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HYPERBOLIC SURFACES OF L_1-2-TYPE

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Abstract. In this paper, we show that an L_1-2-type surface in the three-dimensional hyperbolic space $\mathbb{H}^3 \subset \mathbb{R}^4_1$ either is an open piece of a standard Riemannian product $\mathbb{H}^1(-\sqrt{1+r^2}) \times \mathbb{S}^1(r)$, or it has non constant mean curvature, non constant Gaussian curvature, and non constant principal curvatures.
Keywords: Hyperbolic surface, Cheng-Yau operator, L_1-finite-type surface, L_1-biharmonic surface, Newton transformation.
MSC(2010): Primary: 53B25; Secondary: 53A05, 53C40.

1. Introduction

During the late 1970s, B.Y. Chen introduced finite type submanifolds (that is, submanifolds whose isometric immersion into the Euclidean, or pseudo-Euclidean space is constructed by using eigenfunctions of their Laplacian). The first results on this subject were collected in his book [7], and, in subsequent papers, he has provided a detailed account of recent development on problems and conjectures about this topic, [5, 6]. It is well known that the Laplacian operator Δ can be seen as the first one of a sequence of operators $L_0 = \Delta$, $L_1, \ldots, L_{n-1}$, n being the dimension of the submanifold, where L_k stands for the linearized operator of the first variation of the $(k+1)$-th mean curvature arising from normal variations (see, for instance, [10]).

As might be expected, the notion of finite type submanifold can be defined for any operator L_k, [8], and then it is natural to try to obtain new results and compare them with the classical ones. For example, it is a well known result that the only 2-type surfaces in the hyperbolic space $\mathbb{H}^3$ are open pieces of the Riemannian products $\mathbb{H}^1(-\sqrt{1+r^2}) \times \mathbb{S}^1(r)$, [4, 5].

What can we say about an L_1-2-type surface M^2 in the hyperbolic space $\mathbb{H}^3$? These surfaces are characterized by the following spectral decomposition

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of its position vector $\psi : M^2 \to \mathbb{H}^3 \subset \mathbb{R}^4_1$:

$$\psi = \psi_0 + \psi_1 + \psi_2, \quad L_1\psi_1 = \lambda_1\psi_1, \quad L_1\psi_2 = \lambda_2\psi_2, \quad \lambda_1 \neq \lambda_2,\ \lambda_i \in \mathbb{R},$$

where ψ_0 is a constant vector in $\mathbb{R}^4_1$, and ψ_1, ψ_2 are $\mathbb{R}^4_1$-valued non-constant differentiable functions on M^2. It is easy to see that open pieces of the standard Riemannian products $\mathbb{H}^1(-\sqrt{1+r^2}) \times \mathbb{S}^1(r)$ are surfaces of L_1-2-type (see Example 3.2). Our main theorem is the following local result.

Theorem. *Let $\psi : M^2 \to \mathbb{H}^3 \subset \mathbb{R}^4_1$ be a surface of L_1-2-type. Then either M^2 is an open piece of a standard Riemannian product $\mathbb{H}^1(-\sqrt{1+r^2}) \times \mathbb{S}^1(r)$, or M^2 has non constant mean curvature, non constant Gaussian curvature, and non constant principal curvatures.*

Our conjecture is that (open pieces of) standard Riemannian products $\mathbb{H}^1(-\sqrt{1+r^2}) \times \mathbb{S}^1(r)$ are the only L_1-2-type surfaces in the three-dimensional hyperbolic space.

2. **Preliminaries**

Let $\mathbb{R}^4_1$ denote the 4-dimensional Lorentz-Minkowski space with flat metric given by

$$\langle \cdot, \cdot \rangle = -\mathrm{d}x_1^2 + \mathrm{d}x_2^2 + \mathrm{d}x_3^2 + \mathrm{d}x_4^2,$$

where (x_1, x_2, x_3, x_4) is the usual rectangular coordinate system on $\mathbb{R}^4_1$. We denote by $\mathbb{H}^3$ the (connected) unit hyperbolic space, which has the standard embedding in $\mathbb{R}^4_1$ as

$$\mathbb{H}^3 = \{x \in \mathbb{R}^4_1 \ : \ \langle x, x \rangle = -1, \text{ and } x_1 > 0\}.$$

