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Author(s):

M. Rezaei and Z. Foruzanfar

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CLASSIFICATION OF SOLVABLE GROUPS WITH A GIVEN PROPERTY

M. REZAEI* AND Z. FORUZANFAR

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ABSTRACT. In this paper, we classify all finite solvable groups satisfying the following property P_5: their orders of representatives are set-wise relatively prime for any 5 distinct non-central conjugacy classes.
Keywords: Frobenius group, conjugacy classes, graph, order.
MSC(2010): Primary: 20E45; Secondary: 20D60.

1. **Introduction**

Let G be a finite group and let V be the set of all non-central conjugacy classes of G. From lengths of conjugacy classes, the following class graph $\Gamma(G)'$ was introduced in [1]: its vertex set is the set V and two distinct vertices x^G and y^G are connected with an edge if $(|x^G|, |y^G|) > 1$. The class graph $\Gamma(G)'$ has been studied in some details: see for example [1–3] and [5]. In [5], the authors have studied the structure of a finite group G with the following property: for every prime p, G has at most $n-1$ conjugacy classes whose sizes are multiples of p. In particular, they have classified the finite groups when $n = 5$, extending the result of Fang and Zhang [3]. Similarly, in terms of orders of elements, the authors in [7] have attached a graph $\Gamma(G)$ to G as follows: its vertex set is also the set V and two distinct vertices x^G and y^G are connected with an edge if $(o(x), o(y)) > 1$. Thus a new conjugacy class graph is defined. A finite group G satisfies the property P_n if for every prime integer p, G has at most $n-1$ non-central conjugacy classes whose orders of representatives are multiples of p. Thus $\Gamma(G)$ does not have a subgraph K_n if and only if G satisfies the property P_n. The authors in [7] classified all finite groups that satisfy property P_4. Also in [4], all finite

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*Corresponding author.

non-solvable groups that satisfy property P_5 have been classified. The objective of this paper is to classify all finite solvable groups that satisfy property P_5.

Theorem 1.1. *Let G be a finite solvable group that satisfies property P_5. Then G is isomorphic to one of the following groups:*

(i) *An abelian group;*
(ii) *A Frobenius group with complement of order 2 and kernel $\mathbb{Z}_3$, $\mathbb{Z}_5, \mathbb{Z}_7$, $(\mathbb{Z}_3)^2$ or $\mathbb{Z}_9$;*
(iii) *A Frobenius group with complement of order 3 and kernel $(\mathbb{Z}_2)^2, \mathbb{Z}_7$ or $\mathbb{Z}_{13}$;*
(iv) *A Frobenius group with cyclic complement of order 4 and kernel $\mathbb{Z}_5$, $(\mathbb{Z}_3)^2$, $\mathbb{Z}_{13}$ or $\mathbb{Z}_{17}$;*
(v) *The Frobenius group with complement of order 5 and kernel $\mathbb{Z}_{11}$ and $(\mathbb{Z}_2)^4$;*
(vi) *A Frobenius group with cyclic complement of order 6 and kernel $\mathbb{Z}_7$, $\mathbb{Z}_{13}$, $\mathbb{Z}_{19}$ or $(\mathbb{Z}_5)^2$;*
(vii) D_{20}, Q_{20}, D_{12}, D_8, Q_8 *or* $T = \langle x, y | x^3 = 1, y^4 = 1, xy = yx^{-1}\rangle$.

Conversely, all these groups satisfy property P_5.

2. Preliminaries

Before starting the proof of Theorem 1.1, we give some preliminary results.

Lemma 2.1 ([7, Lemma 1]). *Let G be a finite group. Then G satisfies property P_n if and only if $\Gamma(G)$ has no subgraph K_n.*

Lemma 2.2 ([7, Lemma 2]). *Let G be a finite group that satisfies property P_n. Then property P_n is inherited by quotient groups of G.*

Lemma 2.3 ([6, Lemma 1.3]). *If G possesses an element x with $|C_G(x)| = 4$, then a Sylow 2-subgroup P of G is the dihedral, semi-dihedral or generalized quaternion group. In particular $|\frac{P}{P'}| = 4$ and P has a cyclic subgroup of order $\frac{|P|}{2}$.*

Proposition 2.4 ([6, Proposition 2.1]). *Let N be a normal subgroup of a non-abelian group G. Then $k_G(G-N) = 1$ if and only if G is a Frobenius group with the kernel N of odd order $\frac{|G|}{2}$.*

Theorem 2.5 ([6, Theorem 2.2]). *Let N be a normal subgroup of a non-abelian group G. Then $k_G(G-N) = 2$ if and only if G is one of the following solvable groups.*

