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EXISTENCE AND UNIQUENESS OF WEAK SOLUTIONS FOR A CLASS OF NONLINEAR DIVERGENCE TYPE DIFFUSION EQUATIONS

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Abstract. In this paper, we study the Neumann boundary value problem of a class of nonlinear divergence type diffusion equations. By a priori estimates, difference and variation techniques, we establish the existence and uniqueness of weak solutions of this problem.

Keywords: Existence, uniqueness, weak solution, variation problem, $\mathcal{N}$-function.
MSC(2010): Primary: 35K55; Secondary: 46B20, 35K61.

1. Introduction

The purpose of this paper is to investigate the existence and uniqueness of weak solutions to the following parabolic problem:

$$\begin{cases} u_t - \operatorname{div}(a(|\nabla u|)\nabla u) = 0 & \text{in } \Omega\times(0,T], \\ \frac{\partial u}{\partial \mathbf{n}} = 0 & \text{on } \partial\Omega\times(0,T], \\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded, open domain of $\mathbb{R}^N (N \geq 2)$ with Lipschitz boundary $\partial\Omega$, $\mathbf{n}$ is the outward unit normal vector of $\partial\Omega$, T is a positive number and $u_0 \in L^2(\Omega)$. The function a is such that $\phi : \mathbb{R} \to \mathbb{R}$ defined by

$$\phi(s) = \begin{cases} a(|s|)s & s \neq 0, \\ 0 & s = 0, \end{cases}$$

is an odd increasing homeomorphism from $\mathbb{R}$ onto itself.

Problems of type (1.1) has been motivated by a various range of applications, such as motion of non-Newton fluids, image restoration, elastic materials and mathematical biology. We refer to the bibliographies (see, for example [4, 5, 8, 12, 14, 19]) for more detailed information on the physical situation.

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Parabolic equations that are similar to (1.1) have been studied extensively due to their prominent roles in many modeling phenomena. In 1990, Perona and Malik [13] proposed the Malik-Perona model

$$(1.2) \qquad \begin{cases} u_t - \operatorname{div}(c(|\nabla u|^2)\nabla u) = 0 & \text{in } \Omega \times (0,T], \\ \frac{\partial u}{\partial \mathbf{n}} = 0 & \text{on } \partial\Omega \times (0,T], \\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases}$$

where Ω is an image domain in $\mathbb{R}^2$ and $c(s) > 0$. This model is well-known and has been widely used to denoise and segment images. The equation in (1.2) can be written as

$$(1.3) \qquad u_t = c(|\nabla u|^2)u_{\mathbf{TT}} + b(|\nabla u|^2)u_{\mathbf{NN}},$$

with $b(s) = c(s) + 2sc^{'}(s)$. Thus, the right-hand side of the equation (1.3) may be interpreted as a sum of a diffusion $u_{\mathbf{TT}}$ in the tangent direction ($\mathbf{T}$) plus a diffusion $u_{\mathbf{NN}}$ in the normal ($\mathbf{N} = \frac{\nabla u}{|\nabla u|}$) direction.

Although there are some results about the existence and uniqueness of weak solutions for the Malik-Perona model, the conditions to ensure these results are difficult to check (see [2]).

Therefore, similar to the Perona-Malik model, Wang and Zhou [17] considered a special case of the problem (1.2)

$$(1.4) \qquad u_t - \left(\frac{\Phi^{'}(|\nabla u|)}{|\nabla u|} u_{\mathbf{TT}} + \Phi^{''}(|\nabla u|)u_{\mathbf{NN}} \right) = 0,$$

with $\Phi(s) = slog(1+s)$. They studied the existence and uniqueness of a weak solution of the problem.

Later, Feng and Yin [7] investigated the existence and uniqueness of weak solutions of the more general equation (1.4) with $\Phi(s) = slog(1+\beta(s))$ $(s \geq 0)$, where $\beta(s)$ is a polynomial with the following form:

$\beta(s) = \beta_1 s + \beta_2 s^2 + \cdots + \beta_r s^r$, for some integer $r \geq 1$, and $\beta_1 > 0$, $\beta_r > 0$, $\beta_j \geq 0 \ \ (1 < j < r)$.

