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n-ARRAY JACOBSON GRAPHS

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ABSTRACT. We generalize the notion of Jacobson graphs into n-array columns called n-array Jacobson graphs and determine their connectivities and diameters. Also, we will study forbidden structures of these graphs and determine when an n-array Jacobson graph is planar, outer planar, projective, perfect or domination perfect.
Keywords: Jacobson graph, connectivity, planar graph, outer planar graph, perfect graph.
MSC(2010): Primary: 05C10; Secondary: 05C17, 16P10.

1. **Introduction**

Let R be a commutative ring with a non-zero identity and $J(R)$ be the Jacobson radical of R. The *Jacobson graph* of R, denoted by $\mathfrak{J}_R$, is a graph with $R \setminus J(R)$ as its vertex set and two distinct vertices x and y are adjacent if $1 - xy \notin U(R)$, the set of units of R.

The Jacobson graphs first introduced by Azimi, Erfanian and Farrokhi in [2] where they obtained many graph theoretical properties of these graphs including connectivity, planarity and perfectness (see [1, 3, 4] for further results on Jacobson graphs).

The aim of this paper is to extend the notion of Jacobson graphs from ring elements to n-array vectors with entries as elements of the underlying ring. Our graphs can be considered as a variation of many other known graphs defined on vector spaces, say symplectic graphs, unitary graphs, orthogonal graphs etc (see for instance [8, 10, 11]).

Let R be a commutative ring with a non-zero identity and n be a natural number. Also, let $M_{n\times 1}(R) = \{[r_1 \ \ldots \ r_n]^T : r_1, \ldots, r_n \in R\}$ and $J^n(R) = \{[r_1 \ \ldots \ r_n]^T \in M_{n\times 1}(R) : r_1, \ldots, r_n \in J(R)\}$. Then the *$n$-array Jacobson graph* of R, denoted by $\mathfrak{J}^n_R$, is a graph whose vertex set is $M_{n\times 1}(R) \setminus J^n(R)$

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and two distinct vertices X and Y are adjacent if $1 - X^T \cdot Y \notin U(R)$. Clearly, $\mathfrak{J}_R^1$ is the Jacobson graph of R.

Let $f : R \times R \longrightarrow S$ be a bilinear form of a ring R (vector space V) over a ring S (field F) and $\Lambda \subseteq S$ ($\Lambda \subseteq F$). Then we may define a graph $\Gamma_{f,\Lambda}(R, S)$ whose vertices are elements of R (vectors in V) and two distinct elements (vectors) u and v are adjacent whenever $f(u, v) \in \Lambda$. Now, if $f : M_{n\times 1}(R) \times M_{n\times 1}(R) \longrightarrow R$ is the natural inner product and $\Lambda = R \setminus (1-U(R))$, then $\Gamma_{f,\Lambda}(M_{n\times 1}(R), R)$ is the mentioned n-array generalization of Jacobson graph $\mathfrak{J}_R^n$ associated to R. In particular, if F is a field and V is a vector space of dimension n over F, then $\mathfrak{J}_F^n$ is the same as the graph $\Gamma_{\langle \cdot,\cdot \rangle,\{1\}}(V, F)$ where two distinct vertices are adjacent if their inner products equals 1.

In this paper, we shall study some graph theoretical properties of an n-array Jacobson graph for a natural number n. In Section 2, we discuss the connectivity of this graph and show that an n-array Jacobson graph is connected except when $n = 1$ and the underlying ring is local. In Section 3, we study forbidden structures in n-array Jacobson graphs and determine all planar, outer planar, projective, perfect and domination perfect n-array Jacobson graphs. Throughout this paper, all rings are assumed to be finite commutative rings with a non-zero identity. It is known that such a ring R has a decomposition $R = R_1 \oplus \cdots \oplus R_m$ into local rings R_i, for $i = 1, \ldots, m$ (see [9, Theorem VI.2]). In what follows, $\mathbf{e}_i$ denotes the element $(0, \ldots, 0, 1, 0, \ldots, 0)$ of R with 1 on the ith entry and 0 elsewhere. Also, $\mathbf{1}$ and $\mathbf{0}$ stand for the identity element and the zero element of R, respectively. For $1 \leq i \leq n$ and $1 \leq j \leq m$, the elements $[\mathbf{0} \cdots \mathbf{0}\ \mathbf{1}\ \mathbf{0} \cdots \mathbf{0}]^T$ and $[\mathbf{0} \cdots \mathbf{0}\ \mathbf{e}_j\ \mathbf{0} \cdots \mathbf{0}]^T$ are denoted by $\mathbf{E}_i$ and $\mathbf{E}_{ij}$, respectively, where the non-zero entry lies on the ith row. For convenience, the finite field of order q is denoted by $\mathbb{F}_q$. The union of n disjoint copies of a graph Γ is denoted by $n\Gamma$. The *dot product* of two vertex transitive graphs Γ_1 and Γ_2, denoted by $\Gamma_1 \cdot \Gamma_2$, is the graph obtained from the union of disjoint copies of Γ_1 and Γ_2 by identification of a vertex of Γ_1 with a vertex of Γ_2.

2. Connectedness

In this section, we discuss the connectivity and compute the diameter of n-array Jacobson graphs. Recall that the results are known for Jacobson graphs as in the following theorem.

Theorem 2.1. *Let R be a finite non-local ring. Then $\mathfrak{J}_R^1$ is a connected graph and* $\mathrm{diam}(\mathfrak{J}_R^1) \leq 3$.

Proof. See [2, Theorem 4.1]. □

Now, we consider n-array Jacobson graphs when $n \geq 2$.

Theorem 2.2. *Let R be a finite ring and $n \geq 2$. Then $\mathfrak{J}_R^n$ is connected. Moreover,*

(1) $\operatorname{diam}(\mathfrak{J}_R^n) \leq 4$ *if R is local, and*
(2) $\operatorname{diam}(\mathfrak{J}_R^n) \leq 3$ *if R is not local.*

Proof. (1) Let $\mathfrak{m}$ be the maximal ideal of R and assume that $X = [x_1 \ \ldots \ x_n]^T$ and $Y = [y_1 \ \ldots \ y_n]^T$ are distinct non-adjacent vertices of $\mathfrak{J}_R^n$. If $x_i, y_j \notin \mathfrak{m}$ for some distinct $1 \leq i, j \leq n$, then

$$X \sim x_i^{-1}\mathbf{E}_i \sim x_i\mathbf{E}_i + y_j\mathbf{E}_j \sim y_j^{-1}\mathbf{E}_j \sim Y$$

and we are done. Hence we assume that $x_i, y_i \notin \mathfrak{m}$ for some $1 \leq i \leq n$, and $x_j, y_j \in \mathfrak{m}$ for all $j \neq i$. So

$$X \sim x_i^{-1}\mathbf{E}_i + \mathbf{E}_j \sim \mathbf{E}_j \sim y_i^{-1}\mathbf{E}_i + \mathbf{E}_j \sim Y.$$

Hence $d(X, Y) \leq 4$ so that $\operatorname{diam}(\mathfrak{J}_R^n) \leq 4$. In particular, $\mathfrak{J}_R^n$ is connected.