Let $\psi : M^2 \to \mathbb{H}^3 \subset \mathbb{R}^4_1$ be an isometric immersion of a connected and orientable surface M^2, with Gauss map N. We denote by ∇^0, $\overline{\nabla}$ and ∇ the Levi-Civita connections on $\mathbb{R}^4_1$, $\mathbb{H}^3$ and M^2, respectively. Then the Gauss and Weingarten formulas are given by

$$\nabla^0_X Y = \nabla_X Y + \langle SX, Y\rangle N + \langle X, Y\rangle \psi,$$
$$SX = -\overline{\nabla}_X N = -\nabla^0_X N,$$

for all tangent vector fields $X, Y \in \mathfrak{X}(M^2)$, where $S : \mathfrak{X}(M^2) \to \mathfrak{X}(M^2)$ stands for the shape operator (or Weingarten endomorphism) of M^2, with respect to the chosen orientation N. The mean curvature H and the curvature H_2 of M^2 are defined by $2H = \kappa_1 + \kappa_2$ and $H_2 = \kappa_1\kappa_2$, respectively, κ_1 and κ_2 being the eigenvalues of S (that is, the principal curvatures of the surface). It is well known that the Gaussian curvature K is given by $K = -1 + H_2$.

The Newton transformation of M^2 is the operator $P : \mathfrak{X}(M^2) \to \mathfrak{X}(M^2)$ defined by

$$P = 2HI - S. \tag{2.1}$$

Note that by the Cayley-Hamilton theorem we have $S \circ P = H_2 I$. Observe also that, at any point $m \in M^2$, $S(m)$ and $P(m)$ can be simultaneously diagonalized: if $\{e_1, e_2\}$ are the eigenvectors of $S(m)$ corresponding to the eigenvalues $\kappa_1(m)$ and $\kappa_2(m)$, respectively, then they are also the eigenvectors of $P(m)$ with corresponding eigenvalues $\kappa_2(m)$ and $\kappa_1(m)$.

According to [9, p. 86], the divergence of a vector field X is the differentiable function defined by

$$\mathrm{div}(X) = C(\nabla X) = \langle \nabla_{E_1} X, E_1 \rangle + \langle \nabla_{E_2} X, E_2 \rangle ,$$

$\{E_1, E_2\}$ being any local orthonormal frame of tangent vectors fields. Analogously, for an operator $T : \mathfrak{X}(M^2) \to \mathfrak{X}(M^2)$ the divergence associated to the metric contraction C_{12} will be the vector field $\mathrm{div}(T) \in \mathfrak{X}(M^2)$ defined as

$$\mathrm{div}(T) = C_{12}(\nabla T) = (\nabla_{E_1} T)E_1 + (\nabla_{E_2} T)E_2.$$

The following properties of P are well known (see, for example, [1]).

Lemma 2.1. *The Newton transformation P satisfies:*

(a) $\mathrm{tr}(P) = 2H$.
(b) $\mathrm{tr}(S \circ P) = 2H_2$.
(c) $\mathrm{tr}(S^2 \circ P) = 2HH_2$.
(d) $\mathrm{tr}(\nabla_X S \circ P) = \langle \nabla H_2, X \rangle$, *where ∇H_2 stands for the gradient of H_2.*
(e) $\mathrm{div}(P) = 0$.

Associated to the Newton transformation P, we can define a second-order linear differential operator $L_1 : \mathcal{C}^\infty(M^2) \to \mathcal{C}^\infty(M^2)$ by

$$L_1(f) = \mathrm{tr}\big(P \circ \nabla^2 f\big) = \mathrm{div}(P(\nabla f)), \tag{2.2}$$

where $\nabla^2 f : \mathfrak{X}(M^2) \to \mathfrak{X}(M^2)$ denotes the self-adjoint linear operator metrically equivalent to the Hessian of f, given by $\langle \nabla^2 f(X), Y \rangle = \langle \nabla_X(\nabla f), Y \rangle$. An interesting property of L_1 is the following: for every couple of differentiable functions $f, g \in C^\infty(M^2)$ we have

$$L_1(fg) = gL_1(f) + fL_1(g) + 2 \langle P(\nabla f), \nabla g \rangle . \tag{2.3}$$

Note that the operator L_1 can be naturally extended to vector valued functions.

