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ZERO ELEMENTS AND z-IDEALS IN MODIFIED POINTFREE TOPOLOGY

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Abstract. In this paper, we define and study the notion of zero elements in topoframes; a topoframe is a pair (L, τ), abbreviated L_τ, consisting of a frame L and a subframe τ all of whose elements are complemented elements in L. We show that the f-ring $\mathcal{R}(L_\tau)$, the set of τ-real continuous functions on L, is uniformly complete. Also, the set of all zero elements in a topoframe is closed under the formation of countable meets and finite joins. Also, we introduce the notion of z-filters and z-ideals in modified pointfree topology and make ready some results about them.
Keywords: Topoframe, zero element, z-filter, z-ideal, prime ideal.
MSC(2010): Primary: 06D22; Secondary: 06F25, 54B30, 54G05, 54G10, 18B30.

1. Introduction

In studying relations between topological properties of a space X and algebraic properties of $C(X)$, it is natural to look at the subsets of X of the form

$$Z(f) := \{x \in X \mid f(x) = 0\}$$

called the zero-set of f. Many researchers are interested in the ring of real continuous functions on a set. Hewitt's paper [17] contains basic information about zero-sets. This is the first paper in which zero-sets were exploited in a systematic way in the study of $C(X)$. Zero-sets play a key role in the study of rings of real continuous functions (see, for instance, [1, 4, 12, 16, 18]).

The set-theoretic complement of a zero-set is known as a cozero-set and we label this set by $coz(f)$. B. Banaschewski presented a detailed study of the cozero map from $\mathcal{R}(L)$, the ring of real continuous functions on a frame L, to L which is the pointfree form of assigning to each continuous real function on a space its familiar cozero set, using this in particular to obtain the Stone-Čech

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compactification of a completely regular frame L in the form of a homomorphism from the frame of closed ℓ-ideals of the bounded part of $\mathcal{R}^*(L)$ to L (see [8]). The natural question in pointfree topology is how an explicit definition of zero elements is prepared as duals of cozero elements. An effort to overcome the strangeness of cozero elements and zero ones was expended in [13] to produce a zero set in pointfree topology (frames and their homomorphisms) by defining the trace of an element α of $\mathcal{R}(L)$ on any point p of L, that is a real number which is denoted by $\alpha[p]$, and a zero set of α is defined by

$$Z(\alpha) := \{p \in Pt(L) \mid \alpha[p] = 0\}. \tag{1.1}$$

However, the alternative approach comes into view in this paper; that is we will present "zero elements in modified pointfree topology (topoframes and their homomorphisms)" being actually complements of some cozero elements in pointfree topology. That all cozero elements in pointfree topology have duals (complements) in pointfree topology is yet an unanswered question (see Remark 3.2).

In [14] it has been shown that for a topoframe L_τ the ring of τ-real-continuous functions $\mathcal{R}(L_\tau)$ is isomorphic to a sub-f-ring of the ring of real-valued functions on τ introduced in [19]. It has been also shown that $\mathcal{R}(L_\tau)$ is actually a generalization of $C(X)$, the f-ring of all continuous functions from a space X into the real line set $\mathbb{R}$; in fact $C(X) \cong \mathcal{R}(\mathcal{P}(X)_{\mathcal{O}(X)})$ (see [14, Theorem 3.3]).

The aim of this paper is to show how various facts in zero set topology connected with the real numbers have their counterparts, if not actually their logical antecedents, in modified pointfree topology, that is, in the setting of topoframes and their homomorphisms. The reader can estimate the knowledge required by looking at Section 2.

In Section 3, we define concepts of a zero element and a cozero element in a topoframe (Definition 3.1). We observe that the set

$$\{z(f) \mid f \text{ is a bounded topoframe map in } \mathcal{R}(L_\tau)\}$$

is the same as $z[L_\tau] := \{z(f) \mid f \in \mathcal{R}(L_\tau)\}$, and that $z[L_\tau]$ is closed under the formation of finite joins. We show that for a frame map f in $\mathcal{R}(L_\tau)$, f^{-1} exists if and only if $z(f) = \perp$ (Theorem 3.8).

It is well known that a commutative f-ring with unit A has a natural topology, its uniform topology, with the basic neighborhoods being of the form

$$V_n(a) = \{x \in A \mid |x - a| < \frac{1}{n}\}, \qquad n = 1, 2, \ldots$$

for each $a \in A$.

In Section 4, we show that the f-ring $\mathcal{R}(L_\tau)$ is uniformly complete (Theorem 4.1), and $z[L_\tau]$ is closed under the formation of countable meets (Corollary 4.2).

The basic relations between ideals and z-filters in $C(X)$ are given in [17]. z-ideals, and also the term "z-filter," were introduced by Kohls in [20] (also see [21]). Recently this notion of z-ideals in rings has received a good deal of attention from several authors, for example [2, 3, 5, 6, 11, 22] and many others.

In Section 5, z-filters of L_τ are introduced by using the concept of zero elements. Some natural relations between ideals of $\mathcal{R}(L_\tau)$ and z-filters are explained. Also, we seek some relations among z-ultrafilters and maximal ideals of $\mathcal{R}(L_\tau)$.

In the ring $\mathcal{R}(L)$, the two notions of z-ideal (the algebraic one and the one defined in terms of the cozero map) agree [9, Corollary 3.8]. In [13], a particular case of z-ideals which are called *strongly z-ideals* of $\mathcal{R}(L)$, were defined by introducing zero sets in pointfree topology. The authors studied strongly z-ideals, their relation with z-ideals and the role of spatiality in this relation. For strongly z-ideals, they analyzed prime ideals using the concept of the zero set in Equation (1.1).

Finally, in the last section, z-ideals of $\mathcal{R}(L_\tau)$ are introduced by using the concept of zero elements. It is proved that for a z-ideal I, it is a prime ideal if and only if I contains a prime ideal, if and only if for all $g, h \in \mathcal{R}(L_\tau)$, if $gh = \mathbf{0}$, then $g \in I$ or $h \in I$, if and only if for every $f \in \mathcal{R}(L_\tau)$, there is a zero-element belonging to $z[I]$ on which f does not change sign (Theorem 6.8).

2. Background

A lattice-ordered ring is a ring A with a lattice structure such that for all $a, b, c \in A$,

$$(a \wedge b) + c = (a + c) \wedge (b + c)$$

or, equivalently,

$$(a \vee b) + c = (a + c) \vee (b + c)$$

and

$$0 \leq ab \text{ whenever } 0 \leq a \text{ and } 0 \leq b\,.$$

Further, with the definitions

$$a^+ = a \vee 0,\ a^- = (-a) \vee 0,\ |a| = a \vee (-a)$$

one has the rules

$$0 \leq |a|,\ |a| = a^+ + a^-,\ a = a^+ - a^-,\ a^+ \wedge a^- = 0,$$

$$|a + b| \leq |a| + |b|,\ |ab| \leq |a||b|.$$

An ℓ-ideal in a lattice-ordered ring A is a ring ideal J of A with the added property that $|x| \leq |a|$ and $a \in J$ implies $x \in J$, for any $x, a \in A$.

Now, an f-ring is a lattice-ordered ring A which satisfies any of the following equivalent conditions:

(1) $(a \wedge b)c = (ac) \wedge (bc)$ for any $a,\, b \in A$ and $c \geq 0$ in A.
(2) $|ab| = |a||b|$.

Recall that a *frame* is a complete lattice τ in which the distributive law

$$x \wedge \bigvee S = \bigvee \{x \wedge s \mid s \in S\}$$

holds for all $x \in \tau$ and $S \subseteq \tau$. We denote the top element and the bottom element of τ by $\top$ and $\bot$, respectively. A frame homomorphism is a map between frames which preserves finite meets, including the top element, and arbitrary joins, including the bottom element. The frame of open subsets of a topological space X is denoted by $\mathcal{O}(X)$. The power set of $\mathbb{R}$ is also denoted by $\mathcal{P}(\mathbb{R})$. Let τ be a frame. Recall (from [8]) that the frame of reals is the frame $\mathcal{L}(\mathbb{R})$ generated by all ordered pairs (p, q), with $p, q \in \mathbb{Q}$, subject to the following relations:

(R1) $(p, q) \wedge (r, s) = (p \vee r, q \wedge s)$.
(R2) $(p, q) \vee (r, s) = (p, s)$, whenever $p \leq r < q \leq s$.
(R3) $(p, q) = \bigvee \{(r, s) \mid p < r < s < q\}$.
(R4) $\top = \bigvee \{(p, q) \mid p, q \in \mathbb{Q}\}$.

