpha x \perp \beta y$ for all $\alpha, \beta \in \mathbb{R}$;
- (O4) the Thalesian property: if P is a 2-dimensional subspace of $X, x \in P$ and $\lambda \in \mathbb{R}_+$, then there exists $y_0 \in P$ such that $x \perp y_0$ and $x + y_0 \perp \lambda x - y_0$.

The pair $(X, \perp)$ is called an orthogonality space. By an orthogonality normed space we mean an orthogonality space equipped with a norm.

The relation $\perp$ is called symmetric if $x \perp y$ implies that $y \perp x$ for all $x, y \in X$. It is remarkable to note that a real normed space of dimension greater than 2 is an inner product space if and only if the Birkhoff–James orthogonality is symmetric. There are several orthogonality notions on a real normed space (see e.g. [1] and [2]).

In the recent decades, stability of functional equations have been investigated by many mathematicians (see [17]). The first author treating the stability of the Cauchy equation was D.H. Hyers [12] by proving that if f is a mapping from a normed space X into a Banach space satisfying $\|f(x + y) - f(x) - f(y)\| \leq \epsilon$ for some

$\epsilon > 0$, then there is a unique additive mapping $g : X \rightarrow Y$ such that $\|f(x) - g(x)\| \leq \epsilon$. The stability problem of the Cauchy equation has been extensively investigated by many mathematicians; cf. [13].

R. Ger and J. Sikorska [10] investigated the orthogonal stability of the Cauchy functional equation $f(x+y) = f(x) + f(y)$, namely, they showed that if f is a function from an orthogonality space X into a real Banach space Y and $\|f(x+y) - f(x) - f(y)\| \leq \epsilon$ for all $x, y \in X$ with $x \perp y$ and some $\epsilon > 0$, then there exists exactly one orthogonally additive mapping $g : X \rightarrow Y$ such that $\|f(x) - g(x)\| \leq \frac{16}{3}\epsilon$ for all $x \in X$.

The first author treating the stability of the quadratic equation was F. Skof [25] by proving that if f is a mapping from a normed space X into a Banach space Y satisfying $\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \leq \epsilon$ for some $\epsilon > 0$, then there is a unique quadratic function $g : X \rightarrow Y$ such that $\|f(x) - g(x)\| \leq \frac{\epsilon}{2}$. P. W. Cholewa [3] extended Skof's theorem by replacing X by an abelian group G . Skof's result was later generalized by S. Czerwik [4] in the spirit of Hyers–Ulam. The stability problem of the quadratic equation has been extensively investigated by some mathematicians (see [5], [6], [9], [19]).

The orthogonally quadratic equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y), \quad x \perp y$$

was first investigated by F. Vajzović [29] when X is a Hilbert space, Y is the scalar field, f is continuous and $\perp$ means the Hilbert space orthogonality. Later, H. Drljević [7], M. Fochi [8], M. Moslehian [15, 16] and G. Szabó [27] generalized this result.

Throughout the paper, $\mathbb{R}$ and $\mathbb{R}_+$ denote the sets of real and nonnegative real numbers, respectively.

In [14], S. Lee *et al.* considered the following quartic functional equation

(1.1)

$$f(2x+y) + f(2x-y) = 4f(x+y) + 4f(x-y) + 24f(x) - 6f(y).$$

It is easy to show that the function $f(x) = x^4$ satisfies the functional equation (1.1), which is called a *quartic functional equation*

and every solution of the quartic functional equation is said to be a *quartic mapping*.

Let X be an orthogonality space and Y a real Banach space. A mapping $f : X \rightarrow Y$ is called *orthogonally quartic* if it satisfies the so-called orthogonally quartic functional equation

(1.2)

$$f(2x + y) + f(2x - y) = 4f(x + y) + 4f(x - y) + 24f(x) - 6f(y)$$

for all $x, y \in X$ with $x \perp y$.

In this paper, we prove the Hyers–Ulam stability of the orthogonally quartic functional equation (1.2).

2. Stability of the orthogonally quartic functional equation

Applying some ideas from [10] and [13], we deal with the conditional stability problem for

$$f(2x + y) + f(2x - y) = 4f(x + y) + 4f(x - y) + 24f(x) - 6f(y)$$

for all $x, y \in X$ with $x \perp y$.

Throughout this section, $(X, \perp)$ denotes an orthogonality normed space with norm $\|\cdot\|_X$ and $(Y, \|\cdot\|_Y)$ is a Banach space.

Theorem 2.1. *Let θ and p ($p < 4$) be nonnegative real numbers. Suppose that $f : X \rightarrow Y$ is a mapping fulfilling*

$$\begin{aligned} \|f(2x + y) + f(2x - y) - 4f(x + y) - 4f(x - y) - 24f(x) \\ - 6f(y)\|_Y \leq \theta(\|x\|_X^p + \|y\|_X^p) \end{aligned} \quad (2.1)$$

for all $x, y \in X$ with $x \perp y$. Then there exists a unique orthogonally quartic mapping $T : X \rightarrow Y$ such that

$$(2.2) \quad \|f(x) - T(x)\|_Y \leq \frac{\theta}{32 - 2^{p+1}} \|x\|_X^p$$

for all $x \in X$.