3. First formulas, examples and results

Let $a \in \mathbb{R}^4_1$ be an arbitrary fixed vector. A direct computation shows that the gradient of the height function $\langle \psi, a \rangle$ is given by

$$\nabla \langle \psi, a \rangle = a^\top = a - \langle N, a \rangle N + \langle \psi, a \rangle \psi, \tag{3.1}$$

where $a^\top \in \mathfrak{X}(M^2)$ denotes the tangential component of a. Taking covariant derivative in (3.1), and using the Gauss and Weingarten formulas, we obtain

$$\nabla_X \nabla \langle \psi, a \rangle = \nabla_X a^\top = \langle N, a \rangle SX + \langle \psi, a \rangle X, \tag{3.2}$$

for every vector field $X \in \mathfrak{X}(M^2)$. Finally, by using (2.2) and Lemma 2.1, we find

$$L_1 \langle \psi, a \rangle = \langle N, a \rangle \operatorname{tr}(S \circ P) + \langle \psi, a \rangle \operatorname{tr}(P) = 2H_2 \langle N, a \rangle + 2H \langle \psi, a \rangle \,, \tag{3.3}$$

and therefore

$$L_1 \psi = 2H_2 N + 2H\psi. \tag{3.4}$$

A straightforward computation yields $\nabla \langle N, a \rangle = -Sa^\top$, and then, from the Weingarten formula and (3.2), we get

$$\nabla_X \nabla \langle N, a \rangle = -(\nabla_{a^\top} S)X - \langle N, a \rangle S^2 X - \langle \psi, a \rangle SX,$$

for every tangent vector field X. This equation, jointly with (2.2) and Lemma 2.1, yields

$$\begin{aligned} L_1 \langle N, a \rangle &= -\operatorname{tr}(\nabla_{a^\top} S \circ P) - \langle N, a \rangle \operatorname{tr}(S^2 \circ P) - \langle \psi, a \rangle \operatorname{tr}(S \circ P) \\ &= -\langle \nabla H_2, a \rangle - 2HH_2 \langle N, a \rangle - 2H_2 \langle \psi, a \rangle \,. \end{aligned} \tag{3.5}$$

In other words,

$$L_1 N = -\nabla H_2 - 2HH_2 N - 2H_2 \psi.$$

On the other hand, equations (2.3), (3.3) and (3.5) lead to

$$\begin{aligned} L_1^2 \langle \psi, a \rangle &= 2H_2 L_1 \langle N, a \rangle + 2L_1(H_2) \langle N, a \rangle + 4\big\langle P(\nabla H_2), \nabla \langle N, a \rangle \big\rangle \\ &\quad + 2HL_1 \langle \psi, a \rangle + 2L_1(H) \langle \psi, a \rangle + 4\big\langle P(\nabla H), \nabla \langle \psi, a \rangle \big\rangle, \\ &= -2H_2 \langle \nabla H_2, a \rangle - 4 \langle (S \circ P)(\nabla H_2), a \rangle + 4 \langle P(\nabla H), a \rangle \\ &\quad + \big[2L_1 H_2 - 4HH_2(H_2 - 1)\big] \langle N, a \rangle \\ &\quad + \big[-4H_2^2 + 4H^2 + 2L_1 H\big] \langle \psi, a \rangle \,, \end{aligned}$$

and then we obtain

$$\begin{aligned} L_1^2 \psi &= 4P(\nabla H) - 3\nabla H_2^2 \\ &\quad + 2\big[L_1 H_2 - 2HH_2\big(H_2 - 1\big)\big] N \\ &\quad + 2\big[L_1 H - 2H_2^2 + 2H^2\big] \psi. \end{aligned} \tag{3.6}$$

3.1. **Examples.**

Example 3.1 (Surfaces of L_1-1-type)**.** In this example we present some surfaces $M^2 \subset \mathbb{H}^3$ of L_1-1-type in $\mathbb{R}^4_1$, that is, surfaces whose position vector ψ can be written as $\psi = \psi_0 + \psi_1$, where ψ_0 is a constant vector in $\mathbb{R}^4_1$ and ψ_1 is an $\mathbb{R}^4_1$-valued non-constant differentiable function satisfying $L_1\psi_1 = \lambda\psi_1$, $\lambda \in \mathbb{R}$. In the case $\lambda = 0$, M^2 is said to be an L_1-null-1-type surface or an L_1-harmonic surface.

As is well known, totally umbilical surfaces in $\mathbb{H}^3$ are obtained as the intersection of $\mathbb{H}^3$ with a hyperplane of $\mathbb{R}^4_1$, and the causal character of the hyperplane determines the type of the surface. More precisely, let $a \in \mathbb{R}^4_1$ be a non-zero constant vector with $\langle a, a \rangle \in \{1, 0, -1\}$, and take the differentiable function $f_a : \mathbb{H}^3 \to \mathbb{R}$ defined by $f_a(x) = \langle x, a \rangle$. It is not difficult to see that for every $\tau \in \mathbb{R}$, with $\langle a, a \rangle + \tau^2 = \delta^2 > 0$, the set $M_\tau = f_a^{-1}(\tau)$ is a totally umbilical surface in $\mathbb{H}^3$, with Gauss map $N(x) = (1/\delta)(a + \tau x)$, and shape operator $S = -(\tau/\delta)I$. From here, we get that M_τ has constant mean curvature $H = -\tau/\delta$ and constant Gaussian curvature $K = -\langle a, a \rangle / \delta^2$. Now we will see the different possibilities:

(i) If $\langle a, a \rangle = -1$, then $K = 1/(\tau^2 - 1)$ is positive, and $M_\tau \subset \mathbb{S}^2(\sqrt{\tau^2 - 1})$.
(ii) If $\langle a, a \rangle = 0$, then $K = 0$, and $M_\tau \subset \mathbb{R}^2$. M_τ is said to be a flat totally umbilical surface.
(iii) If $\langle a, a \rangle = 1$, then $K = -1/(\tau^2 + 1)$ is negative, and $M_\tau \subset \mathbb{H}^2(-\sqrt{\tau^2 + 1})$.

Bearing (3.4) in mind, we obtain

$$L_1\psi = \lambda\psi + b, \quad \lambda = \frac{2\tau}{\delta}\Big(\frac{\tau^2}{\delta^2} - 1\Big), \quad b = \frac{2\tau^2}{\delta^3}a.$$

We distinguish three cases:

(i) If $\tau = 0$ (and so $S = 0$), then $L_1\psi = 0$ and $M_\tau \subset \mathbb{H}^2$ is of L_1-null-1-type.
(ii) If $\tau^2 = \delta^2 > 0$, then $L_1\psi = b \neq 0$, $\langle b, b \rangle = 0$, and $M_\tau \subset \mathbb{R}^2$ is of infinite L_1-type.
(iii) If $\lambda \neq 0$, then we write $\psi = \psi_0 + \psi_1$, with $\psi_0 = -(1/\lambda)b$ and $\psi_1 = \psi + (1/\lambda)b$, showing M_τ is of L_1-1-type.

Example 3.2 (Surfaces of L_1-2-type)**.** We will see that the standard Riemannian product $M^2(r) = \mathbb{H}^1(-\sqrt{1 + r^2}) \times \mathbb{S}^1(r)$ is a surface in $\mathbb{H}^3$ of L_1-2-type in $\mathbb{R}^4_1$. Let us suppose $M^2(r) = \{x \in \mathbb{H}^3 : x_3^2 + x_4^2 = r^2\}$, then the Gauss map is given by

$$N(x) = \left(\frac{r}{\sqrt{1 + r^2}}\, x_1, \frac{r}{\sqrt{1 + r^2}}\, x_2, \frac{\sqrt{1 + r^2}}{r}\, x_3, \frac{\sqrt{1 + r^2}}{r}\, x_4\right),$$

and the principal curvatures are computed as

$$\kappa_1 = \frac{-r}{\sqrt{1 + r^2}} \quad \text{and} \quad \kappa_2 = \frac{-\sqrt{1 + r^2}}{r}.$$

Hence, we get

$$H = -\frac{2r^2 + 1}{2r\sqrt{1 + r^2}} \quad \text{and} \quad H_2 = 1.$$

If we put $\psi_1 = (x_1, x_2, 0, 0)$ and $\psi_2 = (0, 0, x_3, x_4)$, then $\psi = \psi_1 + \psi_2$, and from (3.4) we obtain $L_1\psi_1 = \lambda_1\psi_1$ and $L_1\psi_2 = \lambda_2\psi_2$, where

$$\lambda_1 = \frac{-1}{r\sqrt{1 + r^2}} \quad \text{and} \quad \lambda_2 = \frac{1}{r\sqrt{1 + r^2}}.$$

Therefore, $M^2(r)$ is an L_1-2-type surface in $\mathbb{R}^4_1$.

3.2. First results.

Proposition 3.3. *Let $\psi : M^2 \to \mathbb{H}^3 \subset \mathbb{R}^4_1$ be an isometric immersion. Then ψ is of L_1-1-type if and only if M^2 is a non-flat totally umbilical surface in $\mathbb{H}^3$.*

Proof. We have already checked in example 3.1 that non-flat totally umbilical surfaces are of L_1-1-type. Conversely, let us assume that $\psi : M^2 \to \mathbb{H}^3 \subset \mathbb{R}^4_1$ is of L_1-1-type. Then it is easy to get $L_1\psi = \lambda\psi + b$, $b = -\lambda\psi_0$, and by using (3.4) we obtain

$$2H_2N + (2H - \lambda)\psi = b.$$

Taking covariant derivative here, we get

$$-2H_2SX + (2H - \lambda)X + 2X(H_2)N + 2X(H)\psi = 0,$$

for every tangent vector field X. As a consequence, M^2 has constant curvatures H and H_2, that is, M^2 is an isoparametric surface in $\mathbb{H}^3$. Bearing in mind examples 3.1 and 3.2, and the classification of isoparametric surfaces in $\mathbb{H}^3$ [3], we easily obtain the result. □

The following result is also deduced from examples 3.1 and 3.2, taking into account [3].

