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HISTORIC SET CARRIES FULL HAUSDORFF DIMENSION

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Abstract. We prove that the historic set for ratio of Birkhoff average is either empty or full of Hausdorff dimension in a class of one dimensional non-uniformly hyperbolic dynamical systems.
Keywords: Historic set, Moran set, non-uniformly hyperbolic.
MSC(2010): Primary: 37E05; Secondary: 28A80.

1. **Introduction**

Consider a dynamical system with a compact state space X, given by a continuous map $S : X \to X$. We denote by $\mathcal{C}(X, \mathbb{R})$ the collection of continuous functions from X to $\mathbb{R}$. Given $f \in \mathcal{C}(X, \mathbb{R})$, let $S_n f := \sum_{i=0}^{n-1} f \circ S^i$ and $A_n f := \frac{1}{n} S_n f$ be the Birkhoff sum and the Birkhoff average of f. We say that an orbit $\{x, S(x), S^2(x), \cdots\}$ has historic behaviour if for some $f \in \mathcal{C}(X, \mathbb{R})$ the limit $\lim_{n\to\infty} A_n f(x)$ does not exist. This terminology was introduced by Ruelle in [10] and was investigated by Takens in [11]. The absence of limit means that, as time n tends to ∞, the point $S^n x$ keeps having new ideas about what it wants to do. By convention, we call the set of points **historic set** if the orbit $\{x, S(x), S^2(x), ...\}$ of each member x has historic behaviour. Especially, for any given $f \in \mathcal{C}(X, \mathbb{R})$, let $\mathcal{H}(f; X, S) := \{x \in X : \lim_{n\to\infty} A_n f(x) \text{ does not exist}\}$ be the historic set with respect to f.

Historic sets have until very recently been considered of little interest in dynamical systems and geometric measure theory. Indeed, according to folklore, these sets carry no essential information about the underlying structure. However, recent work [2, 8, 4, 5, 12, 13, 1, 14] has changed this point of view: it carries full topological entropy and full Hausdorff dimension in most cases [2, 4, 5, 12, 13, 14], especially in uniformly hyperbolic dynamical systems. The

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full dimension of historic set has been verified in subshift of finite type and conformal repellers[2, 4, 5]. D. Thompson [12, 13] verified the full topological pressure of the historic set for the systems satisfying some weak specification. Barreira et al [1] looked into the historic set from topological viewpoint. They showed that the historic set in subshift with weak specification is either empty or residual. It is interesting to consider the corresponding question in the non-uniformly hyperbolic cases.

In this paper, we study a more general historic set of ratio for Birkhoff average in a class of one dimensional non-uniformly hyperbolic dynamical systems. Given $f, g \in \mathcal{C}(X, \mathbb{R})$ with $\inf_{x\in X} g(x) > 0$, we write the historic set related to the ratio for Birkhoff average of f and g by

$$\mathcal{H}(f, g; X, S) := \left\{ x \in X : \lim_{n\to\infty} \frac{S_n f(x)}{S_n g(x)} \text{ does not exist} \right\}.$$

Clearly, let $g \equiv 1$, $\mathcal{H}(f, g; X, S)$ recovers $\mathcal{H}(f; X, S)$. Now, we introduce the dynamical system considered in this paper.

1.1. **Basic settings.** In this paper, we consider the following model. Denote the unit interval [0,1] by I. Let $T : \bigcup_{i=1}^m I_i \to I$ be a piecewise $C^{1+\rho}$ map with exponent $\rho > 0$, where $\{I_i\}_{i=1}^m$ are m subintervals of I. Moreover, we impose the following assumptions:

(1) $\mathrm{int}(I_i) \cap \mathrm{int}(I_j) = \emptyset$ for $i \neq j$, $\mathrm{int}(I_i)$ is the interior of I_i.
(2) $T|_{I_i} : I_i \to I$, is a $C^{1+\rho}$ continuous and surjective map, for all $1 \leq i \leq m$. There is a unique $x_i \in I_i$ such that $T(x_i) = x_i$ and $T'(x_i) \geq 1$ for $i = 1, 2, \ldots, m$. Point x_i is called parabolic fixed point if $T'(x_i) = 1$, otherwise, we call it expanding fixed point.
(3) $T'(x) > 1$ for $x \notin \{x_1, x_2, \ldots, x_m\}$.