(2) Let $R = R_1 \oplus \cdots \oplus R_m$ $(m \geq 2)$ be a decomposition of R into local rings $(R_i, \mathfrak{m}_i)$, for $i = 1, \ldots, m$. Let $X = [x_1 \ \ldots \ x_n]^T$ and $Y = [y_1 \ \ldots \ y_n]^T$ be distinct non-adjacent vertices of $\mathfrak{J}_R^n$, where $x_i = (x_i^1, \ldots, x_i^m) \notin J(R)$ and $y_j = (y_j^1, \ldots, y_j^m) \notin J(R)$ for some $1 \leq i, j \leq n$. Now, choose s and t as the least indices such that $x_i^s \in U(R_s)$ and $y_j^t \in U(R_t)$. If $i \neq j$ then

$$X \sim (x_i^s)^{-1}\mathbf{E}_{is} + y_j^t\mathbf{E}_{jt} \sim x_i^s\mathbf{E}_{is} + (y_j^t)^{-1}\mathbf{E}_{jt} \sim Y$$

is a path between X and Y. Hence assume that $i = j$. We consider two cases:

Case 1. $s \neq t$. Without loss of generality we assume that $t < s$. Then $X \sim (y_i^t)^{-1}\mathbf{E}_{is} + (x_i^s)^{-1}\mathbf{E}_{is} \sim Y$ is a path connecting vertices X and Y.

Case 2. $s = t$. If $x_i^s = y_i^s$ then $X \sim (x_i^s)^{-1}\mathbf{E}_{is} \sim Y$ is a path connecting vertices X and Y. Also, in the case $x_i^s \neq y_i^s$,

$$\begin{aligned} X &\sim \mathbf{E}_{i1} + \mathbf{E}_{i2} + \cdots + (x_i^s)^{-1}\mathbf{E}_{is} + \cdots + \mathbf{E}_{im} \\ &\sim \mathbf{E}_{i1} + \mathbf{E}_{i2} + \cdots + (y_i^s)^{-1}\mathbf{E}_{is} + \cdots + \mathbf{E}_{im} \\ &\sim Y \end{aligned}$$

is a path between X and Y. Therefore, $\operatorname{diam}(\mathfrak{J}_R^n) \leq 3$ and subsequently $\mathfrak{J}_R^n$ is connected. □

Theorem 2.3. *Let R be a finite ring. Then* $\operatorname{diam}(\mathfrak{J}_R^n) = 2$ *if and only if $R = R_1 \oplus \cdots \oplus R_m$ $(m \geq 2)$ such that $(R_i, \mathfrak{m}_i)$ are local rings with associated fields of order* 2.

Proof. Suppose on the contrary that $R/J(R) \ncong \mathbb{Z}_2 \oplus \cdots \oplus \mathbb{Z}_2$. Hence there exists an element $u \in U(R_i) \setminus \{1\}$ such that $u \notin 1 + \mathfrak{m}_i$ for some $1 \leq i \leq n$. Then $N_{\mathfrak{J}_R^n}(u\mathbf{E}_i) \cap N_{\mathfrak{J}_R^n}(u^{-1}\mathbf{E}_i) = \emptyset$, which contradicts the assumption.

Conversely, let $X = [x_1 \ \ldots \ x_n]^T$ and $Y = [y_1 \ \ldots \ y_n]^T$ be distinct non-adjacent vertices of $\mathfrak{J}_R^n$, where $x_i = (x_i^1, \ldots, x_i^m) \notin J(R)$ and $y_j = (y_1^j, \ldots, y_m^j) \notin J(R)$ for some $1 \leq i, j \leq n$. Now if s and t are the least indices such that $x_i^s \in U(R_s)$ and $y_j^t \in U(R_t)$, then there exists $m_s \in \mathfrak{m}_s$ and

$m_t \in \mathfrak{m}_t$ such that $x_i^s = 1 + m_s$ and $y_j^t = 1 + m_t$. So $X \sim \mathbf{E}_{is} + \mathbf{E}_{jt} \sim Y$ is a path between X and Y. The proof is completed. □

3. Forbidden structures

In this section, we shall study an n-array Jacobson graph, which lacks special subgraphs. This enables us to determine an n-array Jacobson graph that is planar, outer planar, projective, perfect or domination perfect. The following lemma will be used frequently in the sequel.

Lemma 3.1. *The only finite local rings $(R, \mathfrak{m})$ with $|\mathfrak{m}| = p$ are $\mathbb{Z}_{p^2}$ and $\mathbb{Z}_p[x]/(x^2)$.*

Proof. Let $\mathfrak{m} = \{ix : i = 0, \ldots, p-1\}$ and $\alpha + \mathfrak{m}$ be a generator of the multiplicative group of $R/\mathfrak{m}$. Since $\alpha x \in \mathfrak{m}$ and α is a unit, we have that $\alpha x = ix$ for some $1 \leq i \leq p-1$, hence $(\alpha - i)x = 0$. Then $\alpha - i$ is a non-unit element of R, which implies that $\alpha - i \in \mathfrak{m}$. Thus $\alpha - i = jx$ for some $1 \leq j \leq p-1$, from which it follows that $R = \langle 1, x\rangle$. Since R is finite, $J(R) = \mathfrak{m}$ is nilpotent, which implies that $x^2 = 0$. Therefore $R \cong \mathbb{Z}_{p^2}$ or $\mathbb{Z}_p[x]/(x^2)$, as required. □

Remind that a graph is *planar* if it can be drawn in the plane in such a way that two edges intersect only on the endpoints. A well-known theorem of Kuratowski states that a graph is planar if and only if it does not have any subdivision of K_5 or $K_{3,3}$ as a subgraph. The following lemma, as a corollary to Euler's formula, gives a simple criterion for planarity of graphs. Recall that $\delta(\Gamma)$ is the minimum valency of a graph Γ.

Lemma 3.2. *If Γ is a planar graph, then $\delta(\Gamma) \leq 5$.*

Planar Jacobson graphs are completely described in [2] as follows.

Theorem 3.3 ([2, Theorem 4.3])**.** *Let R be a finite ring. Then $\mathfrak{J}_R$ is planar if and only if either R is a field, or R is isomorphic to one of the following rings:*

(i) $\mathbb{Z}_4$, $\mathbb{Z}_2 \oplus \mathbb{Z}_2$, $\mathbb{Z}_2[x]/(x^2)$ *of order* 4,
(ii) $\mathbb{Z}_6$ *of order* 6,
(iii) $\mathbb{Z}_8$, $\mathbb{Z}_2 \oplus \mathbb{Z}_4$, $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$, $\mathbb{Z}_2[x]/(x^3)$, $\mathbb{Z}_4[x]/(2x, x^2)$, $\mathbb{Z}_2 \oplus \mathbb{Z}_2[x]/(x^2)$, $\mathbb{Z}_2 \oplus \mathbb{Z}_2[x]/(x^2+x+1)$, $\mathbb{Z}_4[x]/(2x, x^2-2)$, $\mathbb{Z}_2[x,y]/(x,y)^2$ *of order* 8, *and*
(iv) $\mathbb{Z}_9$, $\mathbb{Z}_3 \oplus \mathbb{Z}_3$, $\mathbb{Z}_3[x]/(x^2)$ *of order* 9.

Theorem 3.4. *Let R be a finite ring and $n \geq 2$. Then $\mathfrak{J}_R^n$ is planer if and only if*

(1) $n = 2$ *and either* $R \cong \mathbb{Z}_2$ *or* $R \cong \mathbb{Z}_3$, *or*
(2) $n = 3$ *and* $R \cong \mathbb{Z}_2$.

