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THE LOWER BOUND FOR THE NUMBER OF 1-FACTORS IN GENERALIZED PETERSEN GRAPHS

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Abstract. In this paper, we investigate the number of 1-factors of a generalized Petersen graph $P(N,k)$ and get a lower bound for the number of 1-factors of $P(N,k)$ when k is odd, which shows that the number of 1-factors of $P(N,k)$ is exponential in this case and confirms a conjecture of Lovász and Plummer (Ann. New York Acad. Sci. 576 (2006), no. 1, 389–398).

Keywords: Generalized Petersen graphs, matching, 1-factors, Fibonacci number.
MSC(2010): Primary: 05C70; Secondary: 05C30, 05C38.

1. Introduction

Let $G=(V(G),E(G))$ be a graph. Hereafter, all graphs are, finite, simple and connected. Also, for the basic terminology not defined here one may refer to [1].

A *matching* in a graph G is a set of pairwise non-adjacent edges. If M is a matching, the two ends of each edge of M are said to be *matched* under M, and each vertex incident with an edge of M is said to be covered by M. A *perfect matching* of a graph G is one which covers every vertex of G, where a perfect matching is also called a *1-factor* of G. Let $\Phi(G)$ be the number of 1-factors of G. Two graphs G and H are *isomorphic*, written $G\cong H$, if there are bijections $\phi: V(G)\to V(H)$ and $\varphi: E(G)\to E(H)$ such that $\psi_G(e)=uv$ if and only if $\psi_H(\varphi(e))=\phi(u)\phi(v)$; such a pair of mappings is called an isomorphism between G and H. A graph G is *n-extendable* if G has a matching of size n, and every such matching extends to (i.e., is contained in) a perfect matching in G. A graph is *factorizable* if it contains a 1-factor. A graph G is called *bicritical* if removing any two vertices of G, there remains a factorizable subgraph. Odd (even) path (cycle) represents a path (cycle) of odd (even) length.

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One of the topics in matching theory is to determine the function $\Phi(G)$. Kasteleyn [6] first introduced Pfaffian method to give the exact value for the number of 1-factors of planar graphs. However, there may exist no uniform formula or efficient algorithm to compute (G) for some graphs G. In particular, Valiant [10] proved that the problem of determining $\Phi(G)$ is NP-hard, even when G is bipartite. This left very little room for finding the exact value of $\Phi(G)$. Naturally, the next move is to find a lower bound for $\Phi(G)$. Up to now, it has obtained many important results for the lower bound (G) of some special graphs G. We present a few classical results in this direction.

Theorem 1.1 ([7])**.** *Let G be a Halin graph. Then $\Phi(G) \geqslant \frac{2}{3}(|V(G)| - 1)$.*

Theorem 1.2 ([12])**.** *Let $G = (X, Y)$ be a bipartite graph with a 1-factor and $d_G(x) \geqslant k$ for every $x \in X$. Then $\Phi(G) \geqslant k!$.*

Theorem 1.3 ([2, 9])**.** *Let G be a k-regular bipartite graph on $2n$ vertices. Then*

$$(\frac{(k-1)^{k-1}}{k^{k-2}})^n \leqslant \Phi(G) \leqslant (k!)^{\frac{n}{k}}.$$

Theorem 1.3 implies that the number of 1-factors of a k-regular bipartite graph is exponential. In addition, some non-bipartite cubic graphs may not have 1-factors. For instance, Sylvester graph has this property.

Theorem 1.4 ([12])**.** *Let G be a k-connected graph with a 1-factor. Then $\Phi(G) \geqslant k!!$. In particular, $\Phi(K_n) = (n-1)!!$. These bounds are sharp when k is odd.*

Theorem 1.5 ([7])**.** *Let G be a k-connected graph with a 1-factor and assume that G is not bicritical. Then $\Phi(G) \geqslant k!$.*

Došlić [4] used ear decomposition theory of 2-connected graphs to establish lower bounds on the number of 1-factors in k-extendable graphs.

Theorem 1.6 ([4])**.** *Let G be a k-extendable graph of n vertices and m edges with maximum degree Δ, where $k \geqslant 1$. Then*

$$\Phi(G) \geqslant \lceil \frac{(k+1)!}{4}(m - n - (k-1)(2\Delta - 3) + 4) \rceil.$$

In 2006, Lovász and Plummer [8] posed a conjecture on the lower bound of 1-factors of 2-edge-connected cubic graphs.

Conjecture 1.7 ([8])**.** Let G be a 2-edge-connected cubic graph. Then there exists a constant number $c > 1$ such that $\Phi(G) \geqslant c^n$.