The appearance of parabolic fixed point makes the picture of dynamical system we considered is quite different with classical uniformly hyperbolic dynamical system. Define the T- invariant repeller as

$$\Lambda := \left\{ x \in \bigcup_{i=1}^m I_i : T^n(x) \in I, \forall n \geq 0 \right\}.$$

The class of non-uniformly hyperbolic maps includes the important example of Manneville-Pomeau map, that is $T : I \to I$ defined by $Tx = x + x^{1+\beta} (\mod 1)$, where $0 < \beta < 1$.

The above system has a natural symbolic codings: let T_i be the inverse map of $T|_{I_i} : I_i \to I$ for $i = 1, 2, \ldots, m$. Let $\mathcal{A} = \{1, 2, \ldots, m\}$ and $\Sigma = \mathcal{A}^{\mathbb{N}}$. Write $\Sigma_n = \{w = w_1 w_2 \ldots w_n : w_i \in \mathcal{A}\}$ as all the words with length n. There is a shift map $\sigma : \Sigma \to \Sigma$ defined by $\sigma((\omega_n)_{n\geq 1}) = (\omega_n)_{n\geq 2}$. Define the coding map $\Pi : \Sigma \to I$ as

$$\Pi(\omega) := \lim_{n\to\infty} T_{\omega_1} \circ T_{\omega_2} \circ \cdots \circ T_{\omega_n}(I).$$

Then $\Pi(\Sigma) = \Lambda$ and $\Pi \circ \sigma(\omega) = T \circ \Pi(\omega)$. In fact, Λ is the attractor of the iterated function system $\{T_i\}_{i=1}^m$, see [3, Section 2.2]. For each $k \geq 1$, let $T_{\omega_1\omega_2\ldots\omega_k}(I)=T_{\omega_1} \circ T_{\omega_2} \circ \ldots \circ T_{\omega_k}(I)$ be the level-k elementary interval and

$$\Lambda_k := \bigcup_{\Sigma_k} T_{\omega_1\omega_2\ldots\omega_k}(I),$$

where the union is over Σ_k. Then, we have

$$\Lambda := \bigcap_{k=1}^{\infty} \Lambda_k.$$

If the iterated function system $\{T_i\}_{i=1}^m$ satisfies the strong separation condition, i.e., $I_i \bigcap I_j = \emptyset$ for all $i \neq j$. Coding map Π is bijection. In this case, Λ can be viewed as the classical Cantor set. If there exist distinct i and j with $T_i \bigcap T_j \neq \emptyset$. In this case, for each $k \geq 1$, only the end points of each level-k elementary intervals may have two codings. Then, the coding map Π is a bijection except for at most countably many points.

Let $S : X \to X$ be a topological dynamical system. Denote by $\mathcal{M}(X, S)$ the set of all S-invariant probability measures and $\mathcal{E}(X, S)$ the set of all ergodic S-invariant probability measures. Given $f, g \in \mathcal{C}(X, \mathbb{R})$ with $\inf_{x \in X} g(x) > 0$, we define the following continuum (compact and connected set)

$$\mathcal{L}(f, g; X, S) := \left\{ \frac{\int f d\mu}{\int g d\mu} : \mu \in \mathcal{M}(X, S) \right\}.$$

The continuum $\mathcal{L}(f, g; X, S)$ has close relation with $\mathcal{H}(f, g; X, S)$.

1.2. **Main result.** In this paper, we prove that the historic set in the non-uniformly hyperbolic dynamical system carries full Hausdorff dimension unless it is empty. The corresponding part in uniformly hyperbolic dynamical systems was established in [2, 4, 5]. We will explain our results more precisely after we present the main result.

Theorem 1.1. *Let $T : \Lambda \to \Lambda$ be $C^{1+\rho}$ continuous and $f, g \in \mathcal{C}(\Lambda, \mathbb{R})$ with $\inf_{x \in \Lambda} g(x) > 0$, then*

(1) $\mathcal{H}(f, g; \Lambda, T) = \emptyset \Longleftrightarrow \mathcal{L}(f, g; \Lambda, T)$ *is a trivial continuum;*
(2) $\mathcal{H}(f, g; \Lambda, T) \neq \emptyset \Longleftrightarrow \dim_H \mathcal{H}(f, g; \Lambda, T) = \dim_H \Lambda$.