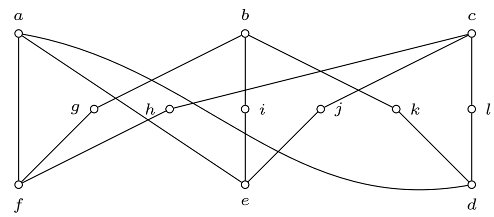


FIGURE 1

Proof. Suppose that $\mathfrak{J}_R^n$ is planar. First assume that R is not a local ring. Let $R = R_1 \oplus \cdots \oplus R_m$ be a decomposition of R into local rings R_i, for $i = 1, \ldots, m$. Then the subgraph induced by

$$\mathbf{E}_{11}, \mathbf{E}_{11} + \mathbf{E}_{21}, \mathbf{E}_{11} + \mathbf{E}_{21} + \mathbf{E}_{22}, \mathbf{E}_{11} + \mathbf{E}_{12}, \mathbf{E}_{11} + \mathbf{E}_{22}$$

is isomorphic to K_5, which is a contradiction. Hence R is local with maximal ideal $\mathfrak{m}$. It is easy to see that $\delta(\mathfrak{J}_R^n) > 5$ when $\mathfrak{m} \neq 0$. Hence, by invoking Lemma 3.2, it follows that $\mathfrak{m} = 0$ so that R is a field. If $n \geq 3$, then the same argument shows that $\delta(\mathfrak{J}_R^n) > 5$ unless $n = 3$ and $R \cong \mathbb{Z}_2$. Finally assume that $n = 2$. If $|R| \geq 4$, then $\mathfrak{J}_R^n$ has a subdivision of $K_{3,3}$ as drawn in Figure 1, in which $a = \mathbf{E}_1$, $b = u\mathbf{E}_1$, $c = v\mathbf{E}_1$, $d = \mathbf{E}_1 + v\mathbf{E}_2$, $e = \mathbf{E}_1 + u\mathbf{E}_2$, $f = \mathbf{E}_1 + \mathbf{E}_2$, $g = u^{-1}\mathbf{E}_1 + (1 - u^{-1})\mathbf{E}_2$, $h = v^{-1}\mathbf{E}_1 + (1 - v^{-1})\mathbf{E}_2$, $i = u^{-1}\mathbf{E}_1 + u^{-1}(1 - u^{-1})\mathbf{E}_2$, $j = v^{-1}\mathbf{E}_1 + u^{-1}(1 - v^{-1})\mathbf{E}_2$, $k = u^{-1}\mathbf{E}_1 + v^{-1}(1 - u^{-1})\mathbf{E}_2$, $l = v^{-1}\mathbf{E}_1 + v^{-1}(1 - v^{-1})\mathbf{E}_2$ and that $u, v \in R \setminus \{0, 1\}$ with $v \neq u$. Therefore, $R \cong \mathbb{Z}_2$ or $\mathbb{Z}_3$. The converse is straightforward by Figures 2, 3 and 4. $\square$

FIGURE 2. $\mathfrak{J}_{\mathbb{Z}_2}^2$

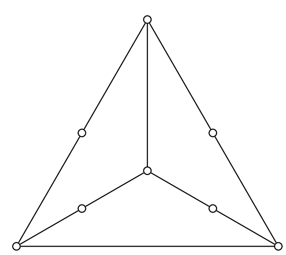


FIGURE 3. $\mathfrak{J}_{\mathbb{Z}_3}^2$

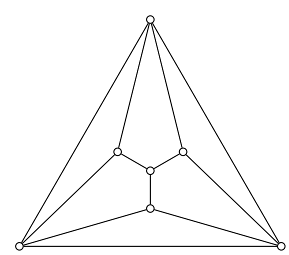


FIGURE 4. $\mathfrak{J}_{\mathbb{Z}_2}^3$

Utilizing the above classifications of planar n-array Jacobson graphs, it is now easy to describe all outer planar n-array Jacobson graphs. Recall that a graph is *outer planar* if it has a planar embedding such that all vertices belong to the outer region.

Corollary 3.5. *Let R be a finite ring. Then $\mathfrak{J}_R^n$ is outer planer if and only if*

(1) *$n = 1$ and R is a field or $R \cong \mathbb{Z}_4$, $\mathbb{Z}_2 \oplus \mathbb{Z}_2$, $\mathbb{Z}_2[x]/(x^2)$, $\mathbb{Z}_6$, $\mathbb{Z}_9$, $\mathbb{Z}_3 \oplus \mathbb{Z}_3$ or $\mathbb{Z}_3[x]/(x^2)$, or*
(2) *$n = 2$ and $R \cong \mathbb{Z}_2$.*

Proof. If $\mathfrak{J}_R^n$ is outer planar, then it is planar and must be one of the rings in Theorems 3.3 or 3.4. Now, a simple verification shows that all rings except those written in the corollary have a subdivision of non-outer planar graphs K_4 or $K_{2,3}$, as required. □

Studying embeddings of n-array Jacobson graphs on surfaces of higher genus is very difficult in general. For this reason, in this paper, we just consider the embedding of n-array Jacobson graphs on the non-orientable surface of genus 1 known as the projective plane. A non-planar graph is said to be *projective* if it can be drawn in the projective plane in such a way that two edges are crossing only at the end vertices. Examples of non-projective graphs are K_7, $2K_5$, $K_{4,4}$, $2K_{3,3}$ and $K_{3,3} \cdot K_{3,3}$ possessing the graphs A_2, A_5, E_{18}, E_{42} and E_1 of [7, pp. 365–369] as subgraph, respectively.

Theorem 3.6. *The graph $\mathfrak{J}_R^n$ is projective if and only if $n = 1$ and $R \cong \mathbb{Z}_{10}$ or $\mathbb{Z}_3 \oplus \mathbb{F}_4$.*

Proof. First suppose that $\mathfrak{J}_R^n$ is a projective graph and $R = R_1 \oplus \cdots \oplus R_m$ is a decomposition of R into local rings $R_1, \ldots, R_m$. If $m, n \geq 2$ then, by Figure 5, $\mathfrak{J}_R^n$ has a subgraph isomorphic to $K_{3,3} \cdot K_{3,3}$, the dot product of two copies of $K_{3,3}$, which is a contradiction. Now, we proceed in two cases:

Case 1. $m = 1$ and $n \geq 2$. Then R is a local ring with a maximal ideal $\mathfrak{m}$. If $|\mathfrak{m}| \geq 3$ and $m, m' \in \mathfrak{m} \setminus \{0\}$, then, by Figure 6, $\mathfrak{J}_R^n$ has a subgraph isomorphic to $K_{3,3} \cdot K_{3,3}$, a contradiction. Now let $|\mathfrak{m}| = 2$ and $m \in \mathfrak{m} \setminus \{0\}$. Then, by Lemma 3.1, $R \cong \mathbb{Z}_4$ or $\mathbb{Z}_2[x]/(x^2)$, hence $\mathfrak{J}_R^n$ has a bipartite subgraph with bipartition

$$\{\mathbf{E}_1 + \mathbf{E}_2, \mathbf{E}_1 + x\mathbf{E}_2, x\mathbf{E}_1 + \mathbf{E}_2, x\mathbf{E}_1 + x\mathbf{E}_2\} \text{ and } \{\mathbf{E}_1, \mathbf{E}_2, x\mathbf{E}_1, x\mathbf{E}_2\},$$

where $x \in R \setminus (J(R) \cup \{1\})$ and this is a contradiction.