The f-ring of all frame maps from $\mathcal{L}(\mathbb{R})$ to τ is denoted by $\mathcal{R}(\tau)$. Due to the fact that the map $j : \mathcal{L}(\mathbb{R}) \longrightarrow \mathcal{O}(\mathbb{R})$, taking any ordered pair (p, q) of $\mathcal{L}(\mathbb{R})$ to the open interval (p, q) of $\mathbb{R}$, is an isomorphism, and for simplicity of notation, we shall denote the set $\mathcal{R}(\tau)$ as the collection of all frame homomorphisms $f : \mathcal{O}(\mathbb{R}) \longrightarrow \tau$, and so $\mathcal{R}(\tau)$ with the operator $\diamond$ defined by

$$(f \diamond g)(U) = \bigvee \{f(U_1) \wedge g(U_2) \mid U_1 \diamond U_2 \subseteq U\},\ U \in \mathcal{O}(\mathbb{R}),$$

where

$$U_1 \diamond U_2 = \{a \diamond b \mid a \in U_1, b \in U_2\},\ \diamond \in \{+, \cdot, \wedge, \vee\},$$

is an f-ring (for more details see [7]). We abbreviate oft-mentioned members of $\mathcal{O}(\mathbb{R})$ as follows:

$$\mathbb{R}^+ = (0, +\infty),\ \ \mathbb{R}^- = (-\infty, 0)\,.$$

We use $b(U, \varepsilon)$ to designate the ε-ball $\bigvee_{r \in U}(r - \varepsilon, r + \varepsilon)$ about $U \in \mathbb{R}$, and in $O(\mathbb{R})$ we also use the notation $U \subseteq_\varepsilon V$ to mean that $b(U, \varepsilon) \subseteq V$.

Lemma 2.1 ([7])**.** *If $\alpha, \beta \in \mathcal{R}(\tau)$ and $\varepsilon > 0$ satisfy $|\alpha - \beta| \leq \varepsilon$, then $\alpha(U) \leq \beta(V)$ for all $U \subseteq_\varepsilon V$ in $\mathcal{O}(\mathbb{R})$.*

Lemma 2.2 ([7])**.** *Let $f, g \in \mathcal{R}(\tau)$. The following statements are equivalent.*

(1) $f \leq g$.
(2) $coz(g - f) = (g - f)(\mathbb{R}^+)$.
(3) $(g - f)(\mathbb{R}^-) = \bot$.
(4) $(g - f)(r, +\infty) = \top$ *for all* $r \in \mathbb{R}^-$.
(5) *For every* $r \in \mathbb{R}$, $f(r, +\infty) \leq g(r, +\infty)$.
(6) *For every* $r \in \mathbb{R}$, $f(-\infty, r) \geq g(-\infty, r)$.

[23] A topoframe, denoted by L_τ, on a frame L is a subframe τ all of whose elements are complementary elements in L. The member of τ are called the open elements of L. The set $\{p' \mid p \in \tau\}$ are called the closed elements of L and will be denoted by τ'.

Definition 2.3. If (L, τ) is a topoframe and $p \in L$, the *closure* of p in L is the element

$$\overline{p} := \bigwedge \{q \in \tau' \mid p \leq q\},$$

and the *interior* of any $p \in L$ is the element

$$p^\circ := \bigvee \{t \in \tau \mid t \leq p\}.$$

We showed in [14] that the ring of "real-continuous functions" $\mathcal{R}(L_\tau)$, consisting of all frame homomorphisms $f : P(\mathbb{R}) \longrightarrow L$ such that $f(\mathcal{O}(\mathbb{R})) \subseteq \tau$, with the operator $\diamond \in \{+, ., \wedge, \vee\}$ defined by

$$(f \diamond g)(X) = \bigvee \{f(Y) \wedge g(Z) \mid Y \diamond Z \subseteq X\},$$

where

$$Y \diamond Z = \{y \diamond z \mid y \in Y, z \in Z\},$$

or, equivalently,

$$(f \diamond g)(X) = \bigvee \{f(\{x\}) \wedge g(\{y\}) \mid x \diamond y \in X\} \tag{2.1}$$

is a sub f-ring of $\mathcal{R}(\tau)$.

Remark 2.4 ([14]). Let τ be a topoframe on a frame L. The frame map $\alpha \in \mathcal{R}\tau$ is called an L-*extendable* real continuous function if and only if for every $r \in \mathbb{R}$,

$$\bigvee_{r \in \mathbb{R}}^{L} (\alpha(-\infty, r) \vee \alpha(r, -\infty))' = \top,$$

or, equivalently,

$$\bigvee_{r \in \mathbb{R}}^{L} \bigwedge_{k \in \mathbb{N}}^{L} \alpha(r - \frac{1}{k}, r + \frac{1}{k}) = \top.$$

If α is L- extendable, the mapping e_α defined by

$$e_\alpha(S) = \bigvee_{x \in S}^{L} (\alpha(-\infty, x) \vee \alpha(x, -\infty))',$$

or, equivalently,

$$e_\alpha(S) = \bigvee_{x \in S}^{L} \bigwedge_{k \in \mathbb{N}}^{L} \alpha(x - \frac{1}{k}, x + \frac{1}{k})$$

is a frame homomorphism from $\mathcal{P}(\mathbb{R})$ into L which $e_\alpha |_{\mathcal{O}(\mathbb{R})} = \alpha$.

A real-valued function on a frame L is a frame homomorphism f from $\mathcal{P}(\mathbb{R})$ to L. The set of all this functions is denoted by $F_{\mathcal{P}}(L)$ and it is an f-ring with the operation defined in Equation (2.1) (see [19]). It is proved that $\mathcal{R}(L_\tau)$ is a sub f-ring of $F_{\mathcal{P}}(L)$ (see [14]).

3. z-elements in modified pointfree topology

In this section we shall introduce the notion of zero-element for the first time in topoframes and we shall discuss some of the basic relations that hold for zero-elements.

Definition 3.1. For every $f \in \mathcal{R}(L_\tau)$, $f(\{0\})$ is called a *zero-element* of f and is denoted by $z(f)$.

Any element in L that is a zero-element of some frame map in $\mathcal{R}(L_\tau)$ is called a zero-element in L_τ. Thus, z is a mapping from the ring $\mathcal{R}(L_\tau)$ onto the set of all zero-elements in L.

For $A \subseteq \mathcal{R}(L_\tau)$, we write $z[A]$ to designate the family of zero-elements $\{z(f) \mid f \in A\}$. This is consistent with our notational convention for the image of a set under a mapping. On the other hand, the family $z[\mathcal{R}(L_\tau)]$ of all zero-elements in L will also be denoted, for simplicity, by $z[L_\tau]$.

Also a *cozero-element* of L_τ is defined by

$$coz(f) := f(-\infty, 0) \vee f(0, +\infty)$$

for some $f \in \mathcal{R}(L_\tau)$. Obviously, $z(f) = (coz(f))'$.

Remark 3.2. For any $f \in \mathcal{R}(L_\tau)$, $f\mid_{\mathcal{O}(\mathbb{R})} \in \mathcal{R}(\tau)$ because its range is included in τ. Trivially, $coz(f) = coz(f\mid_{\mathcal{O}(\mathbb{R})})$, whence for every $f \in \mathcal{R}(L_\tau)$, we have

$$z(f) = (coz(\alpha))'$$

for some $\alpha \in \mathcal{R}(\tau)$.

Remark 3.3. (1) **Constant real functions in** $\mathcal{R}(L_\tau)$: for each $c \in \mathbb{R}$, let $\mathbf{c}$ be defined by

$$\mathbf{c}(X) = \begin{cases} \top_L & \text{if } c \in X\,, \\ \bot_L & \text{if } c \notin X\,, \end{cases}$$

where $X \in \mathcal{P}(\mathbb{R})$. It is clear that $\mathbf{c} \in \mathcal{R}(L_\tau)$. Also, $f + \mathbf{0} = f$ and $f\mathbf{1} = f$ for every $f \in \mathcal{R}(L_\tau)$. Now it is always true that $z(\mathbf{c}) = \bot$, if $c \in \mathbb{R} \setminus \{0\}$ and $z(\mathbf{0}) = \top$.

(2) **Product with a scalar:** for $r \in \mathbb{R}$ and $f \in \mathcal{R}(L_\tau)$ define

$$r.f(X) = \begin{cases} \mathbf{0}(X) & \text{if } r = 0, \\ f(\frac{1}{r}X) & \text{if } r \neq 0, \end{cases}$$

for every $X \in \mathcal{P}(\mathbb{R})$, where $\frac{1}{r}X = \{\frac{x}{r} \mid x \in X\}$. It is straightforward to check that $r.f = \mathbf{r}f$, for every $r \in \mathbb{R}$ and $f \in \mathcal{R}(L_\tau)$.