Remark 1.2. It seems plausible that the $C^{1+\rho}$ regularity can be weakened if we impose some weaker regularity condition near the parabolic fixed point. Indeed, in the uniform hyperbolic case (without parabolic fixed point), C^1 regularity is enough to establish the theorem corresponded to Theorem 1.1. To our best knowledge, it is still an open question that whether we can prove Theorem 1.1 under the assumption of C^1 regularity, and the similar question was also proposed in [6].

2. **Preliminaries**

In this section, we will introduce some notations and include the lemmas needed in the proof of Theorem 1.1.

Recall that $\mathcal{A} = \{1, 2, \dots m\}$ and $\Sigma = \mathcal{A}^{\mathbb{N}}$. For $\omega = \{\omega_n\}_{n=1}^{\infty} \in \Sigma$, write $\omega|_n = \omega_1\omega_2 \dots \omega_n$. For $w \in \Sigma_n$, we define the cylinder $[w] := \{\omega \in \Sigma : \omega|_n = w\}$.

Given $f : \Sigma \to \mathbb{R}$ continuous, define $\|f\| := \sup_{\tau\in\Sigma}|f(\tau)|$. We define $\|f\|$ similarly for $f : \Lambda \to \mathbb{R}$ continuous.

Now consider the coding map $\Pi : \Sigma \to \Lambda$. Let $\tilde{\Lambda} := \{x \in \Lambda : \#\{\Pi^{-1}(x)\} = 2\}$ be the set of points which has two codings, where $\#A$ is the cardinality of the set A, and $\Pi : \Sigma \setminus \Pi^{-1}(\tilde{\Lambda}) \to \Lambda \setminus \tilde{\Lambda}$ is a bijection. For $w = w_1w_2 \dots w_n$, write $I_w := T_{w_1} \circ T_{w_2} \cdots \circ T_{w_n}(I)$. For $\omega \in \Sigma$, write $I_n(\omega) := I_{\omega|_n}$. Let $D_n(\omega) := \mathrm{diam}(I_n(\omega))$ be the diameter of $I_n(\omega)$ and $g(\omega) := -\log T'_{\omega_1}\Pi(\sigma\omega)$. Then $D_n(\omega)$ and $A_ng(\omega)$ can be related by the following lemma:

Lemma 2.1 ([6, 16])**.** *Under the assumption on T, $D_n(\omega)$ converges to 0 uniformly. Moreover*

$$\lim_{n\to\infty} \sup_{\omega\in\Sigma} \left\{ | -\frac{1}{n} \log D_n(\omega) - A_ng(\omega)| \right\} = 0.$$

Given $\mu \in \mathcal{M}(\Sigma, \sigma)$, let $\lambda(\mu, \sigma) := \int g d\mu$ be the Lyapunov exponent of μ and $\Pi_*\mu := \mu \circ \Pi^{-1}$. Similarly, Given $\mu \in \mathcal{M}(\Lambda, T)$, let $\lambda(\mu, T) := \int \log|T'|d\mu$ be the Lyapunov exponent of μ and $h(\mu, T)$ be the metric entropy. The following lemma, which is a combination of [6, Lemma 2 and Lemma 3], is very essential in our proof of Theorem 1.1.

Lemma 2.2. *For any given $\mu \in \mathcal{M}(\Sigma, \sigma)$, there exists a sequence of ergodic σ-invariant measures $\{\mu_n : n \geq 1\}$ such that $\mu_n \to \mu$ in the weak star topology and*

$$h(\mu_n, \sigma) \to h(\mu, \sigma), \quad \lambda(\mu_n, \sigma) \to \lambda(\mu, \sigma).$$

3. **Proof of Theorem 1.1**

Since T is $C^{1+\rho}$ continuous, by [16] we have

$$\dim_H \Lambda = \sup_{\mu\in\mathcal{M}(\Sigma,\sigma)} \left\{ \frac{h(\mu, \sigma)}{\lambda(\mu, \sigma)} : \lambda(\mu, \sigma) > 0 \right\}.$$

Then for any $\epsilon > 0$, there exists $\mu \in \mathcal{M}(\sigma, \Sigma)$ with $\lambda(\mu, \sigma) > 0$ such that

$$\frac{h(\mu, \sigma)}{\lambda(\mu, \sigma)} \geq \dim_H \Lambda - \epsilon. \tag{3.1}$$