Obviously the equation considered in [17] is a special case of [7].

In this paper, we study the existence and uniqueness of weak solutions of problem (1.1), where the equation in (1.1) is associated by $\mathcal{N}$-function $\Phi(s) := \displaystyle\int_0^s \phi(t)dt$. These problems arise in the field of physics, e.g.,

(a) nonlinear elasticity: $\Phi(s) = (1+s^2)^\gamma - 1$, $\gamma > \frac{1}{2}$;

(b) plasticity: $\Phi(s) = s^\alpha \big(log(1+s)\big)^\beta$, $\alpha \geq 1, \beta > 0$;

(c) generalized Newtonian fluids: $\Phi(s) = \displaystyle\int_0^s t^{1-\alpha}(sinh^{-1}t)^\beta dt$.

For details, see [9–11].

We remark that the equation in (1.1) contains the equations proposed in [17] and [7] as particular cases. Here we establish the existence and uniqueness of weak solutions of (1.1) by difference and variation techniques.

We assume that there exist $l, m > 1$ such that

$$l \le \frac{\phi(s)s}{\Phi(s)} \le m \quad \textit{for any} \quad s > 0. \tag{1.5}$$

Denote the cylinder $Q \equiv \Omega \times (0, T]$ and define weak solutions of problem (1.1) as follows.

Definition 1.1. A function $u : \overline{\Omega} \times [0, T] \to \mathbb{R}$ is called a weak solution of problem (1.1) if the following conditions are satisfied:

(i) $u \in C\left([0,T]; L^2(\Omega)\right) \cap L^1\left(0,T; W^{1,1}(\Omega)\right)$ with $\int_0^T \int_\Omega \Phi(|\nabla u|)dxdt < \infty$;

(ii) For every $t \in [0,T]$, we have

$$\int_\Omega u(x,t)dx = \int_\Omega u_0(x)dx;$$

(iii) For any $\varphi \in C^1(\overline{Q})$ with $\varphi(\cdot, T) = 0$, we have

$$-\int_\Omega u_0(x)\varphi(x,0)dx + \int_0^T \int_\Omega [-u\varphi_t + a(|\nabla u|)\nabla u \cdot \nabla\varphi]dxdt = 0.$$

Now we state our main result.

Theorem 1.2. *Under assumption (1.5), the initial-boundary value problem (1.1) admits a unique weak solution.*

This paper is organized as follows. In section 2, we will list and prove some useful Inequalities and Lemmas. In section 3, the proof of Theorem 1.2 will be given.

2. Preliminaries

In this section, we state some basic results that will be used later and utilize the properties of $\mathcal{N}$-function to prove Lemma 2.7 and Lemma 2.10.

Definition 2.1 ([1])**.** $\Phi : \mathbb{R} \to \mathbb{R}$ is called an $\mathcal{N}$-function if it has the following properties:

(1) Φ is even, continuous, convex and $\Phi(0) = 0$;
(2) $\Phi(u) > 0$ for all $u \neq 0$;
(3) $\lim\limits_{u\to 0} \frac{\Phi(u)}{u} = 0$ and $\lim\limits_{u\to\infty} \frac{\Phi(u)}{u} = \infty$.

Proposition 2.2 ([1])**.** *Φ is an $\mathcal{N}$-function iff $\Phi(u) = \int_0^{|u|} \phi(t)dt$, where the right derivative ϕ of Φ satisfies:*

(1) ϕ *is right-continuous and nondecreasing;*
(2) $\phi(t) > 0$ *whenever* $t > 0$;
(3) $\phi(0) = 0$ *and* $\lim_{t\to\infty} \phi(t) = \infty$.

Let ϕ satisfy (1)-(3) of Proposition 2.2. Then we call

$$\psi(s) = sup\{t \geq 0 : \phi(t) \leq s\} = inf\{t \geq 0 : \phi(t) > s\},$$

the right-inverse function of ϕ. Clearly, ψ also satisfies (1)-(3) of Proposition 2.2.