It is also true that $z(r.f) = z(f) = z(\mathbf{r}f)$, for all $r \in \mathbb{R} \setminus \{0\}$; furthermore $z(0.f) = \top = z(\mathbf{0}f)$.

The partial ordering on $\mathcal{R}(L_\tau)$ is to be interpreted as follows.

Proposition 3.4. *For $f, g \in \mathcal{R}(L_\tau)$, the following statements are equivalent.*

(1) $f \leq g$.
(2) $f \mid_{\mathcal{O}(\mathbb{R})} \leq g \mid_{\mathcal{O}(\mathbb{R})}$.
(3) *For every* $r \in \mathbb{R}$, $f[r, +\infty) \leq g[r, +\infty)$.
(4) *For every* $r \in \mathbb{R}$, $f(-\infty, r] \geq g(-\infty, r]$.
(5) *For every* $r \in \mathbb{R}$, $f(r, +\infty) \leq g(r, +\infty)$.
(6) *For every* $r \in \mathbb{R}$, $f(-\infty, r) \geq g(-\infty, r)$.
(7) $coz(g - f) = (g - f)(\mathbb{R}^+)$.
(8) $(g - f)(\mathbb{R}^-) = \perp$.
(9) $(g - f)(r, +\infty) = \top$ *for all* $r \in \mathbb{R}^-$.

Proof. The conditions (1) and (2) are equivalent, because the assignment $f \rightsquigarrow f \mid_{\mathcal{O}(\mathbb{R})}$ from $\mathcal{R}(L_\tau)$ to $\mathcal{R}(\tau)$-or $\mathcal{R}(L)$ is an f-ring monomorphism (see [14]). The conditions (2), $(5 - 9)$ are equivalent, by Lemma 2.2. The conditions (3) and (6) and also (4) and (5) are equivalent, since $f(A)' = f(\mathbb{R} \setminus A)$ for every $f \in \mathcal{R}(L_\tau)$ and $A \subseteq \mathbb{R}$. □

Corollary 3.5. *For $f \in \mathcal{R}(L_\tau)$, the following statements are equivalent.*

(1) $f \geq \mathbf{0}$.
(2) $f \mid_{\mathcal{O}(\mathbb{R})} \geq \mathbf{0}$.
(3) $coz(f) = f(\mathbb{R}^+)$.
(4) $f(\mathbb{R}^-) = \perp$.
(5) $f(r, +\infty) = \top$ *for all* $r \in \mathbb{R}^-$.

Lemma 3.6. *Let $f, g \in \mathcal{R}(L_\tau)$ and $r \in \mathbb{R}$. Then*

(1) $(f \wedge g)(\{r\}) = (f(\{r\}) \wedge g[r, +\infty)) \vee (f[r, +\infty) \wedge g(\{r\}))$.
In particular,

$$(f \wedge g)[r, +\infty) = f[r, +\infty) \wedge g[r, +\infty),\ (f \wedge g)(r, +\infty) = f(r, +\infty) \wedge g(r, +\infty),$$

$$(f \wedge g)(-\infty, r] = f(-\infty, r] \vee g(-\infty, r],\ (f \wedge g)(-\infty, r) = f(-\infty, r) \vee g(-\infty, r),$$

and

$$(f \wedge \mathbf{0})(\{r\}) \quad = \quad \begin{cases} \perp & \text{if } r > 0, \\ f[0, +\infty) & \text{if } r = 0, \\ f\{r\} & \text{if } r < 0. \end{cases}$$

Also, if $f, g \geq 0$, then $(f \wedge g)\{0\} = f\{0\} \vee g\{0\}$.

(2) $(f \vee g)(\{r\}) = (f(\{r\}) \wedge g(-\infty, r]) \vee (f(-\infty, r] \wedge g(\{r\}))$.
In particular,

$$(f \vee g)[r, +\infty) = f[r, +\infty) \vee g[r, +\infty),\ (f \vee g)(r, +\infty) = f(r, +\infty) \vee g(r, +\infty),$$

$(f \vee g)(-\infty, r] = f(-\infty, r] \wedge g(-\infty, r]$, $(f \vee g)(-\infty, r) = f(-\infty, r) \wedge g(-\infty, r)$, *and*

$$(f \vee \mathbf{0})(\{r\}) = \begin{cases} \perp & \textit{if } \ r < 0, \\ f(-\infty, 0] & \textit{if } \ r = 0, \\ f\{r\} & \textit{if } \ r > 0\,. \end{cases}$$

(3) $(fg)[0, +\infty) = (f[0, +\infty) \wedge g[0, +\infty)) \vee (f(-\infty, 0] \wedge g(-\infty, 0])\,.$

(4) *For every* $c \in \mathbb{R}$*, we have* $(f - \mathbf{c})[0, +\infty) = f[c, +\infty)\,.$
In particular, $((f - \mathbf{c}) \wedge \mathbf{0})(\{0\}) = f[c, +\infty)\,.$

(5) *For every* $c \in \mathbb{R}$*, we have* $(f - \mathbf{c})(-\infty, 0] = f(-\infty, c]\,.$
In particluar, $((f - \mathbf{c}) \vee \mathbf{0})(\{0\}) = f(-\infty, c]\,.$

Proof. For $f, g \in \mathcal{R}(L_\tau)$ and $r \in \mathbb{R}$,

(1) By the definition of meet in $\mathcal{R}(L_\tau)$, we have

$$\begin{aligned} (f \wedge g)(\{r\}) &= \bigvee\{f(\{x\}) \wedge g(\{y\}) \mid min\{x, y\} = r\} \\ &= (f(\{r\}) \wedge g[r, +\infty)) \vee (f[r, +\infty) \wedge g(\{r\})). \end{aligned}$$

In particluar,

$$(f \wedge g)[r, +\infty) = \bigvee\{f(\{x\}) \wedge g(\{y\}) \mid min\{x, y\} \geq r\} = f[r, +\infty) \wedge g[r, +\infty)\,,$$

$$\begin{aligned} (f \wedge g)(-\infty, r] &= \bigvee\{f(\{x\}) \wedge g(\{y\}) \mid min\{x, y\} \leq r\} \\ &= (f(\mathbb{R}) \wedge g(-\infty, r]) \vee (f(-\infty, r] \wedge g(\mathbb{R})) \\ &= f(-\infty, r] \vee g(-\infty, r] \end{aligned}$$

and

$$\begin{aligned} (f \wedge \mathbf{0})(\{r\}) &= f(\{r\}) \wedge \mathbf{0}[r, +\infty)) \vee (f[r, +\infty) \wedge \mathbf{0}(\{r\})) \\ &= \begin{cases} \perp & \text{if } \ r > 0, \\ f[0, +\infty) & \text{if } \ r = 0, \\ f\{r\} & \text{if } \ r < 0. \end{cases} \end{aligned}$$

Also, while $f, g \geq 0$

$$\begin{aligned} (f \wedge g)\{0\} &= \bigvee\{f\{x\} \wedge g\{y\} \mid min\{x, y\} = 0\} \\ &= (f\{0\} \wedge g[0, +\infty)) \vee (f[0, +\infty) \wedge g\{0\}) \\ &= (f\{0\} \wedge \top) \vee (\top \wedge g\{0\}) \\ &= f\{0\} \vee g\{0\}. \end{aligned}$$

(2) By the definition of join in $\mathcal{R}(L_\tau)$, we have

$$\begin{aligned} (f \vee g)(\{r\}) &= \bigvee\{f(\{x\}) \wedge g(\{y\}) \mid max\{x, y\} = r\} \\ &= (f(\{r\}) \wedge g(-\infty, r]) \vee (f(-\infty, r] \wedge g(\{r\})). \end{aligned}$$

In particluar,

$$\begin{aligned}(f\vee g)[r,+\infty) &= \bigvee\{f(\{x\})\wedge g(\{y\})\mid max\{x,y\}\geq r\}\\ &= (f(\mathbb{R})\wedge g[r,+\infty))\vee(f[r,+\infty)\wedge g(\mathbb{R}))\\ &= f[r,+\infty)\vee g[r,+\infty)\,,\end{aligned}$$

$$\begin{aligned}(f\vee g)(-\infty,r] &= \bigvee\{f(\{x\})\wedge g(\{y\})\mid max\{x,y\}\leq r\}\\ &= f(-\infty,r]\wedge g(-\infty,r]\end{aligned}$$

and

$$(f\vee\mathbf{0})(\{r\}) = \begin{cases}\perp & \text{if } \ r<0,\\ f(-\infty,0] & \text{if } \ r=0,\\ f\{r\} & \text{if } \ r>0\,.\end{cases}$$