Some partial results are known with regard to this conjecture. For example, Voorhoeve [11] showed that if G is a cubic bipartite graph on $2n$ vertices, then $\Phi(G) \geqslant (\frac{4}{3})^n$. Chudnovsky and Seymour [3] proved that if G is a cubic planar graph with no cut edges, then $\Phi(G) \geqslant 2^{\frac{|V(G)|}{655978752}}$.

Let us fix some notations before presenting the main results.

Let F_{n+1} be the number of the subsets of $\{1, 2, \ldots, n\}$ containing no consecutive integers in $\{1, 2, \ldots, n\}$. Then F_{n+1} is called the *Fibonacci number*. The *Fibonacci sequence* $\{F_n\}$ satisfies the following recurrence relation

$$F_1 = F_2 = 1,$$

$$F_{n+1} = F_n + F_{n-1}.$$

It is known that F_n can be stated as:

$$F_n = \frac{1}{\sqrt{5}}(\sigma^{n+1} - \tau^{n+1}),$$

where $\sigma = \frac{1+\sqrt{5}}{2}$, $\tau = \frac{1-\sqrt{5}}{2}$.

Definition 1.8. A generalized Petersen graph $P(N, k)$ for $N \geqslant 3$ and $1 \leqslant k < \frac{N}{2}$ is a graph on the vertex set

$$V = \{u_i | i = 1, 2, \ldots, N\} \cup \{w_i | i = 1, 2, \ldots, N\},$$

and the edge set

$$E = \{u_i u_{i+1}, u_i w_i, w_i w_{i+k} | i = 1, 2, \ldots, N\},$$

where the subscripts are taken modulo N.

When $N \equiv 0(mod\ 2)$ and $k \equiv 1(mod\ 2)$, $P(N, k)$ is a bipartite graph [5]. Hence, $\Phi(P(N, k))$ is exponential. Neverthless, $P(N, k)$ is non-planar and non-bipartite when $N \equiv 1(mod\ 2)$ and $k \equiv 1(mod\ 2)$. In this paper, we prove that the number of 1-factors of $P(N, k)$ is exponential when $k \equiv 1(mod\ 2)$, which confirms Conjecture 1.7 in this case.

2. Lower bounds for $\Phi(G)$ in generalized Petersen graphs

Lemma 2.1. *Let* $f_m := \sum_{i=0}^{m-1} \binom{m+i}{2i+1}$, $g_m := \sum_{i=0}^{m} \binom{m+i}{2i}$. *Then* $f_m = F_{2m-1}$, $g_m = F_{2m}$, *where* F_{2m-1} *and* F_{2m} *are odd items and even items of Fibonacci sequence* F_m, *respectively.*

Proof. Obviously, f_m and g_m satisfy the following initial condition

$$\begin{cases} f_1 = F_1 = 1, \\ g_1 = F_2 = 1. \end{cases}$$

Now we show that they satisfy the recurrence relations of Fibonacci sequence F_m:

$$\begin{cases} f_m + g_m = F_{2m-1} + F_{2m} = F_{2m+1} = f_{m+1}; \\ g_m + f_{m+1} = F_{2m} + F_{2m+1} = F_{2m+2} = g_{m+1}. \end{cases}$$

In fact,

$$\begin{aligned} f_m + g_m &= \sum_{i=0}^{m-1} \binom{m+i}{2i+1} + \sum_{i=0}^{m} \binom{m+i}{2i} \\ &= \sum_{i=0}^{m} \binom{m+i+1}{2i+1} = f_{m+1}. \end{aligned}$$

$$g_m + f_{m+1} = \sum_{i=0}^{m} \binom{m+i}{2i} + \sum_{i=0}^{m} \binom{m+i+1}{2i+1} = g_{m+1}.$$

The lemma is proved. □

For convenience, let $n = \lfloor \frac{N-1}{k} \rfloor$ and $gcd(a, b)$ be the greatest common divisor of two positive integers a and b.