Definition 2.3 ([1])**.** Let Φ be an $\mathcal{N}$-function, ϕ be the right derivative of Φ, and ψ be the right-inverse function of ϕ. Then we call

$$\Psi(v) = \int_0^{|v|} \psi(s)ds,$$

the complementary $\mathcal{N}$-function of Φ.

Proposition 2.4 ([1])**.** *The relations of* Φ, Ψ, ϕ *and* ψ *are as follows:*

$$uv \leq \Phi(u) + \Psi(v) \quad (u, v \in \mathbb{R}). \quad (Young\ Inequality) \tag{2.1}$$

$$uv = \Phi(u) + \Psi(v) \Leftrightarrow u = \psi(|v|)sign\ v\ or\ v = \phi(|u|)sign\ u \quad (u, v \in \mathbb{R}). \tag{2.2}$$

$$\Phi(u) \leq |u|\phi(|u|) \leq \Phi(2u) \quad (u \in \mathbb{R}). \tag{2.3}$$

$$\psi\left(\phi(t)\right) \geq t, \phi\left(\psi(s)\right) \geq s \quad (s, t \geq 0). \tag{2.4}$$

Definition 2.5 ([1])**.** We say that an $\mathcal{N}$-function Φ satisfies the global Δ_2-condition if there exists $K > 2$ such that

$$\Phi(2u) \leq K\Phi(u), \quad (u \geq 0).$$

In this case, we write $\Phi \in \triangle_2$.

Lemma 2.6 ([15])**.** *The following are equivalent:*
(1) $\Phi \in \Delta_2$.
(2) (1.5) *is satisfied.*
(3) *There exist* $p > 1$ *such that for every* $u > 0$,

$$\frac{u\phi(u)}{\Phi(u)} < p.$$

(4) *There exist* $q = \frac{p}{p-1} > 1$ *such that for every* $v > 0$,

$$\frac{v\psi(v)}{\Psi(v)} > q. \tag{2.5}$$

Lemma 2.7. *For all* $\xi, \eta \in \mathbb{R}^N$, *we have*

$$\big(a(|\xi|)\xi - a(|\eta|)\eta\big) \cdot (\xi - \eta) \geq 0.$$

Proof. If $\xi = \theta, \eta \neq \theta$, then

$$\big(a(|\xi|)\xi - a(|\eta|)\eta\big) \cdot (\xi - \eta) = a(|\eta|)\eta \cdot \eta = a(|\eta|)|\eta|^2 = \phi(|\eta|)|\eta| > 0.$$

If $\xi \neq \theta, \eta = \theta$, then

$$\big(a(|\xi|)\xi - a(|\eta|)\eta\big) \cdot (\xi - \eta) = a(|\xi|)\xi = a(|\xi|)|\xi|^2 = \phi(|\xi|)|\xi| > 0.$$

If $\xi = \theta, \eta = \theta$, then

$$\big(a(|\xi|)\xi \cdot \xi - a(|\eta|)\eta\big) \cdot (\xi - \eta) = 0.$$

If $\xi \neq \theta, \eta \neq \theta$, then

$$\begin{aligned}
&\big(a(|\xi|)\xi - a(|\eta|)\eta\big) \cdot (\xi - \eta)\\
&= a(|\xi|)|\xi|^2 + a(|\eta|)|\eta|^2 - a(|\xi|)\xi \cdot \eta - a(|\eta|)\eta \cdot \xi\\
&= \phi(|\xi|)|\xi| + \phi(|\eta|)|\eta| - \phi(|\xi|)\tfrac{\xi \cdot \eta}{|\xi|} - \phi(|\eta|)\tfrac{\eta \cdot \xi}{|\eta|}\\
&\geq \phi(|\xi|)|\xi| + \phi(|\eta|)|\eta| - \phi(|\xi|)|\eta| - \phi(|\eta|)|\xi|\\
&= (\phi(|\xi|) - \phi(|\eta|))(|\xi| - |\eta|) \geq 0.
\end{aligned}$$

□

Lemma 2.8. *Suppose* Φ *is an* $\mathcal{N}$*-function, then* $\Phi(|\xi|)$ *is a convex function with respect to* $\xi \in \mathbb{R}^N$.