(1) $N = 1$ *and* $G \cong S_3$.
(2) $|\frac{G}{N}| = 3$ *and G is a Frobenius group with the kernel N.*

(3) $|\frac{G}{N}| = 2$ *and* $|C_G(x)| = 4$ *for all* $x \in G - N$. *In particular,* $P \in Syl_2(G)$ *has a cyclic subgroup of order* $\frac{|P|}{2}$; *furthermore, one of the following holds:*

(3.a) *G has a normal and abelian 2-complement.*

(3.b) *G has a normal 2-complement and P is a quaternion group.*

(3.c) *G has an abelian 2-complement and* $P \cong D_8$, *the dihedral group of order 8.*

Theorem 2.6 ([6, Theorem 3.6]). *Let N be a normal subgroup of a non-abelian solvable group G. Then* $G - N = x^G \cup y^G \cup z^G$ *is a union of three conjugacy classes if and only if one of the following is true:*

(1) $N = 1$ *and* $G \cong A_4$ *or* D_{10}.

(2) $\frac{G}{N} \cong S_3$ *and* $G \cong S_4$.

(3) *G is a Frobenius group with the kernel N and a cyclic complement of order 4.*

(4) $G \cong D_8$ *or* Q_8.

(5) $|\frac{G}{N}| = 2$, $|C_G(x)| = |C_G(y)| = |C_G(z)| = 6$. *And in this case, N is of odd order and N has a normal and abelian 3-complement.*

(6) $|\frac{G}{N}| = 2$, $|C_G(x)| = 4$, $|C_G(y)| = 6$ *and* $|C_G(z)| = 12$. *And in this case, either G has a normal 2-complement or* $\frac{G}{O_{2'}(G)} \cong S_4$.

(7) $|\frac{G}{N}| = 2$, $|C_G(x)| = 4$, $|C_G(y)| = |C_G(z)| = 8$. *And in this case, either* $\frac{G}{O_{2'}(G)} \cong GL(2,3)$ *with abelian* $O_{2'}(G)$, *or* $\frac{G}{O_{2'}(G)}$ *is isomorphic to a non-abelian group of order 16.*

3. The proof of Theorem 1.1

It is easy to see that the groups listed in Theorem 1.1 satisfy property P_5. For a finite group G and $A \subseteq G$, let $k_G(A)$ be the number of classes of G contained in A and $\pi_e(G)$ denotes the set of all orders of elements in G. If G is abelian, then G satisfies property P_5. Now suppose that G is a finite non-abelian solvable group that satisfies property P_5 and $M = G^{'}Z(G)$. It is easy to see that $M < G$. Take $xM \in \frac{G}{M}$ such that $o(xM) = p$. Since $\frac{G}{M}$ is abelian, there are at least $p-1$ classes of elements of order p in $\frac{G}{M}$. Note that $o(xM)|o(x)$ and xM, when viewed as a subset of G, is a union of some classes of G. Thus we conclude that G has at least $p-1$ non-central classes whose orders of representatives are multiples of p. Therefore, $p - 1 \leq 4$, i.e., $p = 2, 3$ or 5. Furthermore, $|\frac{G}{M}| = 2, 3, 4, 5$ or 6 and $k_G(G - M) \leq 6$.

1. Suppose that $k_G(G - M) = 1$.
It follows from Proposition 2.4 that G is a Frobenius group with kernel M and M is abelian of odd order $\frac{|G|}{2}$. This implies that $Z(G) = 1$ and $M = G^{'}$. Since G satisfies property P_5, we conclude that $M \in Syl_p(G)$ and thus

$k_G(M-\{1\}) \leq 4$. It follows that $\frac{|M|-1}{2} \leq 4$ and hence $|M| \leq 9$. Therefore G is a Frobenius group with complement of order 2 and kernel $\mathbb{Z}_3$, $\mathbb{Z}_5, \mathbb{Z}_7, (\mathbb{Z}_3)^2$ or $\mathbb{Z}_9$.

2. Suppose that $k_G(G-M) = 2$.
Applying Theorem 2.5, we get the following three cases.
(2.a) $M = 1$ and $G \cong S_3$. In this case $\frac{G}{M} \cong S_3$. Therefore $\frac{G}{M}$ is a non-abelian group, a contradiction.
(2.b) $|\frac{G}{M}| = 3$ and G is a Frobenius group with kernel M.