Theorem 2.2. *Let $P(N, k)$ be a generalized Petersen graph with $gcd(N, k) = 1$ and $k \equiv 1 (mod\ 2)$. Then*

$$\Phi(P(N,k)) > \begin{cases} F_n, & if\ gcd(N,n) \equiv 0 (mod\ 2), \\ F_{n-1}, & if\ gcd(N,n) \equiv 1 (mod\ 2). \end{cases}$$

Proof. We construct a new graph $H = H(V(H), E(H))$:

$$V(H) = \{u_i | i = 1, 2, \dots, N\} \cup \{v_i | i = 0, 1, \dots, N-1\},$$

$$E(H) = \{u_i u_{i+1} | i = 1, 2, \dots, N\} \cup$$
$$\{v_i v_{i+1} | i = 0, 1, \dots, N-1\} \cup \{v_i u_{[ki+1]} | i = 0, 1, \dots, N-1\},$$

where

$$[ki+1] = \begin{cases} ki+1, & 1 \leqslant ki+1 \leqslant N, \\ l, & N < ki+1 = Nr + l. \end{cases}$$

We construct a mapping (f, g) as follows:

$$\begin{cases} f:\ V(H) \to V(G); & \\ u_i \mapsto u_i, & i = 1, 2, , \dots, N, \\ v_i \mapsto w_{[ki+1]}, & i = 0, 1, , \dots, N-1 \end{cases}$$

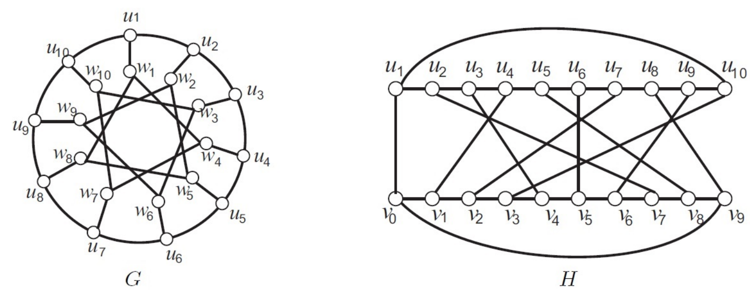
$$\begin{cases} g:\ E(H) \to E(G); & \\ u_i u_{i+1} \mapsto u_i u_{i+1}, & i = 1, 2, , \dots, N, \\ v_i v_{i+1} \mapsto w_{[ki+1]} w_{[k(i+1)+1]}, & i = 0, 1, , \dots, N-1, \\ v_i u_{[ki+1]} \mapsto w_{[ki+1]} u_{[ki+1]}, & i = 0, 1, , \dots, N-1. \end{cases}$$

Note that $P(10, 3)$ has two different drawings, (see Figure 1).

It is easy to see that (f, g) is an isomorphic between G and H when $\gcd(N, k) = 1$. Hence $\Phi(G) = \Phi(H)$. In the following, we evaluate the lower bound of $\Phi(H)$. Let $E_0 = \{e_i = v_i u_{[ki+1]} \in E(H) | i = 0, 1, \dots, N-1\}$ and $F \in E_0$, denote $M_0(F)$ to be the set of 1-factors of H containing F. Two cases must be considered based on the parity of N.

Case 1. $N \equiv 1 (mod\ 2)$.

FIGURE 1. Two drawings of $P(10,3)$



Then $H - E_0$ contains two disjoint odd cycles, denoted by C_1 and C_2, respectively, where

$$C_1 = v_0 v_1 \dots v_{N-1} v_0,$$

$$C_2 = u_1 u_2 \dots u_N u_1.$$

Then H has a 1-factor only for $|M_0(F)| \equiv 1 (mod\ 2)$.

When $|M_0(F)| = 1$, 1-factors of H contain precisely one edge e_i of E_0. Then $C_1 - v_i$ and $C_2 - u_{[ki+1]}$ are two distinct odd paths and each of them has a 1-factor. Thus, H has a 1-factor. Since e_i has N distinct selections, $\Phi(H) = N$.

When $|M_0(F)| = 3$, 1-factors of H contain three edges of E_0. Assume that $M_0(F) = \{e_{i_1}, e_{i_2}, e_{i_3}\}$, then the number of 1-factors of H containing $M_0(F)$ equals to the number of choices of (i_1, i_2, i_3). $H - E_0$ is the set of odd paths since H has a 1-factor in this case. Assume that $C_1 - \{v_{i_1}, v_{i_2}, v_{i_3}\}$ are distinct odd paths. It leads to the parity of i_1, i_2, i_3 $(0 \leqslant i_1 < i_2 < i_3 \leqslant N-1)$ are alternate. Similar to the former, paths of $C_2 - \{u_{[ki_1+1]}, u_{[ki_2+1]}, u_{[ki_3+1]}\}$ are of odd length, and hence the parity of $[ki_1+1]$, $[ki_2+1]$, $[ki_3+1]$ are also alternate. $[ki_j+1]$ has k distinct values for $j = 1,2,3$ as follows:

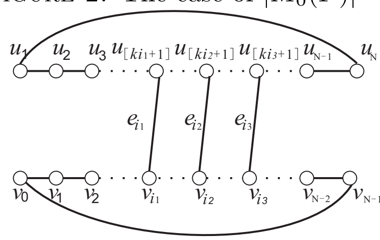
$$[ki_j + 1] = \begin{cases} ki_j + 1, & 1 \leqslant ki_j + 1 \leqslant N; \\ ki_j + 1 - N, & N + 1 \leqslant ki_j + 1 \leqslant 2N; \\ \dots & \dots \\ ki_j + 1 - (k-1)N, & (k-1)N + 1 \leqslant ki_j + 1 \leqslant kN. \end{cases}$$

To guarantee $u_{[ki_1+1]}, u_{[ki_2+1]}, u_{[ki_3+1]}$ on cycle C_2 in this order, we only consider the case that $0 \leqslant ki_j + 1 \leqslant N - 1$. Since $[ki_j + 1] = ki_j + 1$ for $j = 1,2,3$, we have $[ki_1+1], [ki_2+1], [ki_3+1]$ and i_1, i_2, i_3 have the same order, and the following three edges of 1-factors of H are chosen from E_0:

$$\begin{cases} e_{i_1} = v_{i_1} u_{ki_1+1}; \\ e_{i_2} = v_{i_2} u_{ki_2+1}; \\ e_{i_3} = v_{i_3} u_{ki_3+1}, \end{cases}$$

where the order of $e_{i_1}, e_{i_2}, e_{i_3}$ is given in Figure 2.

FIGURE 2. The case of $|M_0(F)| = 3$



Thus,

$$0 \leqslant i_1 < i_2 < i_3 \leqslant \lfloor \frac{N-1}{k} \rfloor = n.$$

It is clear that the number of 1-factors of H with $|M_0(F)| = 3$ equals to the number of the selections of (i_1, i_2, i_3) in $\{0, 1, 2, \dots, n\}$. When $n \equiv 1(mod\ 2)$, we shall consider the parity of i_1. If $i_1 \equiv 1(mod\ 2)$, then

$$\begin{cases} i_1 \equiv i_3 \equiv n \equiv 1(mod\ 2); \\ i_2 \equiv 0(mod\ 2). \end{cases}$$

Let

$$\begin{cases} i_1 - 0 = 2k_1 + 1; \\ i_2 - i_1 = 2k_2 + 1; \\ i_3 - i_2 = 2k_3 + 1; \\ n - i_3 = 2k_4, \end{cases}$$

where k_i $(i = 1, 2, 3, 4)$ is a nonnegative integer. Then

$$k_1 + k_2 + k_3 + k_4 = \frac{n-3}{2}.$$

Observe that the number of the selections of (i_1, i_2, i_3) equals to the number of solutions of the above equation. Therefore, (i_1, i_2, i_3) has $\binom{\frac{n+3}{2}}{3}$ distinct choices. And since $i_1 \equiv 0(mod\ 2)$, (i_1, i_2, i_3) has $\binom{\frac{n+3}{2}}{3}$ distinct selections analogously.

When $n \equiv 0(mod\ 2)$, the number of selections of (i_1, i_2, i_3) is

$$\begin{cases} \binom{\frac{n+2}{2}}{3}, & i_1 \equiv 1(mod\ 2); \\ \binom{\frac{n+4}{2}}{3}, & i_1 \equiv 0(mod\ 2). \end{cases}$$

Since $C_1 - \{v_{i_1}, v_{i_2}, v_{i_3}\}$ and $C_2 - \{u_{[ki_1+1]}, u_{[ki_2+1]}, u_{[ki_3+1]}\}$ are distinct union of odd paths, they have an unique 1-factor. Hence the number of 1-factors of H containing F equals to the number of selections of (i_1, i_2, i_3). Therefore, when $|M_0(F)| = 3$,

$$\Phi(H) \geqslant \begin{cases} \binom{\frac{n+3}{2}}{3}, \ n \equiv 1(mod\ 2); \\ \binom{\frac{n+2}{2}}{3}, \ n \equiv 0(mod\ 2). \end{cases}$$

Similarly, when $|M_0(F)| = 5$,

$$\Phi(H) \geqslant \begin{cases} \binom{\frac{n+5}{2}}{5}, \ n \equiv 1(mod\ 2); \\ \binom{\frac{n+4}{2}}{5}, \ n \equiv 0(mod\ 2). \end{cases}$$