Proposition 3.4. *Let $\psi : M^2 \to \mathbb{H}^3 \subset \mathbb{R}^4_1$ be an isoparametric surface. Then M^2 is of L_1-2-type in $\mathbb{R}^4_1$ if and only if M^2 is an open piece of a standard Riemannian product $\mathbb{H}^1(-\sqrt{1+r^2}) \times \mathbb{S}^1(r)$.*

The following definition appears in a natural way, [2].

Definition 3.5. An isometric immersion $\psi : M^2 \to \mathbb{R}^4_1$ is said to be L_1-biharmonic if $L_1^2\psi = 0$. In the case $L_1^2\psi = 0$ and $L_1\psi \neq 0$, we will say that ψ is a proper L_1-biharmonic immersion.

Observe that, from (3.4), any totally geodesic surface of $\mathbb{H}^3$ is trivially an L_1-biharmonic surface in $\mathbb{R}^4_1$.

Let $\psi : M^2 \to \mathbb{H}^3 \subset \mathbb{R}^4_1$ be an L_1-biharmonic surface. Then (3.6) yields

$$4P(\nabla H) - 3\nabla H_2^2 = 0, \tag{3.7}$$

$$L_1H_2 - 2HH_2(H_2 - 1) = 0, \tag{3.8}$$

$$L_1H + 2(H^2 - H_2^2) = 0. \tag{3.9}$$

If H is constant, then (3.9) yields M^2 is an isoparametric surface in $\mathbb{H}^3$. If K is constant (and so is H_2), by taking divergence in (3.7), we get $L_1H = 0$. Then from (3.9) we also deduce M^2 is an isoparametric surface in $\mathbb{H}^3$. Therefore, bearing [3] in mind, we have obtained the following result.

Proposition 3.6. *Let $\psi : M^2 \to \mathbb{H}^3 \subset \mathbb{R}^4_1$ be an L_1-biharmonic surface. Then one of the following claims holds:*

(a) *M^2 is an open piece of a totally geodesic hyperbolic plane.*

(b) M^2 *is an open piece of a flat totally umbilical surface.*
(c) M^2 *has non constant curvatures* H *and* K.

This result can be improved as follows. If H is an L_1-harmonic function (that is, $L_1 H = 0$), then (3.9) implies again M^2 is an isoparametric surface. The same conclusion is also obtained when H_2 (or K) is an L_1-harmonic function. Indeed, in this case, (3.8) yields

$$HH_2(H_2 - 1) = 0.$$

Let us assume that H is non constant (otherwise, there is nothing to prove) and take the non-empty set $\mathcal{U} = \{p \in M^2 \,|\, \nabla H^2(p) \neq 0\}$. On this set we have $H_2(H_2 - 1) = 0$, and then H_2 (and also K) is constant on $\mathcal{U}$. Hence, Proposition 3.6 implies $\mathcal{U}$ is a totally umbilical surface in $\mathbb{H}^3$, but then the mean curvature H is also constant, a contradiction. So the following result has been proved.

Proposition 3.7. *Let* $\psi : M^2 \to \mathbb{H}^3 \subset \mathbb{R}^4_1$ *be a non totally umbilical* L_1*-biharmonic surface. Then the curvatures* H *and* K *are not* L_1*-harmonic functions.*

4. Main results

The main goal of this section is to improve Proposition 3.4 in several ways. First, we need to do some more computations.

Let us suppose that $\psi : M^2 \to \mathbb{H}^3 \subset \mathbb{R}^4_1$ is an L_1-2-type surface, then we can write $\psi = \psi_0 + \psi_1 + \psi_2$. Since $L_1\psi = \lambda_1\psi_1 + \lambda_2\psi_2$ and $L_1^2\psi = \lambda_1^2\psi_1 + \lambda_2^2\psi_2$, an easy computation shows that $L_1^2\psi = (\lambda_1 + \lambda_2)L_1\psi - \lambda_1\lambda_2(\psi - a)$, and by using (3.4) we obtain

$$\begin{aligned} L_1^2\psi = \lambda_1\lambda_2 a^\top &+ \left[2(\lambda_1 + \lambda_2)H_2 + \lambda_1\lambda_2 \langle N, a\rangle\right] N \\ &+ \left[2(\lambda_1 + \lambda_2)H - \lambda_1\lambda_2 - \lambda_1\lambda_2 \langle \psi, a\rangle\right] \psi. \end{aligned}$$