Similarly, we have $M \in Syl_p(G)$ and $k_G(M-\{1\}) \leq 4$. If M is abelian, then $\frac{|M|-1}{3} \leq 4$ and hence $|M| \leq 13$. Therefore G is a Frobenius group with complement of order 3 and kernel $(\mathbb{Z}_2)^2, \mathbb{Z}_7$ or $\mathbb{Z}_{13}$. If M is non-abelian, then $k_G(Z(M)-\{1\}) \leq 3$. Assume first that $k_G(Z(M)-\{1\}) = 3$. From this we can deduce that $|Z(M)| = 10$, which is not possible. Also assume that $k_G(Z(M)-\{1\}) = 2$. We have $|Z(M)| = 7$ and M is a 7-group. Let $|M| = 7^r$. If $M-Z(M) = \alpha^G$, then it implies successively $|\alpha^G| = 3.7^k$, $7^r = 3.7^k+7$. This equality has no solution. If $M-Z(M) = \alpha^G \cup \beta^G$, then $|\alpha^G| = 3.7^k \leq |\beta^G| = 3.7^s$ and so $7^r = 3.7^k+3.7^s+7$, which forces $(p^k, p^s, p^r) = (7, 7, 49)$. Therefore G is a Frobenius group with complement of order 3 and kernel of order 49. Since this group has at least five non-central conjugacy classes which their orders of representatives are multiples of 7, it does not satisfy property P_5. Now assume that $k_G(Z(M)-\{1\}) = 1$. We have $|Z(M)| = 4$ and M is a 2-group. Let $|M| = 2^r$. If $M-Z(M) = \alpha^G$, then $|\alpha^G| = 3.2^k$ and hence $2^r = 3.2^k+4$, which forces $(p^k, p^r) = (4, 16)$. We conclude that there is an element such that its centralizer in G is of order 4. By Lemma 2.3, M is the dihedral, semi-dihedral or generalized quaternion group. This forces $|Z(M)| = 2$, a contradiction. If $M-Z(M) = \alpha^G \cup \beta^G$, then $|\alpha^G| = 3.2^k \leq |\beta^G| = 3.2^s$ and so $2^r = 3.2^k+3.2^s+4$, which forces $(p^k, p^s, p^r) = (2, 2, 16)$ or $(4, 16, 64)$. If $(p^k, p^s, p^r) = (2, 2, 16)$, then G is a Frobenius group with complement of order 3 and kernel of order 16. Now since this group has exactly five non-central conjugacy classes which their orders of representatives are multiples of 2, it does not satisfy property P_5. If $(p^k, p^s, p^r) = (4, 16, 64)$, then we conclude that there is an element such that its centralizer in G is of order 4. By Lemma 2.3, M is the dihedral, semi-dihedral or generalized quaternion group. This forces $|Z(M)| = 2$, a contradiction. If $M-Z(M) = \alpha^G \cup \beta^G \cup \gamma^G$, then it implies successively $|\alpha^G| = 3.2^k \leq |\beta^G| = 3.2^s \leq |\gamma^G| = 3.2^l$, $2^r = 3.2^k+3.2^s+3.2^l+4$, which forces $(p^k, p^s, p^l, p^r) = (4, 8, 8, 64)$. Therefore G is a Frobenius group with complement of order 3 and kernel of order 64. Now since this group has at least five non-central conjugacy classes which their orders of representatives are multiples of 2, it does not satisfy property P_5.
(2.c) $|\frac{G}{M}| = 2$ and $|C_G(x)| = 4$ for any $x \in G-M$.

Applying Lemma 2.3 and Theorem 2.5, we can see that $Z(G) > 1$. Since $|C_G(x)| = 4$ for any $x \in G-M$, we have $|Z(G)| = 2$. Take $x \in G-M$, we conclude that $o(xZ(G)) = 2$ and $|C_{\frac{G}{Z(G)}}(xZ(G))| = 2$. Thus $xZ(G)$ acts fixed point freely on $\frac{M}{Z(G)}$, so $\frac{G}{Z(G)}$ is a Frobenius group with kernel $\frac{M}{Z(G)}$. Since $\frac{M}{Z(G)}$ is a p-group, we have $\frac{|\frac{M}{Z(G)}|-1}{2} \leq 4$ and hence $|\frac{M}{Z(G)}| = 3, 5, 7$ or 9. Therefore $|G| = 12, 20, 28$ or 36 and G is one of the following groups: D_{12}, $T = \langle x, y | x^3 = 1, y^4 = 1, xy = yx^{-1} \rangle$, D_{20} or Q_{20}.