(3) By the definition of product in $\mathcal{R}(L_\tau)$, we have

$$\begin{aligned}(fg)[0,+\infty) &= \bigvee\{f(\{x\})\wedge g(\{y\})\mid xy\geq 0\}\\ &= \bigvee\{f(\{x\})\wedge g(\{y\})\mid x\geq 0,\ y\geq 0\}\\ &\quad \vee\bigvee\{f(\{x\})\wedge g(\{y\})\mid x\leq 0,\ y\leq 0\}\\ &= (f[0,+\infty)\wedge g[0,+\infty))\vee(f(-\infty,0]\wedge g(-\infty,0]).\end{aligned}$$

(4) For any $c\in\mathbb{R}$, By definition of sum in $\mathcal{R}(L_\tau)$, we have

$$\begin{aligned}(f-\mathbf{c})[0,+\infty) &= \bigvee\{f(\{x\})\wedge -\mathbf{c}(\{y\})\mid x+y\in[0,+\infty)\}\\ &= \bigvee\{f(\{x\})\wedge \mathbf{c}(\{-y\})\mid x+y\geq 0\}\\ &= \bigvee\{f(\{x\})\wedge \mathbf{c}(\{-y\})\mid x\geq -y\}\\ &= \bigvee\{f(\{x\})\wedge \top\mid x\geq c\}\\ &= f[c,+\infty)\,.\end{aligned}$$

Particularly, by statement (1), we have

$$((f-\mathbf{c})\wedge\mathbf{0})(\{0\}) = (f-\mathbf{c})[0,+\infty) = f[c,+\infty)\,.$$

(5) For every $c\in\mathbb{R}$,

$$\begin{aligned}(f-\mathbf{c})(-\infty,0] &= \bigvee\{f(\{x\})\wedge -\mathbf{c}(\{y\})\mid x+y\in(-\infty,0]\}\\ &= \bigvee\{f(\{x\})\wedge \mathbf{c}(\{-y\})\mid x+y\leq 0\}\\ &= \bigvee\{f(\{x\})\wedge \mathbf{c}(\{-y\})\mid x\leq -y\}\\ &= \bigvee\{f(\{x\})\wedge \top\mid x\leq c\}\\ &= f(-\infty,c].\end{aligned}$$

Particularly, by statement (2), we have

$$((f-\mathbf{c})\vee\mathbf{0})(\{0\}) = (f-\mathbf{c})(-\infty,0] = f(-\infty,c].$$

□

Proposition 3.7. *For every $f,g\in\mathcal{R}(L_\tau)$, we have*

(1) *$z(f+g)=z(f)\wedge z(g)$, while $f,g\geq \mathbf{0}$.*
(2) *$z(f\wedge g)=z(f)\vee z(g)$, while $f,g\geq \mathbf{0}$.*
(3) *If $\mathbf{0}\leq f\leq g$, then $z(f)\geq z(g)$.*
(4) *For every $n\in\mathbb{N}$, $z(f)=z(-f)=z(|f|)=z(f^n)$.*
(5) *$z(fg)=z(f)\vee z(g)$.*
(6) *$z(f+g)\geq z(f)\wedge z(g)$.*
(7) *$z(f)\wedge z(g)=z(|f|+|g|)=z(f^2+g^2)$.*
(8) *$z(\mathbf{1})=\perp$ and moreover, $z(f)=\top$ if and only if $f=\mathbf{0}$.*
(9) *For every $n\in\mathbb{N}$, $z(f)=z(|f|\wedge \frac{\mathbf{1}}{\mathbf{n}})$.*
(10) *For every $c,c_1,c_2\in\mathbb{R}$ and $f\in\mathcal{R}(L_\tau)$, we have $z(f-\mathbf{c})=f(\{c\})$.*
(11) *$z((f-\mathbf{c})^+)=f(-\infty,c]$ and $z((f-\mathbf{c})^-)=z((\mathbf{c}-f)^+)=f[c,+\infty)$.*
(12) *$z(f^+)=f(-\infty,0]$ and $z(f^-)=f[0,+\infty)$.*
(13) *$z((f-\mathbf{c_1})^+\wedge(\mathbf{c_2}-f)^+)=f((-\infty,c_1]\cup[c_2,+\infty))$.*
(14) *For every $f,g\geq\mathbf{0}$,*

$$z(f)=\bigwedge_{n\in\mathbb{N}}^{L} z((\mathbf{n}f-g)^+)\,.$$

Proof. Some straightforward computations involving Lemma 3.6 and Equation (2.1) yield all relations in $\mathcal{R}(L_\tau)$; however, we prove the last assertion. First, note that $h\leq k$ implies $h^+\leq k^+$. Thus, for any $f,g\geq\mathbf{0}$

$$\begin{array}{rcll}
\mathbf{n}f\leq \mathbf{n}f+g & \Rightarrow & \mathbf{n}f-g\leq \mathbf{n}f & \\
& \Rightarrow & (\mathbf{n}f-g)^+\leq(\mathbf{n}f)^+=\mathbf{n}f & \\
& \Rightarrow & z(\mathbf{n}f-g)^+\geq z(\mathbf{n}f) & \\
& \Rightarrow & coz(\mathbf{n}f-g)^+\leq coz(\mathbf{n}f) & \\
& \Rightarrow & coz(\mathbf{n}f-g)^+\leq coz(f) & \text{since } coz(\mathbf{n}f)=coz(f).
\end{array}$$

Hence $\bigvee_{n\in\mathbb{N}} coz(\mathbf{n}f-g)^+\leq coz(f)$.
For the reverse relation, note that

$$\begin{array}{rcll}
coz(\mathbf{n}f-g)^+ & = & (\mathbf{n}f-g)^+(0,+\infty) & \text{by Corollary 3.5}\\
& = & (\mathbf{n}f-g)(0,+\infty) & \text{by 3.6(2)}\\
& = & \mathbf{n}(f+\frac{\mathbf{-1}}{\mathbf{n}}g)(0,+\infty) & \\
& = & (f+\frac{\mathbf{-1}}{\mathbf{n}}g)(0,+\infty) & \text{using scalar product}\\
& \geq & \bigvee_{r>0}\{f(r,+\infty)\wedge\frac{\mathbf{-1}}{\mathbf{n}}g(-r,+\infty)\} & \\
& = & \bigvee_{r>0}\{f(r,+\infty)\wedge g(-\infty,nr)\} & \text{using scalar product}
\end{array}$$

and since, for any $r>0$ in $\mathbb{R}$, $\bigvee_{n\in\mathbb{N}} g(-\infty,nr)=g(\mathbb{R})=\top$, we conclude that

$$\begin{array}{rcl} \bigvee_{n\in\mathbb{N}} coz(\mathbf{n}f-g)^{+} & \geq & \bigvee_{n\in\mathbb{N}}\bigvee_{r>0}\{f(r,+\infty)\wedge g(-\infty,nr)\} \\ & = & \bigvee_{r>0}\bigvee_{n\in\mathbb{N}}\{f(r,+\infty)\wedge g(-\infty,nr)\} \\ & = & \bigvee_{r>0}\{f(r,+\infty)\wedge\top\} \\ & = & f(0,+\infty) \\ & = & coz(f) \qquad \text{by Corollary 3.5}. \end{array}$$

Consequently,

$$coz(f)=\bigvee_{n\in\mathbb{N}} coz(\mathbf{n}f-g)^{+}$$

and so

$$z(f)=\bigwedge_{n\in\mathbb{N}} z((\mathbf{n}f-g)^{+}),$$

by complementation. □

Theorem 3.8. *A frame map f is a unit of $\mathcal{R}(L_\tau)$ if and only if $z(f)=\perp$.*

Proof. Suppose that f is a unit of $\mathcal{R}(L_\tau)$, then there exists $g\in\mathcal{R}(L_\tau)$ such that $fg=\mathbf{1}$. So by Proposition 3.7, $\perp=z(1)=z(fg)=z(f)\vee z(g)$, and hence $z(f)=\perp$.