This equation, jointly with (3.6), yield the following equations, that characterize L_1-2-type surfaces in $\mathbb{H}^3$:

$$\lambda_1\lambda_2 a^\top = 4P(\nabla H) - 3\nabla H_2^2, \tag{4.1}$$

$$\lambda_1\lambda_2 \langle N, a\rangle = 2L_1 H_2 - 2H_2\big(2HH_2 - 2H + \lambda_1 + \lambda_2\big), \tag{4.2}$$

$$\lambda_1\lambda_2 \langle \psi, a\rangle = -2L_1 H + 4H_2^2 - 4H^2 + 2(\lambda_1 + \lambda_2)H - \lambda_1\lambda_2. \tag{4.3}$$

Theorem 4.1. *Let* $\psi : M^2 \to \mathbb{H}^3 \subset \mathbb{R}^4_1$ *be a surface of* L_1*-2-type. Then* M^2 *has constant mean curvature if and only if* M^2 *is an open piece of a standard Riemannian product* $\mathbb{H}^1(-\sqrt{1+r^2}) \times \mathbb{S}^1(r)$.

Proof. Let M^2 be a surface of L_1-2-type with constant mean curvature. We will prove that the curvature H_2 is also constant. Otherwise, let us consider the non-empty open set $\mathcal{U}_2 = \left\{p \in M^2 \mid \nabla H_2^2(p) \neq 0\right\}$. By taking covariant derivative in (4.3) we have $\lambda_1\lambda_2 a^\top = 4\nabla H_2^2$. Using this in (4.1) we obtain that H_2 is constant on $\mathcal{U}_2$, a contradiction. Therefore, M^2 is an isoparametric surface in $\mathbb{H}^3$, and then the result follows from Proposition 3.4. $\square$

Theorem 4.2. *Let $\psi : M^2 \to \mathbb{H}^3 \subset \mathbb{R}^4_1$ be a surface of L_1-2-type. Then M^2 has constant Gaussian curvature if and only if M^2 is an open piece of a standard Riemannian product $\mathbb{H}^1(-\sqrt{1+r^2}) \times \mathbb{S}^1(r)$.*

Proof. Let M^2 be an L_1-2-type surface with constant Gaussian curvature K, and consider the open set $\mathcal{U} = \{p \in M^2 \mid \nabla H^2(p) \neq 0\}$. Our goal is to show that $\mathcal{U}$ is empty. If we suppose it is non-empty, by taking covariant derivative in (4.2), and using that H_2 is constant, we obtain

$$\lambda_1\lambda_2 S a^\top = 4H_2(H_2 - 1)\nabla H.$$

From (4.1) and bearing in mind that $S \circ P = H_2 I$, we have $\lambda_1\lambda_2 S a^\top = 4H_2\nabla H$, and therefore

$$H_2(H_2 - 2)\nabla H = 0.$$

Consequently, on $\mathcal{U}$ we have either $H_2 = 2$ or $H_2 = 0$. We will study each case separately.
Case 1: $H_2 = 2$. By applying L_1 to both sides of (4.2), and using (4.3), we get

$$\lambda_1\lambda_2 L_1 \langle N, a\rangle = 4\big[\lambda_1\lambda_2 \langle \psi, a\rangle + 4H^2 - 2(\lambda_1 + \lambda_2)H + \lambda_1\lambda_2 - 16\big].$$

On the other hand, (3.5) leads to

$$\lambda_1\lambda_2 \langle N, a\rangle H + \lambda_1\lambda_2 \langle \psi, a\rangle = -\lambda_1\lambda_2 \langle a, \psi\rangle - 4H^2 + 2(\lambda_1 + \lambda_2)H - \lambda_1\lambda_2 + 16,$$

and using (4.2) we find

$$\lambda_1\lambda_2 \langle \psi, a\rangle = 2H^2 + 3(\lambda_1 + \lambda_2)H - \frac{\lambda_1\lambda_2}{2} + 8. \tag{4.4}$$