3. Suppose that $k_G(G - M) = 3$. Let $G - M = x^G \cup y^G \cup z^G$.
Applying Theorem 2.6, we get the following seven cases.
(3.a) $M = 1$ and $G \cong A_4$ or D_{10}. In this case $\frac{G}{M}$ is a non-abelian group, that is not possible.
(3.b) $\frac{G}{M} \cong S_3$ and $G \cong S_4$. In this case $\frac{G}{M}$ is a non-abelian group, a contradiction.
(3.c) $G \cong D_8$ or Q_8.
(3.d) G is a Frobenius group with kernel M and a cyclic complement of order 4. In this case, arguing as in (1), we have $M \in Syl_p(G)$ and $k_G(M - \{1\}) \leq 4$. It follows that $\frac{|M|-1}{4} \leq 4$ and hence $|M| \leq 17$. We conclude that G is a Frobenius group with cyclic complement of order 4 and kernel $\mathbb{Z}_5, (\mathbb{Z}_3)^2, \mathbb{Z}_{13}$ or $\mathbb{Z}_{17}$.
(3.e)$|\frac{G}{M}| = 2$, $|C_G(x)| = |C_G(y)| = |C_G(z)| = 6$, $o(x) = 2$, $o(y) = 6$ and $z = y^{-1}$. In this case, M is of odd order and M has a normal and abelian 3-complement, say N. Then N is a normal and abelian $\{2, 3\}$-complement of G. Let $|\frac{M}{N}| = 3^n$, where $n \geq 1$. We claim that $|\frac{M}{N}| = 3$. Otherwise, the number of conjugacy classes of $\frac{M}{N}$ is at least 9. Since $|C_G(x)| = 6$, we have $|C_M(x)| = 3$ and thus $\frac{M}{N}$ has at least 6 conjugacy classes which lift to conjugacy classes not contained in $Z(G)$. Since $|\frac{G}{M}| = 2$, the subgroup M contains at least 3 non-central conjugacy classes of G, such that their elements have order divisible by 3. Since also y^G and z^G are such conjugacy classes, which contradicts property P_5. Thus $|\frac{M}{N}| = 3$. If $Z(G) \neq 1$, then $G =< y > N$. So $G^{'} \subseteq N$ and $y^2 \in Z(G)$. For any $a \in N \setminus 1$ we get two further non-central conjugacy classes of 3-elements, namely $(y^2a)^G = \{y^2a, y^2a^x\}$ and $(y^4a)^G = \{y^4a, y^4a^x\}$. Since $N \neq 1$, we have $N \setminus 1 = \{a, a^x\}$ and $|N| = 3$, which is not possible. Thus $Z(G) = 1$. Now we show that $N = 1$. Suppose in contrary that $N > 1$ and $M = HN$, where $H \cong \frac{M}{N}$. Since $(|\frac{M}{N}|, |N|) = 1$, we see that all elements in $M - N$ have the same order 3. It implies that for any element $h \in H - \{1\}$, $C_M(h) = H$. Therefore, M is a Frobenius group with kernel N and cyclic complement H of prime order 3. It implies that $\frac{G}{N} \cong S_3$ and thus G is 2-Froubenius. This forces $6 \notin \pi_e(G)$, a contradiction. Hence $N = 1$ and $|G| = 6$, that is not possible.
(3.f) $|\frac{G}{M}| = 2$, $|C_G(x)| = 4$, $|C_G(y)| = 6$ and $|C_G(z)| = 12$. In this case, M is of even order and either G has a normal 2-complement or $\frac{G}{O_{2'}(G)} \cong S_4$. Let $P \in Syl_2(G)$ and $P \cap M = P_1$. By Lemma 2.3, P is dihedral, semi-dihedral or generalized quaternion. Since $|\frac{G}{M}| = 2$, every element of $G - M$ has an order