Repeat the above discussions again, we may find the lower bound of $\Phi(H)$ for $|M_0(F)| = 7, 9, \ldots, n + \varepsilon_n$, where $\varepsilon_n = 0$ if $n \equiv 1(mod\ 2)$ and ε_n=-1 for otherwise. That is,

$$\Phi(H) > \begin{cases} N + \sum_{i=1}^{\frac{n-1}{2}} \binom{\frac{n+2i+1}{2}}{2i+1}, \ if\ n \equiv 1(mod\ 2); \\ N + \sum_{i=1}^{\frac{n-2}{2}} \binom{\frac{n+2i}{2}}{2i+1}, \ if\ n \equiv 0(mod\ 2). \end{cases}$$

And hence,

$$\Phi(H) > \begin{cases} \sum_{i=0}^{\frac{n-1}{2}} \binom{\frac{n+2i+1}{2}}{2i+1}, \ if\ n \equiv 1(mod\ 2); \\ \sum_{i=0}^{\frac{n-2}{2}} \binom{\frac{n+2i}{2}}{2i+1}, \ if\ n \equiv 0(mod\ 2). \end{cases} \tag{2.1}$$

Case 2. $N \equiv 0(mod\ 2)$.

Then $H - E_0$ contains two even cycles, denoted by C_1 and C_2, respectively, where

$$C_1 = v_0 v_1 \ldots v_{N-1} v_0,$$

$$C_2 = u_1 u_2 \ldots u_N u_1.$$

Therefore, H has a 1-factor when $|M_0(F)| \equiv 0(mod\ 2)$.

When $|M_0(F)| = 0$, 1-factors of H contain no edges of E_0. Hence $H - E_0$ is determined by two even cycles and each of them has two 1-factors. Thus, $\Phi(H) = 4$.

When $|M_0(F)| = 2$, such 1-factors of H have two edges of E_0. Suppose that

$$M_0(F) = \{\{e_{i_1}, e_{i_2}\} | 0 \leqslant i_1 < i_2 \leqslant N - 1\}.$$

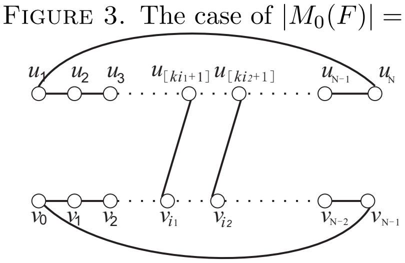
Based on our reasoning so far, the number of 1-factors of H with $|M_0(F)| = 2$ equals to the number of choices of (i_1, i_2). If H contains a 1-factor, then $C_1 - \{v_{i_1}, v_{i_2}\}$ and $C_2 - \{u_{[ki_1+1]}, u_{[ki_2+1]}\}$ are distinct union of odd paths, and hence $\gcd(i_1, i_2) \equiv 1(mod\ 2)$ and $\gcd([ki_1+1], [ki_2+1]) \equiv 1(mod\ 2)$. Therefore, the parity of i_1, i_2 and $[ki_1 + 1], [ki_2 + 1]$ are different. $[ki_j + 1]$ has k distinct values for $j = 1, 2$ as follows:

$$[ki_j + 1] = \begin{cases} ki_j + 1, & 1 \leqslant ki_j + 1 \leqslant N; \\ ki_j + 1 - N, & N + 1 \leqslant ki_j + 1 \leqslant 2N; \\ \cdots & \cdots \\ ki_j + 1 - (k-1)N, & (k-1)N + 1 \leqslant ki_j + 1 \leqslant kN. \end{cases}$$

Now we only consider the case that $0 \leqslant ki_j + 1 \leqslant N - 1, j = 1, 2$, as shown in Figure 3. Then $e_{i_1} = v_{i_1} u_{ki_1+1}$, $e_{i_2} = v_{i_2} u_{ki_2+1}$ with $e_{i_1} \cap e_{i_2} = \emptyset$ and

$$0 \leqslant i_1 < i_2 \leqslant \lfloor \frac{N-1}{k} \rfloor = n.$$

FIGURE 3. The case of $|M_0(F)| = 2$



Now, the number of 1-factors of H with $|M_0(F)| = 2$ equals to the number of the selections of (i_1, i_2). When $n \equiv 1(mod\ 2)$, we consider the parity of i_1. If $i_1 \equiv 1(mod\ 2)$, then $i_2 \equiv 0(mod\ 2)$.