Taking gradients in (4.4), and using (4.1) and (2.1), we obtain

$$\big[4H + 3(\lambda_1 + \lambda_2)\big]\nabla H = 4P(\nabla H) = 8H\nabla H - 4S(\nabla H), \tag{4.5}$$

that is,

$$S(\nabla H) = \frac{4H - 3(\lambda_1 + \lambda_2)}{4}\nabla H.$$

Now, by applying the operator S to both sides of the first equality of (4.5), and bearing in mind that $S \circ P = 2I$, we obtain

$$S(\nabla H) = \frac{8}{4H + 3(\lambda_1 + \lambda_2)}\nabla H.$$

The last two equations for $S(\nabla H)$ imply that H is constant on $\mathcal{U}$, which is a contradiction.
Case 2: $H_2 = 0$. Let us suppose $\kappa_1 = 0$ and $\kappa_2 = 2H \neq 0$ (otherwise, M^2 would be a totally geodesic surface and then of L_1-1-type). Let $\{E_1, E_2\}$ be

a local orthonormal frame of principal directions of S such that $SE_i = \kappa_i E_i$. From Codazzi's equation, we easily obtain

$$\nabla_{E_1} E_1 = 0, \qquad \nabla_{E_1} E_2 = 0,$$

$$\nabla_{E_2} E_1 = -\tfrac{\alpha}{H} E_2, \qquad \nabla_{E_2} E_2 = \tfrac{\alpha}{H} E_1 \qquad [E_1, E_2] = \tfrac{\alpha}{H} E_2,$$

where $\alpha = E_1(H)$. Now, from the definition of curvature tensor [9, p. 74], we get

$$R(E_1, E_2)E_1 = \frac{HE_1(\alpha) - 2\alpha^2}{H^2} E_2,$$

and from the Gauss equation we have $R(E_1, E_2)E_1 = -E_2$. By equating the last two equations we deduce

$$HE_1(\alpha) = 2\alpha^2 - H^2. \tag{4.6}$$

On the other hand, from the definition of L_1, see (2.2), we obtain

$$L_1 H = \kappa_2 \langle E_1, \nabla_{E_1} \nabla H\rangle + \kappa_1 \langle E_2, \nabla_{E_2} \nabla H\rangle = 2HE_1(\alpha). \tag{4.7}$$

By using (4.6) and (4.7), (4.3) can be rewritten as

$$\lambda_1\lambda_2 \langle \psi, a\rangle = 2(\lambda_1 + \lambda_2)H - \lambda_1\lambda_2 - 8\alpha^2.$$

Taking covariant derivative along E_1 here, we have

$$E_1(\lambda_1\lambda_2 \langle \psi, a\rangle) = 2(\lambda_1 + \lambda_2)\alpha - 16\alpha E_1(\alpha). \tag{4.8}$$

On the other hand, from (4.1) we get $\lambda_1\lambda_2 a^\top = 8H\alpha E_1$, and therefore

$$E_1(\lambda_1\lambda_2 \langle \psi, a\rangle) = \left\langle \lambda_1\lambda_2 a^\top, E_1\right\rangle = 8H\alpha.$$

This equation, jointly with (4.8), imply that $(\lambda_1 + \lambda_2)\alpha - 8\alpha E_1(\alpha) = 4H\alpha$. Since $\alpha \neq 0$, see (4.6), we deduce

$$8E_1(\alpha) = -4H + \lambda_1 + \lambda_2.$$

From here and using (4.7) we get $4L_1 H = -4H^2 + (\lambda_1 + \lambda_2)H$. By using this in (4.3), we find

$$\lambda_1\lambda_2 \langle \psi, a\rangle = -2H^2 + \frac{3}{2}(\lambda_1 + \lambda_2)H - \lambda_1\lambda_2.$$

Taking gradient, and using (4.1) and (2.1), we obtain

$$\big[-4H + \frac{3}{2}(\lambda_1 + \lambda_2)\big]\nabla H = 4P(\nabla H) = 8H\nabla H - 4S(\nabla H), \tag{4.9}$$

that is,

$$S(\nabla H) = \big(3H - \frac{3}{8}(\lambda_1 + \lambda_2)\big)\nabla H.$$

On the other hand, by applying the operator S to both sides of the first equality of (4.9), and bearing in mind that $S \circ P = 0$, we obtain

$$\big[-4H + \frac{3}{2}(\lambda_1 + \lambda_2)\big]S(\nabla H) = 0.$$

The last two equations imply that H is constant on $\mathcal{U}$, which is a contradiction.

We have proved that if M^2 is an L_1-2-type surface with constant Gaussian curvature, then its mean curvature is also constant. Then Proposition 3.4 yields the result. $\square$

A surface in $\mathbb{H}^3$ is said to have a *constant principal curvature* if one of its principal curvatures is constant.