divisible by 2. Now since $k_G(G-M)=3$, therefore $G-M$ has at least three non-central conjugacy classes, such that the order of representative of each of which is a multiple of 2. Also since $|Z(G)|||C_G(x)|$, we have $|Z(G)|\leq 2$. Let $|Z(G)|=1$. If $k_G(P_1-\{1\})=1$, then $P_1=1\cup u^G$, for some $u\in P_1$ and P_1 is an elementary abelian normal 2-subgroup of G. Since P_1 has index 2 in P, we conclude that $|P_1|=4$ and $|P|=8$. Also, since P has more than one element of order 2, it must be dihedral. This implies that conjugacy class of u is $P_1-\{1\}$, so the conjugacy class of u would have size 3. If G has a normal 2-complement N, then $M=P_1\times N$. In particular, N centralizes the element u. This implies that the conjugacy class of u in G has size that is a power of 2, this is a contradiction. Therefore, P_1 has at least two non-central conjugacy classes of G, which contradicts property P_5. Now suppose that $G/O_{2'}(G)\cong S_4$. In this case G has a normal subgroup A such that $A/O_{2'}(G)\cong P_1$. Therefore, $A=P_1\times O_{2'}(G)$. In particular, $O_{2'}(G)$ and P_1 centralize the element u. Also P is not a subgroup of $C_G(u)$. This implies that the conjugacy class of u in G has size 2 or 6, which is not possible. Therefore, P_1 has at least two non-central conjugacy classes of G, contradicts by the property P_5. Now suppose that $|Z(G)|=2$ and $a\in Z(G)$ be of order 2. If $|G^{'}\cap Z(G)|=1$, then there are two elements $b,c\in G^{'}-Z(G)$, such that $o(b)=2$ and $o(c)=3$. So b^G and $(ac)^G$ are non-central conjugacy classes of G contained in M, this contradicts property P_5. Now suppose that $|G^{'}\cap Z(G)|=2$. Thus $Z(G)\leq G^{'}$. If $Z(G)=G^{'}$, then $|G|=4$, a contradiction. Suppose that $Z(G)<G^{'}$. Therefore, there is $c\in G^{'}-Z(G)$, such that $o(c)=3$. So $(ac)^G$ is a non-central conjugacy class of G contained in M. Since $P_1\in Syl_2(M)$, $Z(G)$ is contained in P_1. Also since P_1 is a normal subgroup of G, it is a union of some classes of G and so it has a non-central conjugacy class, which contradicts property P_5.
(3.g) $|\frac{G}{M}|=2$, $|C_G(x)|=4$, $|C_G(y)|=|C_G(z)|=8$. Let $P\in Syl_2(G)$ and $P\cap M=P_1$. In this case, P is a non-abelian group of order 16 and P_1 is a non-abelian group of order 8. Since $|\frac{G}{M}|=2$, every element of $G-M$ has an order divisible by 2. Now since $k_G(G-M)=3$, $G-M$ has at least 3 non-central conjugacy classes such that the order of representative of each of which is a multiple of 2. Also since $|Z(G)|||C_G(x)|$, $|Z(G)|\leq 2$. Suppose that $|Z(G)|=1$. Thus $M=G^{'}$. If $k_G(P_1-\{1\})=1$, then P_1 is abelian, which is not possible. Therefore, P_1 has at least two non-central conjugacy classes, this contradicts property P_5. So assume that $|Z(G)|=2$ and $a\in Z(G)$ be of order 2. If $|G^{'}\cap Z(G)|=1$, then there are two elements $b,c\in G^{'}-Z(G)$, such that $o(b)=2$ and $o(c)=p$, where p is an odd prime. So b^G and $(ac)^G$ are non-central conjugacy classes of G contained in M, this contradicts property P_5. Now suppose that $|G^{'}\cap Z(G)|=2$. Thus $Z(G)\leq G^{'}$. If $Z(G)=G^{'}$, then $|G|=4$, a contradiction. If $Z(G)<G^{'}$, then there is $c\in G^{'}-Z(G)$, such that $o(c)=p$, where p is an odd prime. So $(ac)^G$ is a non-central conjugacy class of G contained in M. Since $P_1\in Syl_2(M)$, $Z(G)$ is contained in P_1. Also

since P_1 is a normal subgroup of G, it is a union of some classes of G and has a non-central conjugacy class that contradicts property P_5.