Let

$$\begin{cases} i_1 - 0 = 2k_1 + 1; \\ i_2 - i_1 = 2k_2 + 1; \\ n - i_2 = 2k_3 + 1, \end{cases}$$

where each k_i $(i = 1, 2, 3)$ is a nonnegative integer. Then

$$k_1 + k_2 + k_3 = \frac{n-3}{2}.$$

It is easy to see that the number of the selections of (i_1, i_2) equals to the number of solutions of the above equation. Therefore, (i_1, i_2) has $\binom{\frac{n+1}{2}}{2}$ distinct selections. And as $i_1 \equiv 0(mod\ 2)$, (i_1, i_2) has $\binom{\frac{n+3}{2}}{2}$ distinct choices.

When $n \equiv 0(mod\ 2)$, the number of choices of (i_1, i_2) is $\binom{\frac{n+2}{2}}{2}$. Therefore, when $|M_0(F)| = 2$,

$$\Phi(H) \geqslant \begin{cases} \binom{\frac{n+1}{2}}{2}, & n \equiv 1(mod\ 2); \\ \binom{\frac{n+2}{2}}{2}, & n \equiv 0(mod\ 2). \end{cases}$$

Similar to the above procedure, we may obtain the lower bound of $\Phi(H)$ for $|M_0(F)| = 4, 6, \ldots, n + \varepsilon_n$, where $\varepsilon_n = 0$ if $n \equiv 0(mod\ 2)$ and $\varepsilon_n = -1$ for otherwise, as follows:

$$\Phi(H) > 4 + \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} \binom{\lfloor \frac{n+2i}{2} \rfloor}{2i} \geqslant \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{\lfloor \frac{n+2i}{2} \rfloor}{2i}. \tag{2.2}$$

Set $m = \lceil \frac{n}{2} \rceil$ in inequalities (2.1). Then

$$\Phi(H) > \sum_{i=0}^{m-1} \binom{m+i}{2i+1}.$$

Set $m = \lfloor \frac{n}{2} \rfloor$ in inequalities (2.2). Then

$$\Phi(H) > \sum_{i=0}^{m} \binom{m+i}{2i}.$$

Note that

$$\begin{cases} f_m = \sum_{i=0}^{m-1} \binom{m+i}{2i+1}; \\ g_m = \sum_{i=0}^{m} \binom{m+i}{2i}, \end{cases}$$

by Lemma 2.1, f_m and g_m are odd terms and even terms of Fibonacci sequence F_m, respectively. Then

$$\Phi(H) > \begin{cases} f_m, & N \equiv 1(mod\ 2); \\ g_m, & N \equiv 0(mod\ 2). \end{cases}$$

Since the general form of Fibonacci sequence F_n is

$$F_n = \frac{1}{\sqrt{5}}(\sigma^{n+1} - \tau^{n+1}),$$

where $\sigma = \frac{1+\sqrt{5}}{2}$, $\tau = \frac{1-\sqrt{5}}{2}$, F_n increases exponentially. Hence $\Phi(H)$ also increases exponentially. By the construction of H, $\Phi(H) = \Phi(P(N,k))$. When $\gcd(N,k) = 1$ and $k \equiv 1(mod\ 2)$, the lower bound of $\Phi(P(N,k))$ is some item of Fibonacci sequence, and hence it increases exponentially with order N.

This completes the proof. □

Theorem 2.3. *Let $P(N,k)$ be a generalized Petersen graph. If $gcd(N,k) \neq 1$, $N \equiv 0(mod\ 2)$ and $k \equiv 1(mod\ 2)$, then*

$$\Phi(P(N,k)) > \begin{cases} 2^{t-1}F_n, & if\ n \equiv 1(mod\ 2), \\ 2^{t-1}F_{n-1}, & if\ n \equiv 0(mod\ 2), \end{cases}$$

where $t = gcd(N,k)$.

Proof. For the proof, we construct a new graph H' such that $H' \cong P(N,k)$. It is easy to see that $t \equiv 1(mod\ 2)$ and $\frac{N}{k} \equiv 0(mod\ 2)$. Let $2m = \frac{N}{k}$. Then $P(N,k)$ can be restated as the union of a long cycle of length N, t short cycles of length $2m$ and N edges joining these cycles.