We note that if the historic set $\mathcal{H}(f, g; \Lambda, T)$ is not an empty set, we can construct two measures $\mu, \nu \in \mathcal{M}(\Lambda, T)$ such that $\frac{\int f d\mu}{\int g d\mu} \neq \frac{\int f d\nu}{\int g d\nu}$. These means $\#\mathcal{L}(f, g; \Lambda, T) \geq 2$. Then the "$\Longleftarrow$" part of *1.* follows. Given $f, g \in \mathcal{C}(\Lambda, \mathbb{R})$, we define $F := f \circ \Pi$ and $G := g \circ \Pi$. It is easy to check that $\mathcal{L}(f, g; \Lambda, T) = \mathcal{L}(F, G; \Sigma, \sigma)$. Then (2) and the "$\Longrightarrow$" part of (1) are immediate consequences of the following Lemma.

Lemma 3.1. *For any $\mu, \nu \in \mathcal{M}(\Sigma, \sigma)$ with $\lambda(\mu, \sigma) > 0$, $\lambda(\nu, \sigma) > 0$ and $\frac{\int F d\mu}{\int G d\mu} \neq \frac{\int F d\nu}{\int G d\nu}$, we have*

$$\dim_H \mathcal{H}(f, g; \Lambda, T) \geq \min\left\{\frac{h(\mu, \sigma)}{\lambda(\mu, \sigma)}, \frac{h(\nu, \sigma)}{\lambda(\nu, \sigma)}\right\}.$$

Proof of the remainder of Theorem 1.1.

Let μ be the measure in (3.1) and $\alpha = \frac{\int F d\mu}{\int G d\mu}$. If $\mathcal{H}(f, g; \Lambda, T) \neq \emptyset$, we have $\#\mathcal{L}(f, g; \Lambda, T) \geq 2$, which means $\#\mathcal{L}(F, G; \Sigma, \sigma) \geq 2$. Then we can choose $\nu \in \mathcal{M}(\sigma, \Sigma)$ such that $\beta := \frac{\int F d\nu}{\int G d\nu} \neq \alpha$. We define $\mu_s = s\mu + (1-s)\nu$ with $0 < s < 1$. It is evident that $\mu_s \in \mathcal{M}(\sigma, \Sigma)$, $\lambda(\mu_s, \sigma) > 0$ and $\frac{\int F d\mu_s}{\int G d\mu_s} \neq \alpha$ for any $s \in (0, 1)$. By Lemma 3.1, we have

$$\begin{aligned}\dim_H \mathcal{H}(f, g; \Lambda, T) &\geq \min\left\{\frac{h(\mu, \sigma)}{\lambda(\mu, \sigma)}, \frac{h(\mu_s, \sigma)}{\lambda(\mu_s, \sigma)}\right\} \\ &= \min\left\{\frac{h(\mu, \sigma)}{\lambda(\mu, \sigma)}, \frac{sh(\mu, \sigma) + (1-s)h(\nu, \sigma)}{s\lambda(\mu, \sigma) + (1-s)\lambda(\nu, \sigma)}\right\}\end{aligned}$$

for all $s \in (0, 1)$. Taking s goes to 1, we get

$$\dim_H \mathcal{H}(f, g; \Lambda, T) \geq \frac{h(\mu, \sigma)}{\lambda(\mu, \sigma)} \geq \dim_H \Lambda - \epsilon.$$

By the arbitrary of ϵ, Theorem 1.1 is finished. □

Then we only need to prove Lemma 3.1.

Proof of Lemma 3.1. In the proof, we write $\tilde{\lambda}_n(\omega) = -\log D_n(\omega)/n$ to simplify notation. Our main idea is building a Moran set sitting in historic set and constructing a measure supported on the Moran set. We dived the proof into five steps for clarity.