4. Suppose that $k_G(G-M)=4$ and $G-M=x^G\cup y^G\cup z^G\cup w^G$.
In this case $|\frac{G}{M}|\leq 5$. Let $|\frac{G}{M}|=5$. So all of the elements in each of the four non-trivial cosets of M in G are conjugate. Hence they all have centralizers of order 5. Let $g\in G$ such that gM generates $\frac{G}{M}$. Then g is of order 5 and G is a Frobenius group with kernel M and complement $\langle g\rangle$. This implies that $Z(G)=1$ and $M=G^{'}$. Since G satisfies property P_5, we have $M\in Syl_p(G)$ and $k_G(M-\{1\})\leq 4$. If M is abelian, then $\frac{|M|-1}{5}\leq 4$ and hence $|M|\leq 21$. Therefore, G is a Frobenius group with complement of order 5 and kernel $\mathbb{Z}_{11}$ or $(\mathbb{Z}_2)^4$. If M is non-abelian, then $k_G(Z(M)-\{1\})\leq 3$. Assume first that $k_G(Z(M)-\{1\})=3$. We deduce that $|Z(M)|=16$ and M is a 2-group. Let $|M|=2^r$. Since $M-Z(M)=\alpha^G$ and $|\alpha^G|=5.2^k$, we have $2^r=5.2^k+16$, which has no solution. Now suppose that $k_G(Z(M)-\{1\})=2$. We have $|Z(M)|=11$ and M is a 11-group. Let $|M|=11^r$. If $M-Z(M)=\alpha^G$, then $|\alpha^G|=5.11^k$ and so $11^r=5.11^k+11$, which has no solution. If $M-Z(M)=\alpha^G\cup\beta^G$, then $|\alpha^G|=5.11^k\leq|\beta^G|=5.11^s$ and hence $11^r=5.11^k+5.11^s+11$, which forces $(p^k,p^s,p^r)=(11,11,121)$. Therefore, G is a Frobenius group with complement of order 5 and kernel of order 121. Now since this group has at least five non-central conjugacy classes whose their orders of representatives are multiples of 11, it does not satisfy property P_5. Finally, assume that $k_G(Z(M)-\{1\})=1$. Then $|Z(M)|=6$, a contradiction.
If $|\frac{G}{M}|=4$, then every element of $G-M$ has an order divisible by 2. Since $k_G(G-M)=4$, $G-M$ has at least four non-central conjugacy classes such that the order of representative of each of which is a multiple of 2. Also among these four non-central conjugacy classes, there are two non-central conjugacy classes such that the centralizer of representative of each of which is of order 4. Since $G-M$ possesses an element g with $|C_G(g)|=4$, Lemma 2.3 implies that M is of even order. Also since $|Z(G)|||C_G(g)|$, we have $|Z(G)|\leq 2$. If $|Z(G)|=1$, then M contains at least one non-central conjugacy class of G, such that its representative has order 2, which contradicts property P_5. So assume that $|Z(G)|=2$ and $a\in Z(G)$ be of order 2. If $|G^{'}\cap Z(G)|=1$, then there is $1\neq b\in G^{'}$, such that $(ab)^G$ is a non-central conjugacy class of G contained in M, which contradicts property P_5. Now suppose that $|G^{'}\cap Z(G)|=2$. Thus $Z(G)\leq G^{'}$. If $Z(G)=G^{'}$, then $|G|=8$ and G is isomorphic to D_8 or Q_8, that is impossible. Now let $Z(G)<G^{'}$. Then there is $b\in G^{'}-Z(G)$, such that $(ab)^G$ is a non-central conjugacy class of G contained in M, a contradiction.
Now let $|\frac{G}{M}|=3$. Note that for any $g\in G-M$, $o(g)$ is a multiple of 3 and hence $|C_G(g)|$ is a multiple of 3. Set $|C_G(x)|=3a$, $|C_G(y)|=3b$, $|C_G(z)|=3c$ and $|C_G(w)|=3d$. We conclude that $\frac{1}{3a}+\frac{1}{3b}+\frac{1}{3c}+\frac{1}{3d}+\frac{1}{3}=1$. This equality holds if $a=1$ and $b=c=d=3$, $a=1$, $b=2$ and $c=d=4$ or $a=b=c=d=2$. In