We define a new graph $H' = H'(V(H'), E(H'))$ as follows:

$$\begin{aligned} V(H') &= \{u_i | i = 1, 2, \ldots, N\} \cup \{v_i | i = 0, 1, \ldots, N-1\}, \\ E(H') &= \{v_i v_{i+1} | i = 2(j-1)m, \ldots, 2jm-2, j = 1, 2, \ldots, t\} \\ &\cup \{v_{2m-1}v_0, v_{4m-1}v_{2m}, \ldots, v_{N-1}v_{2(t-1)m}\} \\ &\cup \{v_i u_{[ki+j]} | i = 0, 1, \ldots, N-1, j = 1, 2, \ldots, t\}. \end{aligned}$$

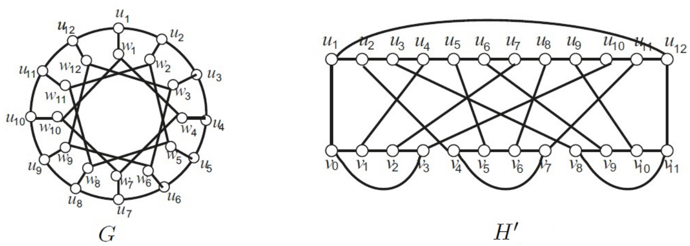
An isomorphic mapping (ϕ, φ) between H' and $P(N,k)$ is defined as:

$$\begin{cases} \phi: \ V(H') \to V(G); & \\ u_i \mapsto u_i, & i = 1, 2, \ldots, N; \\ v_i \mapsto w_{[ki+j]}, & i = 2(j-1)m, \ldots, 2jm-1, j = 1, 2, \ldots, t. \end{cases}$$

$$\begin{cases} \varphi: \ E(H') \to E(G); & \\ u_i u_{i+1} \mapsto u_i u_{i+1}, & i = 1, 2, \ldots, N; \\ v_i v_{i+1} \mapsto w_{[ki+1]} w_{[k(i+1)+1]}, & i = 0, 1, \ldots, 2m-2; \\ v_{2m-1} v_0 \mapsto w_{[k(2m-1)+1]} w_1; & \\ \ldots & \\ v_i v_{i+1} \mapsto w_{[ki+2]} w_{[k(i+1)+2]}, & i = 2m, 2m+1, \ldots, 4m-2; \\ v_{4m-1} v_{2m} \mapsto w_{[k(4m-1)+2]} w_2; & \\ v_i v_{i+1} \mapsto w_{[ki+t]} w_{[k(i+1)+t]}, & i = 2(t-1)m, , \ldots, 2tm-2; \\ v_{N-1} v_{2(t-1)m} \mapsto w_{[k(N-1)+t]} w_2; & \\ v_i u_{[ki+j]} \mapsto w_{[ki+j]} u_{[ki+j]}, & i = 0, 1, \ldots, N-1, j = 1, 2, \ldots, t. \end{cases}$$

Then $P(12,3)$ has two distinct drawings as shown in Figure 4.

FIGURE 4. Two drawings of $P(12,3)$



Since (ϕ,φ) is an isomorphic mapping between G and H', we have $\Phi(G) = \Phi(H')$. Now we start to compute the lower bound of $\Phi(H')$. Let

$$E_0 = \{e_i = v_i u_{[ki+j]} | i = 0, 1, \dots, N-1, j = 1, 2, \dots, t\}.$$

Then $H' - E_0$ contains t distinct short cycles ($C_{1j}, j = 1, 2, \dots, t$) of length $2m$ and a long cycle C_2 of length N, where

$$C_{1j} = v_{2(j-1)m} \dots v_{2jm-1} v_{2(j-1)m}, j = 1, 2, \dots, t;$$

$$C_2 = u_1 u_2 \dots u_N u_1.$$

We still use the definition of F and $M_0(F)$ as before. If a 1-factor of H' contains F, then $|M_0(F)|$ is even. We consider the case that the above edges lying on both C_2 and C_{11}.

When $|M_0(F)| = 0$, 1-factors of H' of this type are from $t+1$ long cycles, and each of them has two independent 1-factors. Then $\Phi(H') = 2^{t+1}$.

When $|M_0(F)| = 2$, 1-factors of H' of this type contain two edges of E_0. Suppose that $M_0(F) = \{e_0, e_i\}$ $(1 \leqslant i \leqslant 2m-1)$. If $i \equiv 1(\mod 2)$, then $C_{11} - \{v_0, v_i\}$ contains a 1-factor. And $C_2 - \{u_1, u_{[ki+1]}\}$ also has a 1-factor for $[ki+1] \equiv 0(mod\ 2)$. Then $[ki+1] = ki+1$ and i have distinct parity for $1 \leqslant ki+1 \leqslant N$ (i.e., $1 \leqslant i \leqslant \lfloor \frac{N-1}{k} \rfloor$). Thus we may only consider the case that $i \equiv 1(mod\ 2), 1 \leqslant i \leqslant n$.