Conversely, assume that $f\in\mathcal{R}(L_\tau)$ and $z(f)=\perp$. Define

$$g(X):=\bigvee\{f(\frac{1}{x})\mid x\in X-\{0\}\}.$$

We show that g belongs to $\mathcal{R}(L_\tau)$ which is the multiplicative inverse of f in $\mathcal{R}(L_\tau)$. The proof consists of five steps to check:
step 1. The first step is verifying that $g(\mathbb{R})=\top$. Since $f\{0\}=\perp$, we have

$$\begin{array}{rcl} g(\mathbb{R}) & = & \bigvee\{f(\frac{1}{x})\mid x\in\mathbb{R}-\{0\}\} \\ & = & \perp\vee\bigvee\{f(\frac{1}{x})\mid x\in\mathbb{R}-\{0\}\} \\ & = & f\{0\}\vee\bigvee\{f(\frac{1}{x})\mid x\in\mathbb{R}-\{0\}\} \\ & = & f(\mathbb{R}) \\ & = & \top. \end{array}$$

step 2. Let $\{X_i\}_{i\in I}\subseteq\mathcal{P}(\mathbb{R})$. If for all i, $X_i=\emptyset$ or $\{0\}$, then obviously,

$$g(\bigcup_{i\in I} X_i)=\perp=\bigvee_{i\in I} g(X_i),$$

or else there is an i which $X_i\neq\emptyset,\{0\}$, then

$$\begin{array}{rcl} g(\bigcup_{i\in I} X_i) & = & \bigvee\{f(\{\frac{1}{x}\})\mid x\in(\bigcup_{i\in I} X_i)-\{0\}\} \\ & = & \bigvee\{f(\{\frac{1}{x}\})\mid x\in\bigcup_{i\in I}(X_i-\{0\})\} \\ & = & \bigvee_{i\in I}\bigvee\{f(\{\frac{1}{x}\})\mid x\in X_i-\{0\}\} \\ & = & \bigvee_{i\in I} g(X_i). \end{array}$$

step 3. Let $X, Y \in P(\mathbb{R})$. If $X, Y \in \{\emptyset, \{0\}\}$, then obviously,

$$g(X \cap Y) = \bot = g(X) \wedge g(Y),$$

or else we have

$$\begin{aligned} g(X \cap Y) &= \bigvee\{f(\{\tfrac{1}{x}\}) \mid x \in (X \cap Y) - \{0\}\} \\ &= \bigvee\{f(\{\tfrac{1}{x}\}) \mid x \in (X - \{0\}) \cap (Y - \{0\})\} \\ &= \bigvee\{f(\{\tfrac{1}{x}\}) \wedge f(\{\tfrac{1}{y}\}) \mid x \in X - \{0\}, y \in Y - \{0\}\} \\ &= \bigvee\{f(\{\tfrac{1}{x}\}) \mid x \in X - \{0\}\} \wedge \bigvee\{f(\{\tfrac{1}{y}\}) \mid y \in Y - \{0\}\} \\ &= g(X) \wedge g(Y)\,. \end{aligned}$$

step 4. We must also prove that g is also continuous. Let (a, b) be an open interval in $\mathbb{R}$.

(1) if $0 \notin (a, b)$ and $a < b < 0$ or $0 < a < b$, then

$$\begin{aligned} g(a,b) &= \bigvee\{f(\{\tfrac{1}{x}\}) \mid x \in (a,b) - \{0\}\} \\ &= \bigvee\{f(\{\tfrac{1}{x}\}) \mid x \in (a,b)\} \\ &= \bigvee\{f(\{\tfrac{1}{x}\}) \mid a < x < b)\} \\ &= \bigvee\{f(\{\tfrac{1}{x}\}) \mid \tfrac{1}{b} < \tfrac{1}{x} < \tfrac{1}{a})\} \\ &= \bigvee\{f(\{\lambda\}) \mid \tfrac{1}{b} < \lambda < \tfrac{1}{a})\} \\ &= f(\tfrac{1}{b}, \tfrac{1}{a}) \end{aligned}$$

and hence g assigns any open set of $\mathbb{R}$ to an open element of L, since f does.

(2) Now, if $0 \in (a, b)$, then

$$\begin{aligned} g(a,b) &= g(a,0) \vee g(\{0\}) \vee g(0,b) \\ &= f(-\infty, \tfrac{1}{a}) \vee \bot \vee f(\tfrac{1}{b}, +\infty) \\ &= f(-\infty, \tfrac{1}{a}) \vee f(\tfrac{1}{b}, +\infty)\,, \end{aligned}$$

so that g assigns an open set to an open element in L.

step 5. In the last step, we show that $fg = \mathbf{1}$.

$$\begin{aligned} (fg)(\{1\}) &= \bigvee\{f(\{x\}) \wedge g(\{y\}) \mid xy = 1\} \\ &= \bigvee\{f(\{x\}) \wedge g(\{\tfrac{1}{x}\}) \mid 0 \neq x \in \mathbb{R}\} \\ &= \bigvee\{f(\{x\}) \wedge f(\{x\}) \mid 0 \neq x \in \mathbb{R}\} \\ &= \bigvee\{f(\{x\}) \mid 0 \neq x \in \mathbb{R}\} \\ &= f(\{0\}) \bigvee\{f(\{x\}) \mid 0 \neq x \in \mathbb{R}\} \\ &= f(\mathbb{R}) \\ &= \top. \end{aligned}$$

and

$$\begin{aligned}(fg)(\{0\}) &= \bigvee\{f(\{x\})\wedge g(\{y\}) \mid xy=0\}\\ &= \bigvee\{f(\{x\})\wedge g(\{y\}) \mid x=0\} \vee \bigvee\{f(\{x\})\wedge g(\{y\}) \mid y=0\}\\ &= \bigvee\{\perp \wedge g(\{y\})\} \vee \bigvee\{f(\{x\})\wedge \perp\}\\ &= \perp.\end{aligned}$$

Also, if $r \neq 0,1$, then

$$\begin{aligned}(fg)(\{r\}) &= \bigvee\{f(\{x\})\wedge g(\{y\}) \mid xy=r\}\\ &= \bigvee\{f(\{x\})\wedge g(\{\tfrac{r}{x}\}) \mid 0\neq x\in\mathbb{R}\}\\ &= \bigvee\{f(\{x\})\wedge f(\{\tfrac{x}{r}\}) \mid x\neq 0\}\\ &= \bigvee\{f(\emptyset) \mid x\neq 0\}\\ &= \perp\end{aligned}$$

and thus $fg = \mathbf{1}$. This completes the proof. □

Corollary 3.9. *Any $f \geq \mathbf{1}$ in $\mathcal{R}(L_\tau)$ has an inverse.*

Proof. Since $\mathbf{0} \leq \mathbf{1} \leq f$, we conclude that $\perp = z(\mathbf{1}) \geq z(f)$, by statement (3) of Proposition 3.7, and hence f has an inverse in $\mathcal{R}(L_\tau)$, by Theorem 3.8. □

Remark 3.10. $f \in \mathcal{R}(L_\tau)$ is called *bounded topoframe map*, if there exists $n \in \mathbb{N}$ such that $|f| < \mathbf{n}$. Put

$$\mathcal{R}^*(L_\tau) := \{f \in \mathcal{R}(L_\tau) \mid \ f \text{ is a bounded topoframe map}\}.$$

Then $z[\mathcal{R}^*(L_\tau)] = z[\mathcal{R}(L_\tau)]$, by statement (9) of Proposition 3.7.

4. The uniform completeness of $\mathcal{R}(L_\tau)$

Recall that a sequence $\{a_n\}_{n\in\mathbb{N}}$ in an f-ring A converges uniformly to $a \in A$, written $a_n \to a$, provided that for any $k \in \mathbb{N}$, there is an $m \in \mathbb{N}$ such that

$$|a_n - a| < \frac{1}{k},$$

for all $n \in \mathbb{N}$ with $n \geq m$. Furthermore, in an f-ring A, a sequence $\{a_n\}_{n\in\mathbb{N}} \subseteq A$ is called Cauchy if for any $k \in \mathbb{N}$ there is an $m \in \mathbb{N}$ such that

$$|a_i - a_j| < \frac{1}{k}$$

for all $i, j \in \mathbb{N}$ with $i, j \geq m$. The f-ring A is said to be uniformly complete if every Cauchy sequence in A converges uniformly to a limit in A. Because $\mathcal{R}(L)$ is uniformly complete (see [7]), it may not be so much surprising that $\mathcal{R}(L_\tau)$ is uniformly complete. This fact appears in the following theorem.