Proof. Putting $y = 0$ in (2.1), we get

$$(2.3) \quad \|2f(2x) - 32f(x)\|_Y \leq \theta \|x\|_X^p$$

for all $x \in X$, since $x \perp 0$. So

$$\|f(x) - \frac{1}{16}f(2x)\|_Y \leq \frac{\theta}{32}\|x\|_X^p$$

for all $x \in X$. Hence

$$(2.4) \quad \left\| \frac{1}{2^{4n}}f(2^n x) - \frac{1}{2^{4m}}f(2^m x) \right\|_Y \leq \frac{\theta}{32} \sum_{k=n}^{m-1} \frac{2^{pk}}{2^{4k}} \|x\|_X^p$$

for all nonnegative integers n, m with $n < m$. Thus $\{\frac{1}{2^{4n}}f(2^n x)\}$ is a Cauchy sequence in Y . Since Y is complete, there exists a mapping $T : X \rightarrow Y$ defined by

$$T(x) := \lim_{n \rightarrow \infty} \frac{1}{2^{4n}}f(2^n x)$$

for all $x \in X$. Letting $n = 0$ and $m \rightarrow \infty$ in (2.4), we get the inequality (2.2). It follows from (2.1) that

$$\begin{aligned} & \|T(2x+y) + T(2x-y) - 4T(x+y) - 4T(x-y) - 24T(x) - 6T(y)\|_Y \\ &= \lim_{n \rightarrow \infty} \frac{1}{2^{4n}} \|f(2^n(2x+y)) + f(2^n(2x-y)) - 4f(2^n(x+y)) \\ &\quad - 4f(2^n(x-y)) - 24f(2^n x) - 6f(2^n y)\|_Y \\ &\leq \lim_{n \rightarrow \infty} \frac{2^{pn}\theta}{2^{4n}} (\|x\|_X^p + \|y\|_X^p) = 0 \end{aligned}$$

for all $x, y \in X$ with $x \perp y$. So

$$T(2x+y) + T(2x-y) - 4T(x+y) - 4T(x-y) - 24T(x) - 6T(y) = 0$$

for all $x, y \in X$ with $x \perp y$. Hence $T : X \rightarrow Y$ is an orthogonally quartic mapping.

Let $Q : X \rightarrow Y$ be another orthogonally quartic mapping satisfying (2.2). Then

$$\begin{aligned} \|T(x) - Q(x)\|_Y &= \frac{1}{2^{4n}} \|T(2^n x) - Q(2^n x)\|_Y \\ &\leq \frac{1}{2^{4n}} (\|f(2^n x) - T(2^n x)\|_Y + \|f(2^n x) - Q(2^n x)\|_Y) \\ &\leq \frac{\theta}{16 - 2^p} \cdot \frac{2^{pn}}{2^{4n}} \|x\|_X^p, \end{aligned}$$

which tends to zero for all $x \in X$. So we have $T(x) = Q(x)$ for all $x \in X$. This proves the uniqueness of T . $\square$

Theorem 2.2. *Let θ and p ($p > 4$) be nonnegative real numbers. Suppose that $f : X \rightarrow Y$ is a mapping fulfilling*

$$\begin{aligned} \|f(2x+y) + f(2x-y) - 4f(x+y) - 4f(x-y) - 24f(x) \\ - 6f(y)\|_Y \leq \theta(\|x\|_X^p + \|y\|_X^p) \end{aligned} \quad (2.5)$$

for all $x, y \in X$ with $x \perp y$. Then there exists a unique orthogonally quartic mapping $T : X \rightarrow Y$ such that

$$(2.6) \quad \|f(x) - T(x)\|_Y \leq \frac{\theta}{2^{p+1} - 32} \|x\|_X^p$$

for all $x \in X$.

Proof. It follows from (2.3) that

$$\|f(x) - 16f(\frac{x}{2})\|_Y \leq \frac{\theta}{2^{p+1}} \|x\|_X^p$$

for all $x \in X$. So

$$(2.7) \quad \|2^{4n}f(\frac{x}{2^n}) - 2^{4m}f(\frac{x}{2^m})\|_Y \leq \frac{\theta}{2^{p+1}} \sum_{k=n}^{m-1} \frac{2^{4k}}{2^{pk}} \|x\|_X^p$$

for all nonnegative integers n, m with $n < m$. Thus $\{2^{4n}f(\frac{x}{2^n})\}$ is a Cauchy sequence in Y . Since Y is complete, there exists a mapping $T : X \rightarrow Y$ defined by

$$T(x) := \lim_{n \rightarrow \infty} 2^{4n}f(\frac{x}{2^n})$$

for all $x \in X$. Letting $n = 0$ and $m \rightarrow \infty$ in (2.7), we get the inequality (2.6).