the first and second case, G possesses an element x of order 3 with $|C_G(x)| = 3$ and thus x acts fixed point freely on M. So G is a Frobenius group with kernel M and complement of order 3. Clearly M is a p-group and $k_G(M - \{1\}) \leq 4$. If M is abelian, then $\frac{|M|-1}{3} \leq 4$ and hence $|M| = 4, 7$ or 13, which is not possible. Suppose that M is not abelian. Thus $k_G(Z(M) - \{1\}) \leq 3$. If $k_G(Z(M)-\{1\}) = 3$, then $|Z(M)| = 10$, that is not possible. Now assume that $k_G(Z(M)-\{1\}) = 2$. We have $|Z(M)| = 7$ and M is a 7-group. Let $|M| = 7^r$. If $M - Z(M) = \alpha^G$, then $|\alpha^G| = 3.7^k$ and so $7^r = 3.7^k + 7$, which has no solution. If $M - Z(M) = \alpha^G \cup \beta^G$, then $|\alpha^G| = 3.7^k \leq |\beta^G| = 3.7^s$ and hence $7^r = 3.7^k+3.7^s+7$, which forces $(p^k, p^s, p^r) = (7, 7, 49)$, a contradiction. Finally assume that $k_G(Z(M) - \{1\}) = 1$. We have $|Z(M)| = 4$ and M is a 2-group. Let $|M| = 2^r$. If $M-Z(M) = \alpha^G$, then $|\alpha^G| = 3.2^k$ and so $2^r = 3.2^k+4$, which forces $(p^k, p^r) = (4, 16)$. We conclude that there is an element such that its centralizer in G is of order 4. By Lemma 2.3, M is a dihedral, semi-dihedral or generalized quaternion group. This forces $|Z(M)| = 2$, a contradiction. In cases $M-Z(M) = \alpha^G\cup\beta^G$ or $M-Z(M) = \alpha^G\cup\beta^G\cup\gamma^G$, by above discussion, we will have a contradiction. In the third case, we have $|C_G(x)| = |C_G(y)| = |C_G(z)| = |C_G(w)| = 6$. So $|Z(G)| \leq 3$. First suppose that $|Z(G)| = 1$. If $3||M|$, then there is an element $b \in M$ of order 3 and b^G is a non-central conjugacy class of G contained in M, a contradiction. Now suppose that $3 \nmid |M|$. Then M is a normal 3-complement of G. Since $(|\frac{G}{M}|, |M|) = 1$, each element in $G - M$ has order 3. Write $G = HM$, where $H \cong \frac{G}{M}$. It implies that for any element $h \in H-\{1\}$, $C_G(h) = H$. Therefore, G is a Frobenius group with kernel M and abelian complement H such that H is a cyclic group of prime order 3. Since G satisfies property P_5, $M \in Syl_p(G)$ and $k_G(M - \{1\}) \leq 4$. If M is abelian, then $\frac{|M|-1}{3} \leq 4$ and hence $|M| = 4, 7$ or 13. But non of the attaining groups satisfy in this case. Suppose that M is not abelian. Thus $k_G(Z(M)-\{1\}) \leq 3$. If $k_G(Z(M) - \{1\}) = 3$, then $|Z(M)| = 10$, which is not possible. Now assume that $k_G(Z(M) - \{1\}) = 2$. We have $|Z(M)| = 7$ and M is a 7-group. Let $|M| = 7^r$. If $M - Z(M) = \alpha^G$, then $|\alpha^G| = 3.7^k$ and so $7^r = 3.7^k + 7$, which has no solution. If $M - Z(M) = \alpha^G \cup \beta^G$, then $|\alpha^G| = 3.7^k \leq |\beta^G| = 3.7^s$ and hence $7^r = 3.7^k+3.7^s+7$, which forces $(p^k, p^s, p^r) = (7, 7, 49)$, a contradiction. Finally assume that $k_G(Z(M) - \{1\}) = 1$. We have $|Z(M)| = 4$ and M is a 2-group. If $M-Z(M) = \alpha^G$, then $|\alpha^G| = 3.2^k$. Let $|M| = 2^r$. Then $2^r = 3.2^k+4$, which forces $(p^k, p^r) = (4, 16)$. We conclude that there is an element such that its centralizer in G is of order 4. By Lemma 2.3 and above discussion, we will have a contradiction. Now suppose that $|Z(G)| = 2$ and $a \in Z(G)$ be of order 2. Since for every $g \in G - M$, $|C_G(g)| = 6$, we have $o(g) = 3$ or 6. If there is an element $h \in G - M$, such that $o(h) = 6$, then $ah \notin C_G(h)$ and so $C_G(h) \subset C_G(ah)$. Thus $|C_G(ah)| \geq 12$, $ah \notin G-M$ and $(ah)^G$ is a non-central conjugacy class of G contained in M, a contradiction. Therefore for every $g \in G - M$, $o(g) = 3$. Now since $o(ag) = 6$, therefore $(ag)^G$ is a non-central conjugacy class of G contained in M, a contradiction. Finally, suppose that $|Z(G)| = 3$ and $a \in Z(G)$ has order 3. Let $|G^{'} \cap Z(G)| = 1$ and $1 \neq b \in G^{'}$ be of order 2. Then $(ab)^G$ is a non-central conjugacy class of G contained in M, this contradicts property P_5. Now suppose that $|G^{'} \cap Z(G)| = 3$. Thus $Z(G) \leq G^{'}$. If $Z(G) = G^{'}$, then $|G| = 9$, a contradiction. So assume that $Z(G) < G^{'}$. Then there is $b \in G^{'} - Z(G)$ of order 2, such that $(ab)^G$ is a non-central conjugacy class of G contained in M, a contradiction. Finally, let $|\frac{G}{M}| = 2$. Note that for any $g \in G - M$, $o(g)$ is even and hence $|C_G(g)|$ is a multiple of 2. Set $|C_G(x)| = 2a$, $|C_G(y)| = 2b$, $|C_G(z)| = 2c$ and $|C_G(w)| = 2d$. Since $k_G(G-M) = 4$, therefore $\frac{1}{2a} + \frac{1}{2b} + \frac{1}{2c} + \frac{1}{2d} + \frac{1}{2} = 1$. This equality holds for $a = 2$ and $b = c = d = 6$, $a = 2, b = 4$ and $c = d = 8$ or $a = b = c = d = 4$. In the first and second case, since G possesses an element x with $|C_G(x)| = 4$, Lemma 2.3 implies that M is of even order. Also since $|Z(G)|||C_G(x)|$, $|Z(G)| \leq 2$. If $|Z(G)| = 1$, then M contains at least one non-central conjugacy class of G, such that the order of representative of it is 2, this contradicts property P_5. Now suppose that $|Z(G)| = 2$ and $a \in Z(G)$ be of order 2. Let $|G^{'} \cap Z(G)| = 1$ and $1 \neq b \in G^{'}$. Then $(ab)^G$ is a non-central conjugacy class of G contained in M, this contradicts property P_5. Now suppose that $|G^{'} \cap Z(G)| = 2$. Thus $Z(G) \leq G^{'}$. If $Z(G) = G^{'}$, then $|G| = 4$, which is not possible. So assume that $Z(G) < G^{'}$. Then there is $b \in G^{'} - Z(G)$, such that $(ab)^G$ is a non-central conjugacy class of G contained in M, a contradiction. In the third case, $|C_G(x)| = |C_G(y)| = |C_G(z)| = |C_G(w)| = 8$. Since $|Z(G)|||C_G(x)|$, $|Z(G)| = 1, 2$ or 4. Also since $|C_G(x)|||G|$, $|G|$ is a multiple of 8. If $|Z(G)| = 1$, then $M = G^{'}$ and M contains at least one non-central conjugacy class of G, such that the order of its representative is 2, this contradicts property P_5. Now assume that $|Z(G)| = 2$ and $a \in Z(G)$ be of order 2. Let $|G^{'} \cap Z(G)| = 1$. Then there is $1 \neq b \in G^{'}$, such that $(ab)^G$ is a non-central conjugacy class of G contained in M, this contradicts property P_5. Now suppose that $|G^{'} \cap Z(G)| = 2$. Thus $Z(G) \leq G^{'}$. Now using the argument mentioned before, we get a contradiction. Finally suppose that $|Z(G)| = 4$ and $a \in Z(G)$ be of order 2. If $|G^{'} \cap Z(G)| = 1$, then there is $1 \neq b \in G^{'}$, such that $(ab)^G$ is a non-central conjugacy class of G contained in M, which contradicts property P_5. Now assume that $|G^{'} \cap Z(G)| = 2$. We know that $|G^{'}|$ is a multiple of 2. If $|G^{'}| = 2$, then $|G| = 8$ and therefore G is isomorphic to D_8 or Q_8. But non of these groups satisfy in this case. So $|G^{'}| \geq 4$ and therefore there is an element $b \in G^{'} - Z(G)$, such that $(ab)^G$ is a non-central conjugacy class of G contained in M, which contradicts property P_5. Finally suppose that $|G^{'} \cap Z(G)| = 4$. Thus $Z(G) \leq G^{'}$. Using the discussion mentioned before, we get a contradiction again.