Now suppose that $\mathfrak{m} = 0$, hence R is a finite field. If $n \geq 4$ then $\mathfrak{J}_R^n$ has a subgraph isomorphic to the non-projective graph G (see [7, p. 370]) as drawn in Figure 7, where $a, b, c, d, e, f, g, h, i, j, k, l$ denote $\mathbf{E}_1+\mathbf{E}_4$, $\mathbf{E}_1$, $\mathbf{E}_1+\mathbf{E}_2+\mathbf{E}_4$, $\mathbf{E}_2$, $\mathbf{E}_1+\mathbf{E}_2$, $\mathbf{E}_2+\mathbf{E}_3$, $\mathbf{E}_1+\mathbf{E}_3$, $\mathbf{E}_3+\mathbf{E}_4$, $\mathbf{E}_2+\mathbf{E}_3+\mathbf{E}_4$, $\mathbf{E}_3$, $\mathbf{E}_2+\mathbf{E}_4$, $\mathbf{E}_1+\mathbf{E}_2+\mathbf{E}_3+\mathbf{E}_4$, respectively, a contradiction. Hence $n \leq 3$.

Suppose $n = 3$. If R has an element α different from $0, \pm 1$, then, by Figure 8, $\mathfrak{J}_R^n$ is not projective, a contradiction. Also, if $R \cong \mathbb{Z}_3$ then, by Figure 9, $\mathfrak{J}_R^n$ is not projective from which it follows that $R \cong \mathbb{Z}_2$. This implies that $\mathfrak{J}_R^n$ is planar, which is a contradiction. Finally, assume $n = 2$. First, observe that by Figures 10 and 11, the graph $\mathfrak{J}_R^n$ is not projective when $|R| = 4$ and 5, respectively. Note that $a, b, c, d, e, f, g, h, i, j, k, l, m, n$ denote the vertices

$$\begin{bmatrix}1\\\theta^2\end{bmatrix}, \begin{bmatrix}\theta^2\\\theta^2\end{bmatrix}, \begin{bmatrix}\theta^2\\1\end{bmatrix}, \begin{bmatrix}1\\\theta\end{bmatrix}, \begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}\theta^2\\\theta\end{bmatrix}, \begin{bmatrix}0\\\theta^2\end{bmatrix}, \begin{bmatrix}\theta\\\theta\end{bmatrix}, \begin{bmatrix}\theta\\1\end{bmatrix}, \begin{bmatrix}\theta^2\\0\end{bmatrix}, \begin{bmatrix}\theta\\\theta^2\end{bmatrix}, \begin{bmatrix}0\\\theta\end{bmatrix}, \begin{bmatrix}\theta\\0\end{bmatrix}$$

in Figure 10 where θ is the multiplicative generator of R and denote the vertices

$$\begin{bmatrix}4\\2\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}4\\1\end{bmatrix}, \begin{bmatrix}4\\0\end{bmatrix}, \begin{bmatrix}4\\3\end{bmatrix}, \begin{bmatrix}2\\1\end{bmatrix}, \begin{bmatrix}3\\0\end{bmatrix}, \begin{bmatrix}2\\2\end{bmatrix}, \begin{bmatrix}0\\3\end{bmatrix}, \begin{bmatrix}0\\2\end{bmatrix}, \begin{bmatrix}2\\3\end{bmatrix}, \begin{bmatrix}2\\4\end{bmatrix}, \begin{bmatrix}1\\2\end{bmatrix}$$

in Figure 11. Hence we must have $|R| \geq 7$. Now, by [6, Proposition 3], we have $|E(\mathfrak{J}_R^n)| \leq 3|V(\mathfrak{J}_R^n)| - 6$, which is impossible for

$$\deg_{\mathfrak{J}_R^n}\left(\begin{bmatrix}a\\b\end{bmatrix}\right) = \begin{cases}|R|, & (a,b) \neq (\pm1, 0) \text{ and } (0, \pm1),\\ |R|-1, & (a,b) = (\pm1,0) \text{ or } (0,\pm1).\end{cases}$$

Case 2. $n = 1$. If R is a finite local ring, then $|R| = |\mathfrak{m}||F|$ is a prime power in which $\mathfrak{m}$ and F are the maximal ideal and the associated field of R, respectively. By [2, Theorem 2.2], $\mathfrak{J}_R^1 \cong (1+\varepsilon_F)K_{|\mathfrak{m}|} \cup (|F|-2-\varepsilon_F)/2K_{|\mathfrak{m}|,|\mathfrak{m}|}$, where ε_F is the parity of $|F|$. Since $\mathfrak{J}_R^1$ has no subgraphs isomorphic to $K_{4,4}$, $2K_{3,3}$ and $2K_5$, it follows that $|\mathfrak{m}| \leq 4$ and $|F| \leq 3$ so that $\mathfrak{J}_R^1$ is planar, a contradiction. Hence R is a finite non-local ring. Let $R = R_1 \oplus \cdots \oplus R_m$ ($m \geq 2$) be a decomposition of R into local rings $(R_i, \mathfrak{m}_i)$ with associated fields F_i, for $i = 1, \ldots, m$. If $\mathfrak{m}$ is a maximal ideal of R, then $|\mathfrak{m}| \leq 6$ for otherwise $1 + \mathfrak{m}$ induces a complete subgraph with $|\mathfrak{m}| \geq 7$ vertices, a contradiction. Hence

$$\frac{|R|}{|F_i|} = |R_1 \oplus \cdots \oplus R_{i-1} \oplus \mathfrak{m}_i \oplus R_{i+1} \oplus \cdots \oplus R_m| \leq 6.$$

Moreover, $|F_i| \leq 5$ since $|F_i|$ is a prime power and $|F_i| \leq |R|/|F_j| \leq 6$ for any $j \neq i$. On the other hand, none of the graphs of $\mathbb{Z}_5 \oplus \mathbb{Z}_5$, $\mathbb{Z}_5 \oplus \mathbb{F}_4$, $\mathbb{F}_4 \oplus \mathbb{F}_4$ and $\mathbb{Z}_5 \oplus \mathbb{Z}_3$ is projective for they have subgraphs isomorphic to $2K_5$, $2K_5$, $K_{3,3} \cdot K_{3,3}$ and $2K_5$, respectively. Hence, by using [2, Theorem 4.3], R is isomorphic to one of the rings $\mathbb{Z}_5 \oplus \mathbb{Z}_2$, $\mathbb{Z}_4 \oplus \mathbb{Z}_3$, $\mathbb{Z}_2[x]/(x^2) \oplus \mathbb{Z}_3$, $\mathbb{F}_4 \oplus \mathbb{Z}_3$, $\mathbb{Z}_4 \oplus \mathbb{Z}_4$, $\mathbb{Z}_4 \oplus \mathbb{Z}_2[x]/(x^2)$, $\mathbb{Z}_4 \oplus \mathbb{F}_4$, $\mathbb{Z}_2[x]/(x^2) \oplus \mathbb{Z}_2[x]/(x^2)$, $\mathbb{Z}_2[x]/(x^2) \oplus \mathbb{F}_4$ and $\mathbb{Z}_3 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$. By Figure 12, $\mathfrak{J}^1_{\mathbb{Z}_3 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2}$ is not projective, where a, b, c, d, e, f, g denote $\mathbf{e}_2$, $\mathbf{e}_1 + \mathbf{e}_2$, $-\mathbf{e}_1 + \mathbf{e}_2$, $-\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3$, $\mathbf{e}_2 + \mathbf{e}_3$, $\mathbf{e}_1 + \mathbf{e}_3$, $-\mathbf{e}_1 + \mathbf{e}_3$, $\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3$, respectively.