Theorem 4.1. *The f-ring $\mathcal{R}(L_\tau)$ is uniformly complete.*

Proof. Suppose that $\{f_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{R}(L_\tau)$ and take $\alpha_n := f_n\mid_{\mathcal{O}(\mathbb{R})}$. Since $\mathcal{R}(\tau)$ is complete in its uniform topology, $\{\alpha_n\}_{n\in\mathbb{N}} \subseteq \mathcal{R}(L)$ converges to a real continuous function $\alpha \in \mathcal{R}(L)$. Hence for any $k \in \mathbb{N}$, there exists an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$,

$$|\alpha_n - \alpha| \leq \frac{1}{2k}\,,$$

and then Lemma 2.1 gives the inequality $\alpha_n(U) \leq \alpha(V)$ for all $U \subseteq_{\frac{1}{2k}} V$ in $\mathcal{O}(\mathbb{R})$; particularly, for all $r \in \mathbb{R}$,

$$\alpha_n(r - \frac{1}{2k}, r + \frac{1}{2k}) \leq \alpha(r - \frac{1}{k}, r + \frac{1}{k})$$

(because $(r - \frac{1}{2k}, r + \frac{1}{2k}) \subseteq_{\frac{1}{2k}} (r - \frac{1}{k}, r + \frac{1}{k})$). Thus

$$\top = \bigvee_{r\in\mathbb{R}}^{L} \bigwedge_{k\in\mathbb{N}}^{L} \alpha_n(r - \frac{1}{2k}, r + \frac{1}{2k}) \leq \bigvee_{r\in\mathbb{R}}^{L} \bigwedge_{k\in\mathbb{N}}^{L} \alpha(r - \frac{1}{k}, r + \frac{1}{k})\,,$$

because every α_n is L-extendable, and hence α is L-extendable. Consequently, there exists an $f \in \mathcal{R}(L_\tau)$ such that $f\mid_{\mathcal{O}(\mathbb{R})} = \alpha$, by Remark 2.4. So for any $k \in \mathbb{N}$, there exists an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ and $r \in \mathbb{R}$,

$$|f_n - f|(\{r\}) \leq |f_n - f|(p, q) = |\alpha_n - \alpha|(p, q) \leq \frac{1}{2k}(p, q)\,,$$

where $p, q \in \mathbb{Q}$ with $\frac{1}{2k} < p < r < q$ or $p < r < q < \frac{1}{2k}$ or $p < r = \frac{1}{2k} < q$. This completes the proof. $\square$

Corollary 4.2. *Let $\{f_n\}_{n\in\mathbb{N}} \subseteq \mathcal{R}(L_\tau)$. Then there is an $f \in \mathcal{R}(L_\tau)$ such that*

$$\bigwedge_{n=1}^{\infty} z(f_n) = z(f)\,.$$

Proof. Take $\phi_n := |f_n| \wedge \mathbf{2^{-n}}$. Then, by the proof of Theorem 4.1, the Cauchy sequence $\{\phi_1 + \cdots + \phi_n \mid n \in \mathbb{N}\}$ has a supremum f in the poset $\mathcal{R}(L_\tau)$ and the Cauchy sequence $\{\phi_1\mid_{\mathcal{O}(\mathbb{R})} + \cdots + \phi_n\mid_{\mathcal{O}(\mathbb{R})} \mid n \in \mathbb{N}\}$ has the supremum $f\mid_{\mathcal{O}(\mathbb{R})}$ in the poset $\mathcal{R}(\tau)$. Thus

$$\begin{aligned}
\bigvee\nolimits_{n\in\mathbb{N}} coz(f_n) &= \bigvee\nolimits_{n\in\mathbb{N}} coz(\phi_n)\\
&= \bigvee\nolimits_{n\in\mathbb{N}} coz(\phi_n\mid_{\mathcal{O}(\mathbb{R})})\\
&= coz(f\mid_{\mathcal{O}(\mathbb{R})}) \qquad \text{by the proof of [10, Proposition 3.6]}\\
&= coz(f)\,,
\end{aligned}$$

so that $\bigwedge_{n\in\mathbb{N}} z(f_n) = z(f)$ as desired. $\square$

5. ideals and z-filters in modified pointfree topology

The join of two zero elements of L_τ is also a zero element and so is the meet of two zero elements, by Theorem 3.7. This means that $z[L_\tau]$ is a sub-lattice of L. Continuing our study of the relations between algebraic properties of $\mathcal{R}(L_\tau)$ and lattice properties of L_τ, we now examine the special features of a family of zero-elements in $z[L_\tau]$.

Definition 5.1. A proper filter of $z[L_\tau]$ is called a *z-filter* on L_τ. Therefore, if $\mathcal{F}$ is a z-filter on L_τ, then

(i) $\perp \notin \mathcal{F} \subseteq z[L_\tau]$ and $\top \in \mathcal{F}$,
(ii) for every $a, b \in \mathcal{F}$, there exists $\perp \neq c \in \mathcal{F}$ that $c \leq a \wedge b$, and
(iii) if $b \in \mathcal{F}$, $a \in L$, and $b \leq a$, then $a \in \mathcal{F}$.

Let $a \in A$ and $\mathcal{F} \subseteq z[L_\tau]$. We say a meets $\mathcal{F}$ if and only if $a \wedge b \neq \perp$, for all $b \in \mathcal{F}$. A *z-ultrafilter* $\mathcal{F}$ on L_τ is maximal element of collection of all z-filters on L_τ with inclusion relation. It is evident that:

(1) A z-filter $\mathcal{F}$ on L_τ is a z-ultrafilter if and only if a meets $\mathcal{F}$ implies $a \in \mathcal{F}$, for every $a \in z[L_\tau]$.
(2) If $\mathcal{F}$ and $\mathcal{G}$ are disjoint z-ultrafilter on L_τ, then there is elements $a \in \mathcal{F}$ and $b \in \mathcal{G}$ such that $a \wedge b = \perp$.
(3) If $\{\mathcal{F}_i\}_{i \in I}$ is a nonempty collection of z-filters on L_τ, then $\bigcap_{i \in I} \mathcal{F}_i$ is a z-filter on L_τ.
(4) Every z-filter on L_τ is contained in a z-ultrafilter on L_τ .

Proposition 5.2. *In $\mathcal{R}(L_\tau)$, the following statements hold.*

(1) *If I is a proper ideal in $\mathcal{R}(L_\tau)$, then the family*
$$z[I] = \{z(f) \mid f \in I\}$$
is a z-filter on L_τ.

(2) *If $\mathcal{F}$ is a z-filter on L_τ, then the family*
$$z^{-1}[\mathcal{F}] = \{f \mid z(f) \in \mathcal{F}\}$$
is a proper ideal in $\mathcal{R}(L_\tau)$.

Proof. (1) It suffices to check that conditions (i)-(iii) of Definition 5.1 hold for $z[I]$.
(i) Since I contains no unit, we conclude from Theorem 3.8 that $\perp \notin z[I]$.
(ii) Let $z_1, z_2 \in z[I]$. Then there exist $f, g \in I$ such that $z_1 = z(f)$ and $z_2 = z(g)$. Since I is an ideal, $f^2 + g^2 \in I$. Hence
$$z_1 \wedge z_2 = z(f^2 + g^2) \in z[I].$$
(iii) Let $z \in z[I]$, $\zeta \in z[L_\tau]$ with $z \leq \zeta$ and let $f \in I$ and $g \in \mathcal{R}(L_\tau)$ satisfy $z = z(f), \zeta = z(g)$. Since I is an ideal, we have $fg \in I$. Hence if $z \leq \zeta$, then
$$\zeta = z \vee \zeta = z(fg) \in z[I].$$

(2) Let $J = z^{-1}[\mathcal{F}]$. By Definition 5.1 and Theorem 3.8, J contains no unit. Let $f, g \in J$, and let $h \in \mathcal{R}(L_\tau)$. Then

$$z(f-g) = z(f+(-g)) \geq z(f) \wedge z(-g)) \geq z(f) \wedge z(g) \in \mathcal{F}$$

and hence $z(f-g) \in \mathcal{F}$, by Definition 5.1. Therefore $f-g \in z^{-1}[\mathcal{F}]$. Moreover,

$$z(hf) = z(h) \vee z(h) \geq z(f) \in \mathcal{F},$$

and hence $z(fh) \in \mathcal{F}$, by Definition 5.1. Therefore $fh \in z^{-1}[\mathcal{F}]$. This completes the proof that J is a proper ideal in $\mathcal{R}(L_\tau)$. □

Proposition 5.3. *In $\mathcal{R}(L_\tau)$, the following statements hold.*

(1) *If M is a maximal ideal in $\mathcal{R}(L_\tau)$, then $z[M]$ is a z-ultrafilter on L_τ.*
(2) *If $\mathcal{F}$ is a z-ultrafilter on L_τ, then $z^{-1}[\mathcal{F}]$ is a maximal ideal in $\mathcal{R}(L_\tau)$.*
The mapping z is one-one from the set of all maximal ideals in $\mathcal{R}(L_\tau)$ onto the set of all z-ultrafilters on L_τ.