Step 1: Constructing large block in the odd level

By Lemma 2.1, we can choose a decreasing sequence $\epsilon_i \downarrow 0$ such that for all $n \geq 2i-1$, we have

$$\begin{cases} \operatorname{var}_n A_n g < \epsilon_{2i-1}, \\ \max_{\omega \in \Sigma} |\tilde{\lambda}_n(\omega) - A_n g(\omega)| < \epsilon_{2i-1}. \end{cases} \tag{3.2}$$

By Lemma 2.2, we can choose a sequence of $\mu_{2i-1} \in \mathcal{E}(\Sigma, \sigma)$, such that

$$\begin{cases} |\frac{\int F d\mu_{2i-1}}{\int G d\mu_{2i-1}} - \alpha| < \epsilon_{2i-1}, \\ |h(\mu_{2i-1}, \sigma) - h(\mu, \sigma)| < \epsilon_{2i-1}, \\ |\lambda(\mu_{2i-1}, \sigma) - \lambda(\mu, \sigma)| < \epsilon_{2i-1}. \end{cases} \tag{3.3}$$

We write $\frac{\int F d\mu_{2i-1}}{\int G d\mu_{2i-1}} = \alpha_{2i-1}$. Since μ_{2i-1} is ergodic, by Birkhoff ergodic Theorem, Shannon-Mcmillan-Breiman Theorem and Egoroff's Theorem, for any given $\delta > 0$, there exists $\Omega'(2i-1) \subset \Sigma$ with $\mu_{2i-1}(\Omega'(2i-1)) > 1-\delta$ and there exists $l_{2i-1} \geq 2i-1$, such that for all $n \geq l_{2i-1}$ and $\omega \in \Omega'(2i-1)$, we have

$$\begin{cases} |S_n F(\omega) - \alpha_{2i-1} S_n G(\omega)| < n||G||\epsilon_{2i-1}, \\ |A_n g(\omega) - \lambda(\mu_{2i-1}, \sigma)| < \epsilon_{2i-1}, \\ |-\frac{\log \mu_{2i-1}[\omega|_n]}{n} - h(\mu_{2i-1}, \sigma)| < \epsilon_{2i-1}. \end{cases} \tag{3.4}$$

For the Birkhoff ergodic Theorem, readers can refer to [15], for the Shannon-Mcmillan-Breiman Theorem readers can refer to [9]. The Egoroff's Theorem can be found in any text book of measure theory.

Let

$$\Sigma(2i-1) = \{\omega|_{l_{2i-1}} \mid \omega \in \Omega'(2i-1)\} \text{ and } \Omega(2i-1) = \bigcup_{w \in \Sigma(2i-1)} [w].$$

Then

$$\mu_{2i-1}(\Omega(2i-1)) \geq \mu_{2i-1}(\Omega'(2i-1) \geq 1-\delta.$$

Step 2: Constructing large block in the even level

Similarly, for all $n \geq 2i$, we have

$$\begin{cases} \mathrm{var}_n\, A_n g < \epsilon_{2i}, \\ \max_{\omega \in \Sigma} |\tilde{\lambda}_n(\omega) - A_n g(\omega)| < \epsilon_{2i}. \end{cases} \tag{3.5}$$

We can choose a sequence of $\mu_{2i} \in \mathcal{E}(\Sigma, \sigma)$, such that

$$\begin{cases} |\beta_{2i} - \beta| < \epsilon_{2i}, \\ |h(\mu_{2i}, \sigma) - h(\nu, \sigma)| < \epsilon_{2i}, \\ |\lambda(\mu_{2i}, \sigma) - \lambda(\nu, \sigma)| < \epsilon_{2i} \end{cases} \tag{3.6}$$

where in (3.6), $\beta_{2i} = \frac{\int F d\mu_{2i}}{\int G d\mu_{2i}}$. Given any $\delta > 0$, there exists $\Omega'(2i) \subset \Sigma$ with $\mu_{2i}(\Omega'(2i)) > 1-\delta$ and there exists $l_{2i} \geq 2i$, such that for all $n \geq l_{2i}$ and $\omega \in \Omega'(2i)$, we have

$$
(3.7) \qquad \begin{cases} |S_nF(\omega) - \beta_{2i}S_nG(\omega)| < n||G||\epsilon_{2i}, \\ |A_ng(\omega) - \lambda(\mu_{2i}, \sigma)| < \epsilon_{2i}, \\ |-\dfrac{\log \mu_{2i}[\omega|_n]}{n} - h(\mu_{2i}, \sigma)| < \epsilon_{2i}. \end{cases}
$$