On the other hand, if $A = \mathbb{Z}_4$ or $\mathbb{Z}_2[x]/(x^2)$, $B = \mathbb{Z}_4$ or $\mathbb{Z}_2[x]/(x^2)$ or $\mathbb{F}_4$, and $A \setminus J(A) = \{a, b\}$, then $\mathfrak{J}_{A \oplus B}$ has a bipartite subgraph isomorphic to $K_{4,4}$ whose parts are $a \oplus B$ and $b \oplus B$. Therefore, $R \cong \mathbb{Z}_5 \oplus \mathbb{Z}_2 \cong \mathbb{Z}_{10}$ and $\mathbb{F}_4 \oplus \mathbb{Z}_3$, as required.

Conversely, from Figures 13, 14 and 15, the graphs $\mathfrak{J}^1_{\mathbb{Z}_2[x]/(x^2) \oplus \mathbb{Z}_3} \cong \mathfrak{J}^1_{\mathbb{Z}_3 \oplus \mathbb{Z}_4}$, $\mathfrak{J}^1_{\mathbb{Z}_5 \oplus \mathbb{Z}_2}$ and $\mathfrak{J}^1_{\mathbb{F}_4 \oplus \mathbb{Z}_3}$ are projective, where $a, b, c, d, e, f, g, h, i, j$ denote $\mathbf{e}_2$, $-\mathbf{e}_2$, $\mathbf{e}_1 + \mathbf{e}_2$, $\mathbf{e}_1 - \mathbf{e}_2$, $-\mathbf{e}_1 - \mathbf{e}_2$, $-\mathbf{e}_1 + \mathbf{e}_2$, $\mathbf{e}_1$, $\mathbf{e}_1 + 2\mathbf{e}_2$, $-\mathbf{e}_1$, $-\mathbf{e}_1 + 2\mathbf{e}_2$ in Figure 13 with $R = \mathbb{Z}_3 \oplus \mathbb{Z}_4$, $a, b, c, d, e, f, g, h, i$ denote $\mathbf{e}_2$, $\mathbf{e}_1 + \mathbf{e}_2$, $2\mathbf{e}_1 + \mathbf{e}_2$, $-2\mathbf{e}_1 + \mathbf{e}_2$, $-\mathbf{e}_1 + \mathbf{e}_2$, $\mathbf{e}_1$, $-2\mathbf{e}_1$, $2\mathbf{e}_1$, $-\mathbf{e}_1$ in Figure 14, and $a, b, c, d, e, f, g, h, i, j, k$ denote

$\mathbf{e}_2,\ \alpha\mathbf{e}_1+\mathbf{e}_2,\ \alpha^{-1}\mathbf{e}_1,\ \alpha\mathbf{e}_1-\mathbf{e}_2,\ \mathbf{e}_1-\mathbf{e}_2,\ -\mathbf{e}_2,\ \alpha^{-1}\mathbf{e}_1-\mathbf{e}_2,\ \alpha\mathbf{e}_1,\ \alpha^{-1}\mathbf{e}_1+\mathbf{e}_2,$ $\mathbf{e}_1+\mathbf{e}_2,\ \mathbf{e}_1$ in Figure 15 with $\alpha\in\mathbb{F}_4\setminus\{0,1\}$. The proof is complete. □

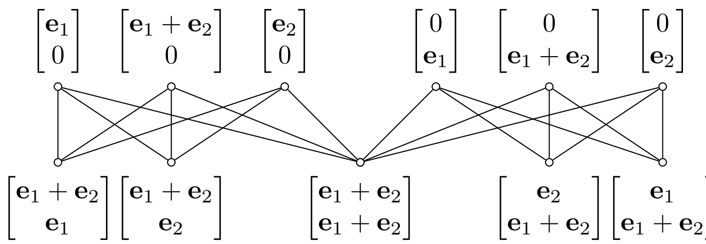


FIGURE 5

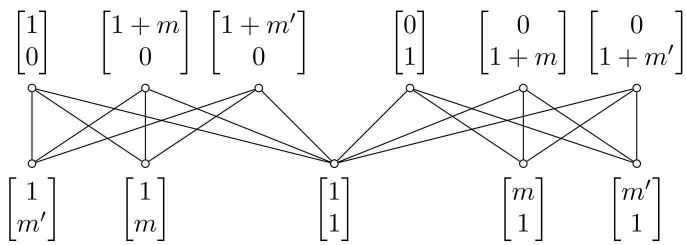


FIGURE 6

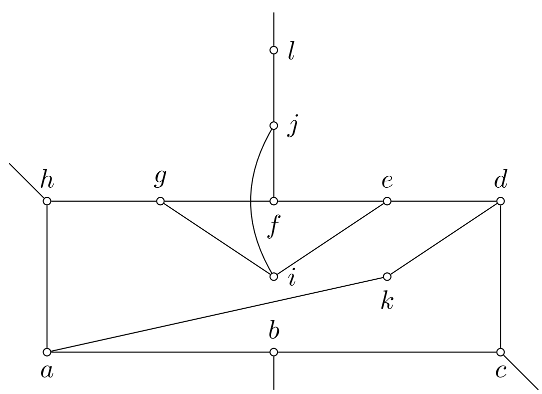


FIGURE 7

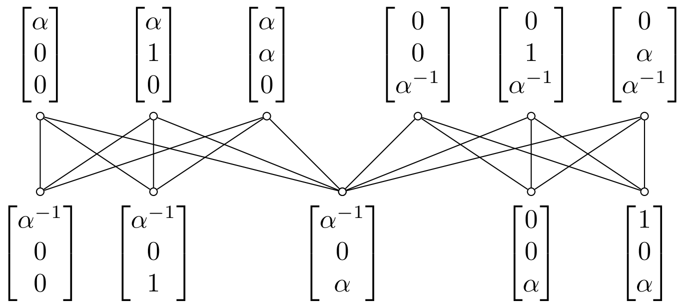


FIGURE 8

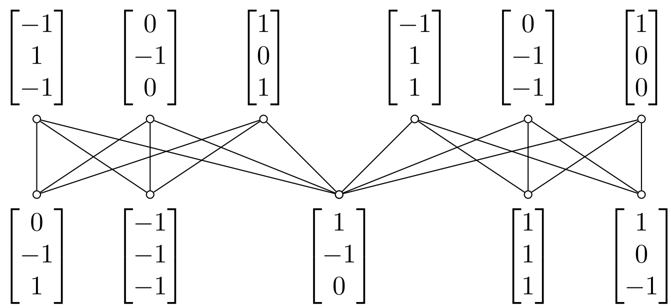


FIGURE 9

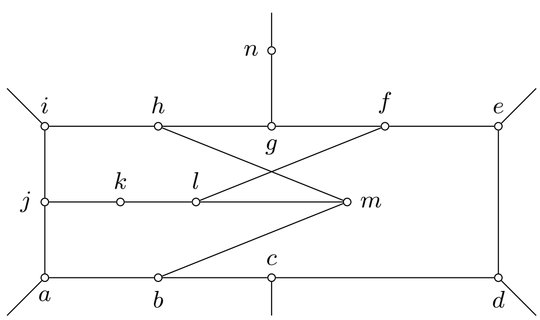


FIGURE 10

In what follows we consider two problems of different nature which can be state in terms of forbidden subgraphs. A *proper coloring* of a graph is an assignment of some colors to its vertices in such a way that adjacent vertices have distinct colors. The minimum number of colors required to color a graph properly is called the *chromatic number* of the graph. A graph is *perfect* if the