Theorem 4.3. *Let $\psi : M^2 \to \mathbb{H}^3 \subset \mathbb{R}^4_1$ be a surface of L_1-2-type. Then M^2 has a constant principal curvature if and only if M^2 is an open piece of a standard Riemannian product $\mathbb{H}^1(-\sqrt{1+r^2}) \times \mathbb{S}^1(r)$.*

Proof. Let M^2 be a surface of L_1-2-type with constant curvature κ_1; and assume that κ_1 is a nonzero constant (otherwise, $H_2 = 0$ and Theorem 4.2 applies). Consider the open set $\mathcal{U} = \{p \in M^2 \mid \nabla\kappa_2^2(p) \neq 0\}$. Our goal is to show that $\mathcal{U}$ is empty. Otherwise, equations (4.1)–(4.3) can be rewritten in terms of κ_2 as follows,

$$\lambda_1\lambda_2 a^\top = [-6\kappa_1^2\kappa_2 + 2(\kappa_1+\kappa_2)]\nabla\kappa_2 - 2S(\nabla\kappa_2), \tag{4.10}$$

$$\lambda_1\lambda_2 \langle N, a\rangle = 2\kappa_1 L_1\kappa_2 - 2\kappa_1\kappa_2\big[(\kappa_1+\kappa_2)(\kappa_1\kappa_2 - 1) + \lambda_1 + \lambda_2\big], \tag{4.11}$$

$$\begin{aligned}\lambda_1\lambda_2 \langle \psi, a\rangle = {}& -L_1\kappa_2 + 4\kappa_1^2\kappa_2^2 - (\kappa_1+\kappa_2)^2 \\ & + (\lambda_1+\lambda_2)(\kappa_1+\kappa_2) - \lambda_1\lambda_2.\end{aligned} \tag{4.12}$$

From (4.11) and (4.12) we find

$$\begin{aligned}\lambda_1\lambda_2 \langle N, a\rangle = {}& -2\kappa_1\lambda_1\lambda_2\langle \psi, a\rangle \\ & + 2\kappa_1\Big[-\kappa_1^2 + (\lambda_1+\lambda_2)\kappa_1 - \lambda_1\lambda_2 - 3\kappa_1\kappa_2 + 3\kappa_1^2\kappa_2^2 - \kappa_1\kappa_2^3\Big],\end{aligned}$$

and by taking gradient, we obtain

$$-\lambda_1\lambda_2 S a^\top = -2\kappa_1\lambda_1\lambda_2 a^\top + 2\kappa_1^2\Big[-3 + 6\kappa_1\kappa_2 - 3\kappa_2^2\Big]\nabla\kappa_2. \tag{4.13}$$

On the other hand, by using $S \circ P = H_2 I$ and (4.1), we get

$$\lambda_1\lambda_2 S a^\top = -6\kappa_1^2\kappa_2 S(\nabla\kappa_2) + 2\kappa_1\kappa_2\nabla\kappa_2. \tag{4.14}$$

Now, from (4.10), (4.13) and (4.14), we deduce

$$(3\kappa_1\kappa_2 - 2)S(\nabla\kappa_2) = (-3\kappa_1\kappa_2^2 + (9\kappa_1^2 - 1)\kappa_2 - \kappa_1)\nabla\kappa_2.$$

Since $3\kappa_1\kappa_2 - 2 \neq 0$ (otherwise, κ_2 would be constant), we deduce

$$S(\nabla\kappa_2) = f(\kappa_1,\kappa_2)\nabla\kappa_2, \qquad f(\kappa_1,\kappa_2) = \frac{-3\kappa_1\kappa_2^2 + (9\kappa_1^2 - 1)\kappa_2 - \kappa_1}{3\kappa_1\kappa_2 - 2}.$$

This equation implies that either $f(\kappa_1,\kappa_2) = \kappa_1$ or $f(\kappa_1,\kappa_2) = \kappa_2$. In any case, it follows that κ_2 is constant on $\mathcal{U}$, a contradiction. $\square$

As a consequence of theorems 4.1, 4.2 and 4.3, we have the following characterization of L_1-2-type surfaces in $\mathbb{H}^3$.

Theorem 4.4. *Let $\psi : M^2 \to \mathbb{H}^3 \subset \mathbb{R}^4_1$ be a surface of L_1-2-type. Then either M^2 is an open piece of a standard Riemannian product $\mathbb{H}^1(-\sqrt{1+r^2}) \times \mathbb{S}^1(r)$, or M^2 has non constant mean curvature, non constant Gaussian curvature, and non constant principal curvatures.*

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