The rest of the proof is simialr to the proof of Theorem 2.1. $\square$

REFERENCES

- [1] J. Alonso and C. Benítez, Orthogonality in normed linear spaces: a survey I. Main properties, *Extracta Math.* **3** (1988), 1–15.

- [2] J. Alonso and C. Benítez, Orthogonality in normed linear spaces: a survey II. Relations between main orthogonalities, *Extracta Math.* **4** (1989), 121–131.
- [3] P.W. Cholewa, Remarks on the stability of functional equations, *Aequationes Math.* **27** (1984), 76–86.
- [4] S. Czerwik, On the stability of the quadratic mapping in normed spaces, *Abh. Math. Sem. Univ. Hamburg* **62** (1992), 59–64.
- [5] S. Czerwik, *Functional Equations and Inequalities in Several Variables*, World Scientific Publishing Company, New Jersey, London, Singapore and Hong Kong, 2002.
- [6] S. Czerwik, *Stability of Functional Equations of Ulam–Hyers–Rassias Type*, Hadronic Press, Palm Harbor, Florida, 2003.
- [7] F. Drljević, On a functional which is quadratic on A -orthogonal vectors, *Publ. Inst. Math. (Beograd)* **54** (1986), 63–71.
- [8] M. Fochi, Functional equations in A -orthogonal vectors, *Aequationes Math.* **38** (1989), 28–40.
- [9] G. L. Forti, Comments on the core of the direct method for proving Hyers–Ulam stability of functional equations, *J. Math. Anal. Appl.* **295** (2004), 127–133.
- [10] R. Ger and J. Sikorska, Stability of the orthogonal additivity, *Bull. Polish Acad. Sci. Math.* **43** (1995), 143–151.
- [11] S. Gudder and D. Strawther, Orthogonally additive and orthogonally increasing functions on vector spaces, *Pacific J. Math.* **58** (1975), 427–436.
- [12] D. H. Hyers, On the stability of the linear functional equation, *Proc. Nat'l. Acad. Sci. U.S.A.* **27** (1941), 222–224.
- [13] D. H. Hyers, G. Isac and Th. M. Rassias, *Stability of Functional Equations in Several Variables*, Birkhäuser, Basel, 1998.
- [14] S. H. Lee, S. M. Im and I. S. Hwang, Quartic functional equations, *J. Math. Anal. Appl.* **307** (2005), 387–394.
- [15] M. S. Moslehian, On the orthogonal stability of the Pexiderized quadratic equation, *J. Differ. Equations Appl.* (to appear).
- [16] M. S. Moslehian, On the stability of the orthogonal Pexiderized Cauchy equation, *J. Math. Anal. Appl.* (to appear).
- [17] L. Paganoni and J. Rätz, Conditional function equations and orthogonal additivity, *Aequationes Math.* **50** (1995), 135–142.
- [18] A. G. Pinsker, Sur une fonctionnelle dans l'espace de Hilbert, C. R. (Dokl.) *Acad. Sci. URSS, n. Ser.* **20** (1938), 411–414.
- [19] Th. M. Rassias, On the stability of the linear mapping in Banach spaces, *Proc. Amer. Math. Soc.* **72** (1978), 297–300.
- [20] Th. M. Rassias, The problem of S.M. Ulam for approximately multiplicative mappings, *J. Math. Anal. Appl.* **246** (2000), 352–378.

- [21] Th. M. Rassias, On the stability of functional equations in Banach spaces, *J. Math. Anal. Appl.* **251** (2000), 264–284.
- [22] Th. M. Rassias, *Functional Equations, Inequalities and Applications*, Kluwer Academic Publishers, Dordrecht, Boston and London, 2003.
- [23] J. Rätz, On orthogonally additive mappings, *Aequationes Math.* **28** (1985), 35–49.
- [24] J. Rätz and Gy. Szabó, On orthogonally additive mappings IV, *Aequationes Math.* **38** (1989), 73–85.
- [25] F. Skof, Proprietà locali e approssimazione di operatori, *Rend. Sem. Mat. Fis. Milano* **53** (1983), 113–129.
- [26] K. Sundaresan, Orthogonality and nonlinear functionals on Banach spaces, *Proc. Amer. Math. Soc.* **34** (1972), 187–190.
- [27] Gy. Szabó, Sesquilinear-orthogonally quadratic mappings, *Aequationes Math.* **40** (1990), 190–200.
- [28] S. M. Ulam, *Problems in Modern Mathematics*, Wiley, New York, 1960.
- [29] F. Vajzović, Über das Funktional H mit der Eigenschaft: $(x, y) = 0 \Rightarrow H(x+y) + H(x-y) = 2H(x) + 2H(y)$, *Glasnik Mat. Ser. III* **2** (22) (1967), 73–81.

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