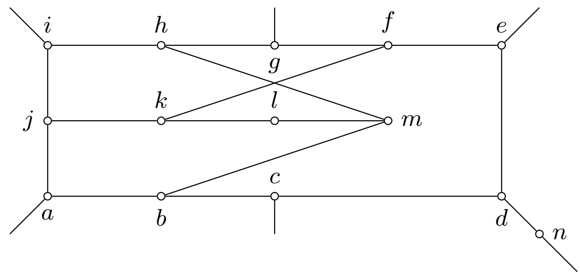


FIGURE 11

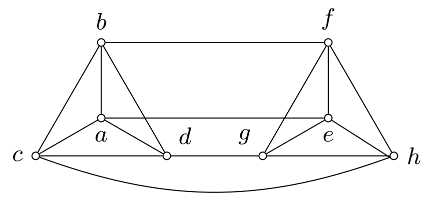


FIGURE 12

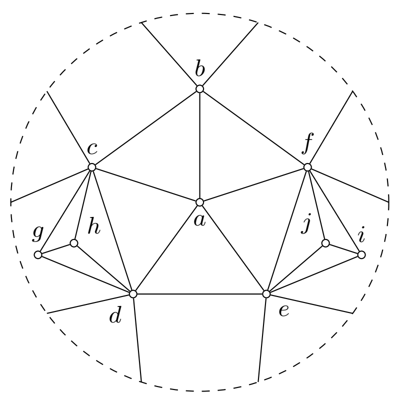


FIGURE 13. $\mathfrak{J}^1_{\mathbb{Z}_2[x]/(x^2)\oplus\mathbb{Z}_3} \cong \mathfrak{J}^1_{\mathbb{Z}_3\oplus\mathbb{Z}_4}$

chromatic and clique number of its induced subgraphs are the same. The following theorem of Chudnovsky, Robertson, Seymour and Thomas characterizes all perfect graphs.

Theorem 3.7 (Strong perfect graph theorem [5])**.** *A graph* Γ *is perfect if and only if neither* Γ *nor* $\overline{\Gamma}$ *contains an induced odd cycle of length* ≥ 5.

The perfect Jacobson graphs are already classified as follows.

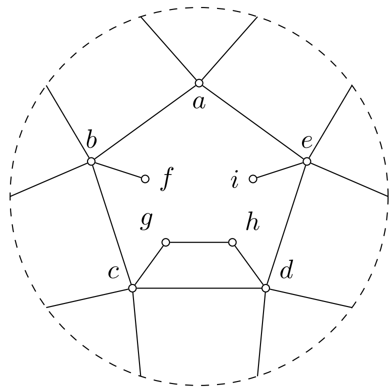


FIGURE 14. $\mathfrak{J}^1_{\mathbb{Z}_5\oplus\mathbb{Z}_2}$

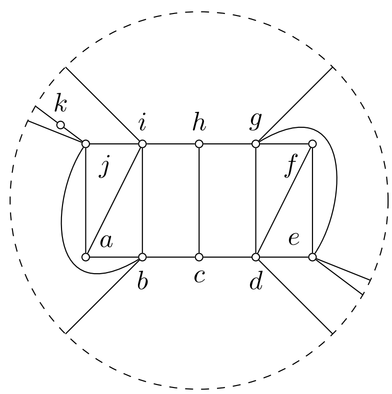


FIGURE 15. $\mathfrak{J}^1_{\mathbb{F}_4\oplus\mathbb{Z}_3}$

Theorem 3.8 ([2, Theorem 4.6]). *Let R be a finite ring. Then $\mathfrak{J}_R$ is perfect if and only if*

(1) *R is a local ring,*
(2) *$R/J(R) \cong \mathbb{Z}_3 \oplus \mathbb{Z}_3$ or $\mathbb{Z}_2 \oplus F$ or $\mathbb{Z}_3 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$ or $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$ or $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$,*

where F is a finite field.

In Theorem 3.10, we complete the classification of perfect n-array Jacobson graphs. Our proof uses the following lemma, which can be proved in the same way as in [2, Lemma 4.5].

Lemma 3.9. *Let R be a finite ring. Then $\mathfrak{J}^n_R$ is perfect if and only if $\mathfrak{J}^n_{R/J(R)}$ is perfect.*

Theorem 3.10. *Let R be a finite ring and $n \geq 2$. Then $\mathfrak{J}_R^n$ is perfect if and only if*

(1) *$n \leq 4$ and R is a local ring with associated field of order 2, or*
(2) *$n = 2$ and $R/J(R) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$.*

Proof. Let R be a finite ring with perfect n-array Jacobson graph. By Lemma 3.9, we may assume that $J(R) = 0$. First suppose that R is not local and that $R = F_1 \oplus \cdots \oplus F_m$ $(m \geq 2)$ is a decomposition of R into fields F_i. The five vertices

$$\mathbf{E}_{11} + \mathbf{E}_{13}, \mathbf{E}_{11} + \mathbf{E}_{12}, \mathbf{E}_{12} + \mathbf{E}_{21} + \mathbf{E}_{23}, \mathbf{E}_{21} + \mathbf{E}_{22}, \mathbf{E}_{13} + \mathbf{E}_{22}$$

induce a five-cycle in $\mathfrak{J}_R^n$ when $m \geq 3$. Hence $m = 2$. If $0, 1 \neq \alpha \in U(F_i)$ and $j \neq i$, then the five vertices

$$\mathbf{E}_{ii} + \mathbf{E}_{ij} + \mathbf{E}_{ji}, \alpha\mathbf{E}_{ii} + \mathbf{E}_{ij}, \alpha^{-1}\mathbf{E}_{ii}, \alpha\mathbf{E}_{ii} + \mathbf{E}_{jj}, \mathbf{E}_{ji} + \mathbf{E}_{jj}$$

induce a five-cycle in $\mathfrak{J}_R^n$, which is impossible. Hence $R \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$. Moreover, $n = 2$ for otherwise the five vertices

$$\mathbf{E}_{12} + \mathbf{E}_{31}, \mathbf{E}_{11} + \mathbf{E}_{21} + \mathbf{E}_{31}, \mathbf{E}_{11}, \mathbf{E}_{11} + \mathbf{E}_{32}, \mathbf{E}_{12} + \mathbf{E}_{22} + \mathbf{E}_{32}$$

induce a five-cycle in $\mathfrak{J}_R^n$, which is a contradiction.