Proof. Since z and z^{-1} preserve inclusion, the result follows at once from Proposition 5.2. □

Proposition 5.4. *For any $f, g \in \mathcal{R}(L_\tau)$, the following statements are equivalent.*

(1) $\langle f, g\rangle \neq \mathcal{R}(L_\tau)$.
(2) $z(f) \wedge z(g) \neq \perp$.
(3) *$f^2 + g^2$ and $|f| + |g|$ are not units of $\mathcal{R}(L_\tau)$.*

Proof. (1) implies (3): Since $f^2 + g^2 \in \langle f, g\rangle \neq \mathcal{R}(L_\tau)$, we infer that $f^2 + g^2$ is not unit and hence $|f| + |g|$ is not unit too, because $z(f^2 + g^2) = z(|f| + |g|)$.

(3) implies (2): Assume that $z(f) \wedge z(g) = \perp$, consequently

$$z(f^2 + g^2) = z(|f| + |g|) = z(f) \wedge z(g) = \perp,$$

so that $f^2 + g^2$ and $|f| + |g|$ are unit which is in contradiction to the condition (3).

(2) implies (1): Let $h \in \langle f, g\rangle$. Then there exist $h_1, h_2 \in \mathcal{R}(L_\tau)$ such that $h = h_1 f + h_2 g$ and so

$$\begin{aligned} z(h_1 f + h_2 g) &\geq z(h_1 f) \wedge z(h_2 g) \\ &= (z(h_1) \vee z(f)) \wedge (z(h_2) \vee z(g)) \\ &\geq z(f) \wedge z(g) \neq \perp. \end{aligned}$$

Thus $z(h) \neq \perp$. Hence h is not unit.

□

Proposition 5.5. *Let M be a maximal ideal in $\mathcal{R}(L_\tau)$. If $z(f)$ meets every member of $Z(M]$, then $f \in M$.*

Proof. The set $z[M]$ is a z-ultrafilter on L_τ, by 5.3(1) and so, if $z(f)$ meets every member of $z[M]$, then $z(f) \in z[M]$. Therefore $f \in z^{-1}[z[M]]$; moreover, $M \subseteq z^{-1}[z[M]]$, and M is a maximal ideal, so that $f \in M = z^{-1}[z[M]]$. □

The properties stated in the foregoing proposition are, in fact, characteristic of maximal ideals and z-filters: if a z-filter $\mathcal{F}$ contains every zero-element that meets all members of $\mathcal{F}$, then, clearly, $\mathcal{F}$ is a z-ultrafilter. Like any mapping, Z satisfies for $\mathcal{F} \subseteq z[L_\tau]$,

$$z[z^{-1}[\mathcal{F}]] = \mathcal{F} \text{ and } z^{-1}[z[I]] \supseteq I.$$

The first relation implies that every z-filter is of the form $z[J]$ for some ideal J in $\mathcal{R}(L_\tau)$. In the second relation, the inclusion may be proper, as the example shown in [15, p. 26], whenever we identify $C(X)$ with $\mathcal{R}(P(X)_{\mathcal{O}(X)})$, by the following theorem proved in [14].

Theorem 5.6. *The assignment $f \mapsto f^{-1}$ from $C(X)$ to $\mathcal{R}(\mathcal{P}(X)_{\mathcal{O}(X)})$ is an f-ring isomorphism, where $f^{-1}(A) = \{x \mid f(x) \in A\}$, for all $A \in \mathcal{P}(X)$.*

6. z-ideals and prime ideals in modified pointfree topology

We recall the notion of a z-ideal of a ring A as was introduced by Mason in [21]. In the lattice theory this notion is known as "z-ideals à la Mason". Denote by $Max(A)$ the set of all maximal ideals of a ring A. For $a \in A$ and $S \subseteq A$, let

$$\mathfrak{M}(a) = \{M \in Max(A) \mid a \in M\} \text{ and } \mathfrak{M}(S) = \{M \in Max(A) \mid S \subseteq M\}.$$

Note that, since an ideal contains an element if and only if it contains the principal ideal generated by the element, we have that $\mathfrak{M}(a) = \mathfrak{M}(\langle a \rangle)$. An ideal I of a ring A is called a z-ideal à la Mason if whenever $\mathfrak{M}(a) \subseteq \mathfrak{M}(b)$ and $a \in I$, then $b \in I$. We shall define a z-ideal of $\mathcal{R}(L_\tau)$ "topologically".

Definition 6.1. An ideal I of a ring $\mathcal{R}(L_\tau)$ is called a z-ideal if whenever $z(f) \leq z(g)$, $f \in I$ and $g \in \mathcal{R}(L_\tau)$, then $g \in I$.

Proposition 6.2. *Every z-ideal à la Mason of $\mathcal{R}(L_\tau)$ is a z-ideal.*

Proof. For the proof, it suffices to show that $z(f) \leq z(g)$ implies $\mathfrak{M}(f) \subseteq \mathfrak{M}(g)$. Assume on the contrary that $\mathfrak{M}(f) \nsubseteq \mathfrak{M}(g)$. Then there exists a maximal ideal M such that $f \in M$ but it does not contain g, and consequently $z(f) \nleq z(g)$, by definition. This contradicts the hypothesis. □

If $\mathcal{F}$ is a z-filter, then $z^{-1}[\mathcal{F}]$ is a z-ideal (since $z[z^{-1}[\mathcal{F}]] = \mathcal{F}$). Hence if J is any ideal in $\mathcal{R}(L_\tau)$, then $I = z^{-1}[z[J]]$ is a z-ideal; clearly, I is the smallest z-ideal containing J. It is evident that every maximal ideal is a z-ideal and the intersection of an arbitrary (nonempty) family of z-ideals is a z-ideal. We now see some examples of z-ideals and give some properties of them.

Definition 6.3. Let L be a lattice. The element $\perp < p \in L$ is called a *particle* if and only if $p \leq \bigvee_i a_i$, whenever $\bigvee_i a_i$ exists, implies $p \leq a_i$ for some i.

A straightforward calculation shows that every atom in a frame is a particle. Furthermore, it follows at once that any particle is co-prime.

Proposition 6.4. *Let L_τ be a topoframe. Then the following statements hold.*

(1) *If $S \subseteq L$, then $M_S = \{f \in \mathcal{R}(L_\tau) \mid s \leq z(f)$ for each $s \in S)\}$ is a z-ideal.*
(2) *For each $a \in L$, $O_a = \{f \in \mathcal{R}(L_\tau) \mid a \leq (z(f))^\circ\}$ is a z-ideal.*
(3) *For each particle $p \in L$, if it exists, $M_p = \{f \in \mathcal{R}(L_\tau) \mid p \leq z(f)\}$ is a maximal z-ideal.*

Proof. (1) Let $f, g \in M_S$. Then for every $s \in S$, $s \leq z(f)$ and $s \leq z(g)$, so that

$$s \leq z(f) \wedge z(g) \leq z(f - g)$$

and hence $f - g \in M_S$. Let $f \in M_S$, and $h \in \mathcal{R}(L_\tau)$. Then

$$z(hf) = z(h) \vee z(f) \geq z(f) \geq s$$

for all $s \in S$. Hence $fh \in M_S$. Thus, M_S is an ideal in $\mathcal{R}(L_\tau)$. Clearly, M_S is a z-ideal.

(2) Let $f, g \in O_a$. Then $a \leq z(f)^\circ$, and $a \leq z(f)^\circ$. Hence

$$(z(f - g))^\circ \geq (z(f) \wedge z(g))^\circ = (z(f))^\circ \wedge (z(g))^\circ \geq a\,.$$

Therefore $f - g \in O_a$. Suppose now that $f \in O_a$ and $h \in \mathcal{R}(L_\tau)$. Then

$$(z(hf))^\circ = (z(h) \vee z(f))^\circ \geq (z(f))^\circ \geq a\,.$$

Hence $fh \in O_a$. Thus, O_a is an ideal in $\mathcal{R}(L_\tau)$. Obviously, O_a is a z-ideal.