Let

$$
\Sigma(2i) = \{\omega|_{l_{2i}} \mid \omega \in \Omega'(2i)\} \text{ and } \Omega(2i) = \bigcup_{w \in \Sigma(2i)} [w].
$$

Then

$$
\mu_{2i}(\Omega(2i)) \geq \mu_{2i}(\Omega'(2i)) \geq 1 - \delta.
$$

Step 3: Constructing Moran set by gluing repeated blocks

Let $N_0 = 1$, $N_i = 2^{l_{i+2}+N_{i-1}}$ for $i \geq 1$. Here N_i is the **repeated number** of $\Sigma(i)$ in the i-th level. We define the Moran set

$$
M = \prod_{i=1}^{\infty}\prod_{j=1}^{N_i}\Sigma(i).
$$

Here, we only point out that the repeat number N_i is very important in the proof to get arbitrarily large dimension argument. It seems that if we only take the length of $\Omega(i)$ very large is not enough to prove Lemma 3.1.

Step 4: The Moran set is contained in historic set

In this step, we will prove in the constructed Moran set, Birkhoff average oscillates from α to β. By the definition of $\Sigma(i)$, clearly we have $M \cap \Pi^{-1}\tilde{\Lambda} = \emptyset$. Now we will show that $\Pi M \subset \mathcal{H}(f, g, \Lambda, T)$. Noting that

$$
\lim_{n\to\infty} \frac{S_nF(\omega)}{S_nG(\omega)} = \alpha \quad \text{iff} \quad \lim_{n\to\infty} A_n(F(\omega) - \alpha G(\omega)) = 0
$$

and

$$
\lim_{j\to\infty} \frac{l_2N_2 + l_4N_4 + \ldots + l_{2j}N_{2j}}{l_1N_1 + l_2N_2 + \ldots + l_{2j+1}N_{2j+1}} = 0.
$$

Let $n_j = \sum_{i=1}^{j} l_iN_i$ and fix $\omega \in M$, we can check the following result:

$$
\lim_{j\to\infty} \frac{S_{n_{2j+1}}F(\omega)}{S_{n_{2j+1}}G(\omega)} = \alpha, \qquad \lim_{j\to\infty} \frac{S_{n_{2j}}F(\omega)}{S_{n_{2j}}G(\omega)} = \beta.
$$

These implies that $\Pi M \subset \mathcal{H}(f, g; \Lambda, T)$.

Step 5: Estimation of the lower bound of the Hausdorff dimension for the Moran set M

We can construct a measure η supported on M, and we call it Moran measure. The Moran measure is exactly the products of measures $\mu_i\lfloor\Sigma(i)$, to be more precise

$$\eta := \prod_{i=1}^{\infty}\prod_{j=1}^{N_i}\mu_i\lfloor\Sigma(i),$$

where, $\mu_i\lfloor\Sigma(i)$ is the restriction of μ_i on $\Sigma(i)$.

By (3.2), (3.3), (3.4), (3.5), (3.6), (3.7), repeating similar estimation as in our earlier work [7], for all $x \in \Pi M$, we can show that

$$\liminf_{r\downarrow 0}\frac{\log \Pi_*\eta(B(x,r))}{\log r} \geq \min\left\{\frac{h(\mu,\sigma)}{\lambda(\mu,\sigma)}, \frac{h(\nu,\sigma)}{\lambda(\nu,\sigma)}\right\}.$$

Then we have

$$\dim_H \Pi M \geq \dim_H \Pi_*\eta \geq \min\left\{\frac{h(\mu,\sigma)}{\lambda(\mu,\sigma)}, \frac{h(\nu,\sigma)}{\lambda(\nu,\sigma)}\right\},$$

where $\dim_H \Pi_*\eta := \sup\{\dim_H E : \Pi_*\eta(E) > 0\}$ is the Hausdorff dimension of measure $\Pi_*\eta$, the reader can refer to [3] for the inequality. Noting that $\Pi M \subset \mathcal{H}(f,g;\Lambda,T)$, then it is obviously that

$$\dim_H \mathcal{H}(f,g;\Lambda,T) \geq \dim_H \Pi M \geq \min\left\{\frac{h(\mu,\sigma)}{\lambda(\mu,\sigma)}, \frac{h(\nu,\sigma)}{\lambda(\nu,\sigma)}\right\}.$$

Thus, the result follows. □

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