Now suppose that R is a local ring. Then R is a field. If $|R| \geq 4$, then the five vertices

$$u\mathbf{E}_1, u^{-1}\mathbf{E}_1 + u\mathbf{E}_2, u^{-1}\mathbf{E}_2, u\mathbf{E}_2, u^{-1}\mathbf{E}_1 + u^{-1}\mathbf{E}_2$$

induce a five-cycle in $\mathfrak{J}_R^n$ whenever $u \in R \setminus \{0, \pm 1\}$, hence we must have $|R| \leq 3$. Also, if $|R| = 3$ then the five vertices

$$\mathbf{E}_1, \mathbf{E}_1 - \mathbf{E}_2, -\mathbf{E}_1 + \mathbf{E}_2, \mathbf{E}_2, \mathbf{E}_1 + \mathbf{E}_2$$

induce a five-cycle in $\mathfrak{J}_R^n$, which is a contradiction. Therefore $R \cong \mathbb{Z}_2$. On the other hand, if $n \geq 5$ then the five vertices

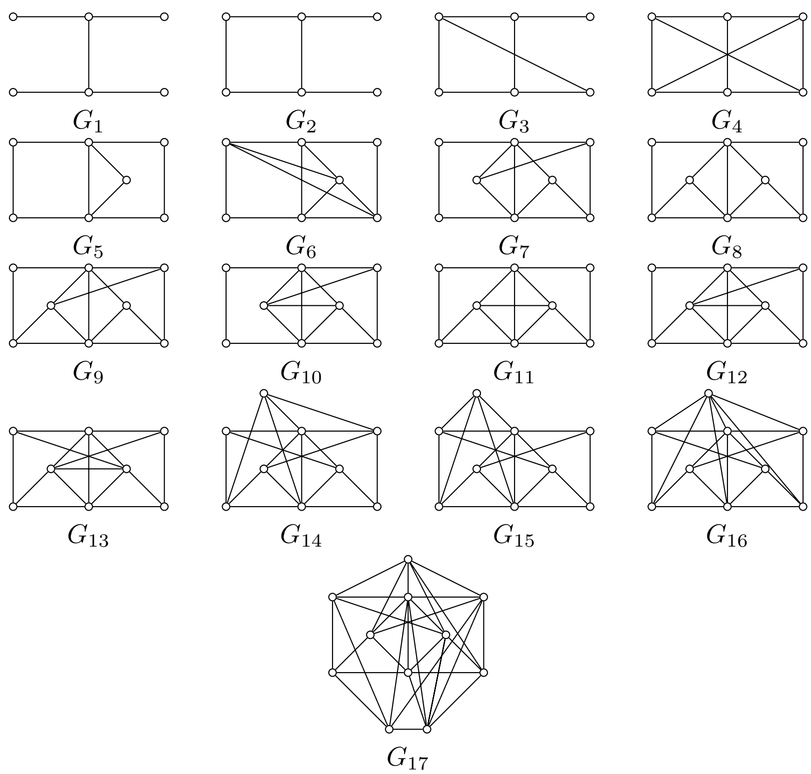
$$\mathbf{E}_1 + \mathbf{E}_2, \mathbf{E}_2 + \mathbf{E}_3, \mathbf{E}_3 + \mathbf{E}_4, \mathbf{E}_4 + \mathbf{E}_5, \mathbf{E}_5 + \mathbf{E}_1$$

induce a five-cycle in $\mathfrak{J}_R^n$, which is impossible. Hence $n \leq 4$, as required. The converse is straightforward. □

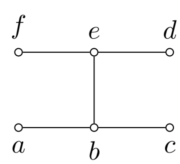
We conclude this paper with studying a variation of the notion of perfect graphs. A graph Γ is said to be *domination perfect* provided that $\gamma(S) = \iota(S)$ for every induced subgraph S of Γ, where $\gamma(S)$ is the domination number of S and $\iota(S)$ is the minimum cardinality among all maximal independent sets of S.

The following theorem of I.E. Zverovich and V.E. Zverovich is crucial in our investigation, so we mention it here for convenience.

Theorem 3.11 (Zverovich and Zverovich, [12, Theorem 11])**.** *A graph Γ is domination perfect if and only if it does not have the following graphs as an induced subgraph.*



We note that in the above theorem $G_4 \cong K_{3,3}$. In what follows the labeled graph G_1 shown below is called an H-graph and it is denoted by $\mathcal{H}(a, b, c, d, e, f)$.



To prove the next key lemma, we use the fact that the map $v \mapsto N_{G_i}[v]$, the closed neighborhood of v in G_i, is injective for all $1 \leq i \leq 17$.

Lemma 3.12. *Let $R = R_1 \oplus \cdots \oplus R_m$ be a decomposition of the ring R into local rings R_i with associated fields F_i ($|F_i| \leq 3$), for $i = 1, \ldots, m$. Then $\mathfrak{J}_R$ is domination perfect if and only if $\mathfrak{J}_{R/J(R)}$ is domination perfect.*

Proof. Clearly, $\mathfrak{J}_{R/J(R)}$ is domination perfect if $\mathfrak{J}_R$ is donimation perfect. Hence, assume that $\mathfrak{J}_{R/J(R)}$ is domination perfect. If $\mathfrak{J}_R$ has an induced subgraph S isomorphic to G_i for some $1 \leq i \leq 17$, then there must exist distinct vertices $x, y \in V(S)$ such that $x + J(R) = y + J(R)$. Since $|F_i| \leq 3$ for all

$1 \leq i \leq m$, it follows that x and y are adjacent so that x and y have the same closed neighborhood in S, which is a contradiction. □

Theorem 3.13. *The graph $\mathfrak{J}_R^n$ is domination perfect if and only if*

(1) $n = 3$ *and* $R \cong \mathbb{Z}_2$,
(2) $n = 2$ *and* $R/J(R) \cong \mathbb{Z}_2$,
(3) $n = 1$ *and* $R/J(R) \cong \mathbb{Z}_2$, $\mathbb{Z}_3$, $\mathbb{Z}_2 \oplus \mathbb{Z}_2$, $\mathbb{Z}_3 \oplus \mathbb{Z}_2$, $\mathbb{Z}_3 \oplus \mathbb{Z}_3$, $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$, $\mathbb{Z}_3 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$ *or* $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$, *or* $R = S \oplus \mathbb{Z}_2$, *where* S *is a local ring.*

Proof. We first suppose that $\mathfrak{J}_R^n$ is a domination perfect graph. Let $R = R_1 \oplus \cdots \oplus R_m$ be a decomposition of R into local rings $R_1, \ldots, R_m$. If $m, n \geq 2$ then $\mathfrak{J}_R^n$ has an induced H-subgraph

$$\mathcal{H}\left(\mathbf{E}_{11}, \mathbf{E}_{11} + \mathbf{E}_{12} + \mathbf{E}_{22}, \mathbf{E}_{12}, \mathbf{E}_{22}, \mathbf{E}_{21} + \mathbf{E}_{22}, \mathbf{E}_{21}\right),$$

from where it is not domination perfect. Now, we proceed in two cases:

Case 1. $m = 1$ and $n > 1$. Then R is a local ring with a maximal ideal $\mathfrak{m}$. Observe that $n \leq 3$ for otherwise $\mathfrak{J}_R^n$ has an induce H-subgraph

$$\mathcal{H}\left(\mathbf{E}_1, \mathbf{E}_1 + \mathbf{E}_2 + \mathbf{E}_3, \mathbf{E}_2, \mathbf{E}_3, \mathbf{E}_3 + \mathbf{E}_4, \mathbf{E}_4\right),$$

which is a contradiction.