5. Suppose that $k_G(G-M) = 5$. It is easy to see that $|\frac{G}{M}| = 6$. In this case, all elements in each of five non-trivial cosets of M in G are conjugate. Hence they all have centralizers of order 6. Let $g \in G$ such that gM generates $\frac{G}{M}$. Then g is of order 6, and G is a Frobenius group with kernel M and complement $\langle g \rangle$. This implies that $Z(G) = 1$ and $M = G^{'}$. Since G satisfies property P_5, we have $M \in Syl_p(G)$ and $k_G(M - \{1\}) \leq 4$. It follows that $\frac{|M|-1}{6} \leq 4$ and hence $|M| \leq 25$. Therefore, G is a Frobenius group with complement of order 6 and kernel $\mathbb{Z}_7, \mathbb{Z}_{13}, \mathbb{Z}_{19}$ or $(\mathbb{Z}_5)^2$.

6. Finally suppose that $k_G(G - M) = 6$. It is easy to see that $|\frac{G}{M}| = 6$. In this case, there is an element $g \in G - M$ of order 6, such that $|C_G(g)| = 6$. It implies that g acts fixed point freely on M. Thus G is a Frobenius group with kernel M and complement $\langle g \rangle$. Since G satisfies property P_5, we have $M \in Syl_p(G)$ and $k_G(M - \{1\}) \leq 4$. It follows that $\frac{|M|-1}{6} \leq 4$ and hence $|M| \leq 25$. Therefore G is a Frobenius group with complement of order 6 and kernel $\mathbb{Z}_7, \mathbb{Z}_{13}, \mathbb{Z}_{19}$ or $(\mathbb{Z}_5)^2$, but non of these groups satisfy in this case.

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(Mehdi Rezaei) Department of Mathematics, Buein Zahra Technical University, Buein Zahra, Qazvin, Iran.
E-mail address: `m_rezaei@bzte.ac.ir;mehdrezaei@gmail.com`

(Zeinab Foruzanfar) Department of Engineering Sciences and Physics, Buein Zahra Technical University, Buein Zahra, Qazvin, Iran.
E-mail address: `zforouzanfar@gmail.com`