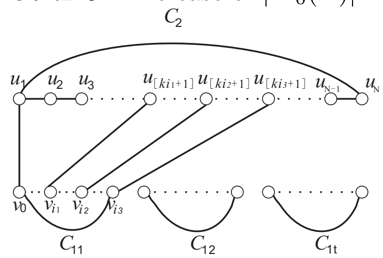
When $n \equiv 1(mod\ 2)$ (or $n \equiv 0(mod\ 2)$), similar to the discussions we used before, the choices of i are $\frac{n+1}{2}$ (or $\frac{n}{2}$). If e_0 is not fixed, then the first edge e_0 has exactly $n+1$ choices and once repeated, hence the two edges have $\frac{(n+1)n}{2}$ selections. Since the subgraphs determined by the left $t-1$ short cycles have 2^{t-1} distinct 1-factors,

$$\Phi(H') \geqslant \begin{cases} 2^{t-1}\frac{n+1}{2}\frac{n+1}{2}, & n \equiv 1(mod\ 2); \\ 2^{t-1}\frac{n+1}{2}\frac{n}{2}, & n \equiv 0(mod\ 2). \end{cases}$$

When $|M_0(F)| = 4$, let $M_0(F) = \{\{e_0, e_{i_1}, e_{i_2}, e_{i_3}\} | 1 \leqslant i_1 \leqslant i_2 \leqslant i_3 \leqslant 2m-1\}$. Then both of $C_{11} - \{v_0, v_{i_1}, v_{i_2}, v_{i_3}\}$ and $C_2 - \{u_1, u_{[ki_1+1]}, u_{[ki_2+1]},$

$u_{[ki_3+1]}\}$ have 1-factors if $i_1 \equiv 1(mod\ 2)$, $i_2 \equiv 0(mod\ 2)$, $i_3 \equiv 1(mod\ 2)$ and $[ki_1+1] \equiv 0(mod\ 2)$, $[ki_2+1] \equiv 1(mod\ 2)$, $[ki_3+1] \equiv 0(mod\ 2)$ (see Figure 5).

FIGURE 5. The case of $|M_0(F)| = 4$



As we have shown before, if $1 \leqslant i \leqslant n$, $[ki+1] = ki+1$ and i have distinct parity, then the selections of $\{e_0, e_{i_1}, e_{i_2}, e_{i_3}\}$ equal to the choices of (i_1, i_2, i_3). If e_0 is also not fixed, then it has $n+1$ choices and four times repeated. Hence the contribution of $M_0(F)$ is at least

$$\begin{cases} \frac{n+1}{4}\binom{\frac{n+1}{2}+1}{3}, & n \equiv 1(mod\ 2); \\ \frac{n+1}{4}\binom{\frac{n}{2}+1}{3}, & n \equiv 0(mod\ 2). \end{cases}$$

Since the remaining $t-1$ cycles have 2^{t-1} distinct 1-factors,

$$\Phi(H') \geqslant \begin{cases} 2^{t-1}\frac{n+1}{4}\binom{\frac{n+1}{2}+1}{3}, & n \equiv 1(mod\ 2); \\ 2^{t-1}\frac{n+1}{4}\binom{\frac{n}{2}+1}{3}, & n \equiv 0(mod\ 2). \end{cases}$$

By the same method as above, the lower bound for $\Phi(H')$ with $|M_0(F)| = 6, 8, \ldots, 2m$ is

$$\begin{cases} 2^{t+1} + 2^{t-1}\dfrac{n+1}{2}\dfrac{n+1}{2} + 2^{t-1}\displaystyle\sum_{i=1}^{\frac{n-1}{2}}\dfrac{n+1}{2i+2}\binom{\frac{n+1}{2}+i}{2i+1}, & n \equiv 1(mod\ 2), \\ 2^{t+1} + 2^{t-1}\displaystyle\sum_{i=1}^{\frac{n}{2}}\dfrac{n+1}{2i}\binom{\frac{n}{2}+i-1}{2i-1}, & n \equiv 0(mod\ 2). \end{cases}$$

Therefore,

$$\Phi(H') > \begin{cases} 2^{t-1}\displaystyle\sum_{i=1}^{\frac{n-1}{2}}\binom{\frac{n+1}{2}+i}{2i+1} = 2^{t-1}F_n, & n \equiv 1(mod\ 2), \\ 2^{t-1}\displaystyle\sum_{i=1}^{\frac{n}{2}-1}\binom{\frac{n}{2}+i}{2i+1} = 2^{t-1}F_{n-1}, & n \equiv 0(mod\ 2), \end{cases}$$

which completes the proof. □

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