If $\alpha \in R \setminus (\mathfrak{m} + \{0, 1\})$, then $\mathfrak{J}_R^n$ has an H-subgraph

$$\mathcal{H}\left(\mathbf{E}_1, \mathbf{E}_1 + \mathbf{E}_2, \mathbf{E}_2, \alpha^{-1}\mathbf{E}_1, \alpha\mathbf{E}_1 + (1-\alpha)\mathbf{E}_2, (1-\alpha)^{-1}\mathbf{E}_2\right),$$

which is a contradiction. Thus $R = \mathfrak{m} + \{0, 1\}$, that is, $R/\mathfrak{m} \cong \mathbb{Z}_2$. If $n = 3$ and $\mathfrak{m} \neq 0$, then $\mathfrak{J}_R^n$ has an induced subgraph isomorphic to $K_{3,3}$ with bipartition

$$\{\mathbf{E}_1 + \mathbf{E}_2, \mathbf{E}_1 + \mathbf{E}_2 + x\mathbf{E}_3, \mathbf{E}_1 + (1+x)\mathbf{E}_2\},$$
$$\{\mathbf{E}_2 + \mathbf{E}_3, x\mathbf{E}_1 + \mathbf{E}_2 + \mathbf{E}_3, (1+x)\mathbf{E}_2 + \mathbf{E}_3\}$$

for any $x \in \mathfrak{m} \setminus \{0\}$, a contradiction. Hence, either $n = 2$, or $n = 3$ and $R = \mathbb{Z}_2$.

Case 2. $n = 1$ and $m > 1$. Let R be a finite non-local ring and $R = R_1 \oplus \cdots \oplus R_m$ be a decomposition of R into local rings R_i with maximal ideals $\mathfrak{m}_i$ for $i = 1, 2, \ldots, m$. Suppose without loss of generality that $|R_1/\mathfrak{m}_1| \geq \cdots \geq |R_m/\mathfrak{m}_m|$. If $m \geq 5$ then $\mathfrak{J}_R^n$ has an induced H-subgraph

$$\mathcal{H}\left(\mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3, \mathbf{e}_2, \mathbf{e}_4, \mathbf{e}_3 + \mathbf{e}_4 + \mathbf{e}_5, \mathbf{e}_5\right),$$

which is impossible. Hence we must have $m \leq 4$. Let $T = R_2 \oplus \cdots \oplus R_m$. We proceed in two cases:

(1) $|R_1/\mathfrak{m}_1| \geq 4$. If $|T| = 2$ then we have noting to prove. Thus assume that $|T| \geq 3$. Let $\alpha \in R_1 \setminus (\mathfrak{m}_1 + \{0, \pm 1\})$ and $a, b \in T \setminus \{0\}$ be distinct elements such that $1 - ab \in U(T)$. Then $\mathfrak{J}_R^n$ has an induced subgraph isomorphic to $K_{3,3}$ with bipartition

$$\{\alpha\mathbf{e}_1, \alpha\mathbf{e}_1 + a\mathbf{e}_2, \alpha\mathbf{e}_1 + b\mathbf{e}_2\} \quad \text{and} \quad \{\alpha^{-1}\mathbf{e}_1, \alpha^{-1}\mathbf{e}_1 + a\mathbf{e}_2, \alpha^{-1}\mathbf{e}_1 + b\mathbf{e}_2\}$$

which is impossible.

(2) $|R_1/\mathfrak{m}_1| \leq 3$. By Lemma 3.12, we may assume that $J(R) = 0$. If $|R_1| = |R_2| = 3$ and $m \geq 3$, then $\mathfrak{J}_R^n$ has an induced H-subgraph

$$\mathcal{H}\left(\mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3, \mathbf{e}_2, -\mathbf{e}_2, -\mathbf{e}_1 - \mathbf{e}_2 + \mathbf{e}_3, -\mathbf{e}_1\right),$$

which is a contradiction. Also, if $R \cong \mathbb{Z}_3 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$, then $\mathfrak{J}_R^n$ has an induced H-subgraph

$$\mathcal{H}\left(\mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_4, \mathbf{e}_4, -\mathbf{e}_1, -\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3, \mathbf{e}_3\right),$$

a contradiction.

Conversely, suppose that R is one of the rings in the theorem. By Figures 2 and 4, each of the graphs $\mathfrak{J}_{\mathbb{Z}_2}^2$ and $\mathfrak{J}_{\mathbb{Z}_2}^3$ is domination perfect, respectively. Now consider the case $n = 2$ and R is a local ring with associated field of order 2. Since any element of an independence set of size three of $\mathfrak{J}_R^n$ has invertible entries, this graph does not have any induced subgraph isomorphic to $K_{3,3}$. On the other hand, $\mathfrak{J}_R^n$ does not have any induced path of length three. Therefore, $\mathfrak{J}_R^n$ is a domination perfect graph for any of the graphs $G_1, \ldots, G_{17}$ is either isomotphic to $K_{3,3}$ or has an induced path of lenght three. Next assume that $n = 1$. Let $R = S \oplus \mathbb{Z}_2$ where S is a local ring. The graph $\mathfrak{J}_R^n$ has no induced subgraph isomorphic to $K_{1,3}$, and since any of the graphs $G_1, \ldots, G_{17}$ has an induce subgraph isomorphic to $K_{1,3}$, so $\mathfrak{J}_R^n$ is a domination perfect graph. Now assume that R is one of the remained rings. Then, by Lemma 3.12, we may assume that $J(R) = 0$. Clearly, $\mathfrak{J}_R^n$ is domination perfect when $R \cong \mathbb{Z}_3 \oplus \mathbb{Z}_3$ (see [2, Figure 4]). Suppose $R \cong \mathbb{Z}_3 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. If $\mathfrak{J}_R^n$ has a graph G_i as an induced subgraph for some i, $1 \leq i \leq 17$, then it contains an induced subgraph isomorphic to $K_{1,3}$ whose center must be $c = \pm\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3$. But c is adjacent to all vertices expect for $\mp\mathbf{e}_1$ contradicting the fact that G_i is an induced subgraph of $\mathfrak{J}_R^n$. Hence $\mathfrak{J}_R^n$ is domination perfect. If $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ and $\mathfrak{J}_R^n$ has an induced subgraph isomorphic to $K_{1,3}$, then its center must be $\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3$, $\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_4$, $\mathbf{e}_1 + \mathbf{e}_3 + \mathbf{e}_4$, $\mathbf{e}_2 + \mathbf{e}_3 + \mathbf{e}_4$ or $\mathbf{1}$, which are adjacent to all but at most one vertex. Hence the same argument as before shows that $\mathfrak{J}_R^n$ is domination perfect. Finally, if $R \cong \mathbb{Z}_2$, $\mathbb{Z}_3$, $\mathbb{Z}_2 \oplus \mathbb{Z}_2$, $\mathbb{Z}_3 \oplus \mathbb{Z}_2$ or $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$, then $\mathfrak{J}_R^n$ is isomorphic to an induced subgraph of $\mathfrak{J}^n_{\mathbb{Z}_3 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2}$ or $\mathfrak{J}^n_{\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2}$, whence it is domination perfect. The proof is complete. □

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