(3) The set M_p is just a special case of M_S in the statement (1), and hence it is a z-ideal. Since $p \neq \perp$, Theorem 3.8 shows that M_p is a proper ideal. To verify that M_p is maximal, consider $f \in \mathcal{R}(L_\tau) \setminus M_p$. Then there exists a real number r such that $p \leq f(\{r\})$, since p is a particle and

$$p \leq \top = f(\mathbb{R}) = \bigvee_{r \in \mathbb{R}} f(\{r\}).$$

Note that $r \neq 0$, since $f \notin M_p$. In order to show that M_p is a maximal ideal , it suffices to prove that $1 - \frac{\mathbf{1}}{\mathbf{r}} f \in M_p$. For this,

$$\begin{aligned}
z(\mathbf{1} - \tfrac{\mathbf{1}}{\mathbf{r}} f) &= \bigvee\{\mathbf{1}(\{a\}) \wedge (-\tfrac{\mathbf{1}}{\mathbf{r}} f)(\{b\}) \mid a + b = 0\} \\
&= \bigvee\{\mathbf{1}(\{1\}) \wedge (-\tfrac{\mathbf{1}}{\mathbf{r}} f)(\{b\}) \mid 1 + b = 0\} \\
&= (-\tfrac{\mathbf{1}}{\mathbf{r}} f)(\{-1\}) \\
&= \bigvee\{-\tfrac{\mathbf{1}}{\mathbf{r}}(\{c\}) \wedge f(\{d\}) \mid cd = -1\} \\
&= \bigvee\{-\tfrac{\mathbf{1}}{\mathbf{r}}(\{-\tfrac{1}{r}\}) \wedge f(\{d\}) \mid -\tfrac{1}{r} d = -1\} \\
&= f(\{r\}) \\
&\geq p\,.
\end{aligned}$$

□

Proposition 6.5. *Every z-ideal in $\mathcal{R}(L_\tau)$ is an intersection of prime ideals.*

Proof. For all $n \in \mathbb{N}$ and $f \in \mathcal{R}(L_\tau)$, $z(f^n) = z(f)$, by Theorem 3.7. Hence if I is any z-ideal, then $f^n \in I$ implies $f \in I$, and so $I = \sqrt{I}$. This means that I is the intersection of all the prime ideals containing it. □

We have already observed that every intersection of maximal ideals is a z-ideal. The converse is not true, however. To see a z-ideal that is not an intersection of maximal ideals, consider the example in [15, p. 28], whenever, by Theorem 5.6, we identify $C(X)$ with $\mathcal{R}(P(X)_{\mathcal{O}(X)})$.

Definition 6.6. Let $f \in \mathcal{R}(L_\tau)$ and $z \in z[L_\tau]$. We shall say that f is non-negative on z if $z \wedge f(-\infty, 0) = \bot$. Likewise, we say that f is non-positive on z if $z \wedge f(0, +\infty) = \bot$. We say that f does not change sign on z when f is non-negative on z or f is non-positive on z.

The following lemma will be useful in the proof of the next theorem that clarifies a relation between prime ideals and z-ideals.

Lemma 6.7. *Let $g, h \in \mathcal{R}(L_\tau)$. Then*

$$(|g| - |h|)(-\infty, 0) \geq z(g) \wedge coz(h)\,.$$

Proof. By Lemma 3.6, we have

$$\begin{aligned}
&(|g| - |h|)(-\infty, 0) \\
&\qquad \geq \bigvee_{t\in\mathbb{Q}}(|g|(-\infty, t) \wedge (-|h|)(-\infty, -t)) \\
&\qquad = \bigvee_{t\in\mathbb{Q}}(|g|(-\infty, t) \wedge |h|(t, +\infty)) \\
&\qquad = \bigvee_{t\in\mathbb{Q}}(g \vee (-g))(-\infty, t) \wedge (h \vee (-h))(t, +\infty)) \\
&\qquad = \bigvee_{t\in\mathbb{Q}}(g(-\infty, t) \wedge (-g)(-\infty, t)) \wedge (h(t, +\infty) \vee (-h)(t, +\infty)) \\
&\qquad = \bigvee_{t\in\mathbb{Q}}((g(-\infty, t) \wedge g(-t, +\infty)) \wedge (h(t, +\infty) \vee h(-\infty, -t))) \\
&\qquad = \bigvee_{t>0}(g(-t, t)) \wedge h((t, +\infty) \vee (-\infty, -t))) \\
&\qquad \geq g(\{0\}) \wedge \bigvee_{t>0}(h((t, +\infty) \vee (-\infty, -t))) \\
&\qquad = z(g) \wedge coz(h).
\end{aligned}$$

The proof is now complete. □

Theorem 6.8. *For any z-ideal I in $\mathcal{R}(L_\tau)$, the following are equivalent.*

(1) *I is a prime ideal.*
(2) *I contains a prime ideal.*
(3) *For all $g, h \in \mathcal{R}(L_\tau)$, if $gh = \mathbf{0}$, then $g \in I$ or $h \in I$.*
(4) *For every $f \in \mathcal{R}(L_\tau)$, there is a zero-element of $z[I]$ on which f does not change sign.*

Proof. (1) implies (2): trivial.

(2) implies (3): if I contains a prime ideal P, and $gh = \mathbf{0}$, then $gh \in P$, whence either g or h is in P and hence in I.

(3) implies (4): observe that for every $f \in \mathcal{R}(L_\tau)$,

$$(f \vee \mathbf{0})(f \wedge \mathbf{0}) = f^+(-f^-) = \mathbf{0}$$

(see [8, p.80]). Then, by hypothesis, either $f \vee \mathbf{0}$ or $f \wedge \mathbf{0}$ is in I , and hence $z(f \vee \mathbf{0})$ or $z(f \wedge \mathbf{0})$ is in $z[I]$, however, f does not change sign on them, since

$$z(f \wedge \mathbf{0}) \wedge f(-\infty, 0) = f[0, +\infty) \wedge f(-\infty, 0) = \perp$$

and

$$z(f \vee \mathbf{0}) \wedge f(0, +\infty) = f(-\infty, 0] \wedge f(0, +\infty) = \perp .$$

(4) implies (1): given $gh \in I$, consider the function $|g| - |h|$. By hypothesis, there is a zero-element $z[I]$ of I on which $|g| - |h|$ is non-negative, say, i.e., $z \wedge (|g| - |h|)(-\infty, 0) = \perp$. Then, by Lemma 6.7,

$$\perp = z \wedge (|g| - |h|)(-\infty, 0) \geq z \wedge z(g) \wedge coz(h),$$

which follows that $z \wedge z(g) \leq z(h)$. Hence $z \wedge z(gh) \leq z(h)$, so that $z(h) \in z[I]$. Since I is a z-ideal, $h \in I$, and thus I is prime. □

Remark 6.9. An obvious consequence of the latter theorem is that every prime ideal in $\mathcal{R}(L_\tau)$ is contained in a unique maximal ideal; for, if M and M' are distinct maximal ideals, their intersection is a z-ideal (since M and $M'$are z-ideals), but it is not prime (since If J and J' are ideals, neither containing each other, then $J \cap J'$ is not prime), and hence by Theorem 6.8, $M \cap M'$ contains no prime ideal.

Definition 6.10. By a *prime z-filter*, we shall mean a z-filter $\mathcal{F}$ with the following property: whenever the join of two zero-elements belongs to $\mathcal{F}$, then at least one of them belongs to $\mathcal{F}$.

Proposition 6.11. *In $\mathcal{R}(L_\tau)$, the following statements hold.*

(1) *If P is a prime ideal in $\mathcal{R}(L_\tau)$, then $z[P]$ is a prime z-filter.*
(2) *If $\mathcal{F}$ is a prime z-filter, then $z^{-1}[\mathcal{F}]$ is a prime z-ideal.*

Proof. (1) Let $Q = z^{-1}[z[P]]$. Then $z[Q] = z[P]$, and Q is a z-ideal containing the prime ideal P. By Theorem 6.8, Q is prime. Suppose, now, that $z(f) \vee z(g) \in z[P]$. This implies that $z(fg) \in z[Q]$; therefore fg belongs to the z-ideal Q. Since Q is prime, it contains f, say. Then $z(f) \in z[Q] = z[P]$.

(2) Suppose that $fg \in P = z^{-1}[\mathcal{F}]$. Then $z(fg) = z(f) \vee z(g) \in z[P] = \mathcal{F}$. By hypothesis, $z(f)$, say, belongs to $z[P]$. Then f belongs to P, since the ideal $P = z^{-1}[\mathcal{F}]$ is a z-ideal. □

Remark 6.12. It is immediate from the preceding theorem and Theorem 6.9 that a prime z-filter is contained in a unique z-ultrafilter. In addition, since every maximal ideal in $\mathcal{R}(L_\tau)$ is prime, every z-ultrafilter is a prime z-filter.

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