Proof. For every pair of $\xi_1, \xi_2 \in \mathbb{R}^N$ and every $\lambda \in [0, 1]$, we have

$$\Phi\big(|\lambda\xi_1 + (1-\lambda)\xi_2|\big) \leq \Phi\big(\lambda|\xi_1| + (1-\lambda)|\xi_2|\big) \leq \lambda\Phi(|\xi_1|) + (1-\lambda)\Phi(|\xi_2|). \quad \square$$

Lemma 2.9 ([3, 16, The Biting Lemma])**.** *Let* $\Omega \subset \mathbb{R}^N$ *be measurable with finite Lebesgue measure* $\mu(\Omega)$ *and suppose that* $\{f_n\}$ *is a bounded sequence in* $L^1(\Omega; \mathbb{R}^N)$. *Then there exist a subsequence* $\{f_{n_j}\} \subset \{f_n\}$, *a function* $f \in L^1(\Omega; \mathbb{R}^N)$ *and a decreasing family of measurable sets* E_k *such that* $\mu(E_k) \to 0$ *as* $k \to \infty$ *and for any* k, $f_{n_j} \rightharpoonup f$ *weakly in* $L^1(\Omega \setminus E_k; \mathbb{R}^N)$ *as* $j \to \infty$.

Lemma 2.10. *Suppose* Φ *is an* $\mathcal{N}$*-function. Let* $\Omega \subset \mathbb{R}^N$ *be measurable with finite Lebesgue measure* $\mu(\Omega)$ *and suppose that* $\{f_n\} \subset L^1(\Omega; \mathbb{R}^N)$ *satisfies that*

$$\int_\Omega \Phi(|f_n|)dx \leq C,$$

where C *is a positive constant. Then there exist a subsequence* $\{f_{n_i}\} \subset \{f_n\}$,

a function
$f \in L^1(\Omega;\mathbb{R}^N)$ *such that*

$$f_{n_i} \rightharpoonup f \text{ weakly in } L^1(\Omega;\mathbb{R}^N) \text{ as } i \to \infty,$$

with

$$\int_\Omega \Phi(|f|)dx \le C.$$

Proof. Φ is an $\mathcal{N}$-function, so there exists $s_0 > 0$, such that for $s > s_0$,

$$\frac{\Phi(s)}{s} \ge 1, \text{ i.e., } s \le \Phi(s).$$

Hence

$$\begin{aligned}\int_\Omega |f_n|dx &= \int_{\Omega\cap\{|f_n|\le s_0\}} |f_n|dx + \int_{\Omega\cap\{|f_n|>s_0\}} |f_n|dx \\ &\le s_0\mu(\Omega) + \int_{\Omega\cap\{|f_n|>s_0\}} \Phi(|f_n|)dx \\ &\le s_0\mu(\Omega) + \int_\Omega \Phi(|f_n|)dx \\ &\le s_0\mu(\Omega) + C.\end{aligned}$$

So $\{f_n\}$ is a bounded sequence in $L^1(\Omega;\mathbb{R}^N)$. Thank to Lemma 2.9, we can find a subsequence $\{f_{n_i}\} \subset \{f_n\}$, a function $f \in L^1(\Omega;\mathbb{R}^N)$ and decreasing family of measurable sets $\{E_k\}$ such that $\mu(E_k) \to 0$ as $k \to \infty$ and for any k,

$$f_{n_i} \rightharpoonup f \text{ weakly in } L^1(\Omega \setminus E_k;\mathbb{R}^N) \text{ as } i \to \infty. \tag{2.6}$$

For every $\varphi \in L^\infty(\Omega;\mathbb{R}^N)$, for every fixed k, we write

$$\int_\Omega (f_{n_i} - f)\cdot\varphi dx = \int_{\Omega\setminus E_k} (f_{n_i} - f)\cdot\varphi dx + \int_{E_k} f_{n_i}\cdot\varphi dx - \int_{E_k} f\cdot\varphi dx.$$

Using (2.6), we have

$$\begin{aligned}&\overline{\lim_{i\to\infty}} \left|\int_\Omega (f_{n_i} - f)\cdot\varphi dx\right| \\ &\le \overline{\lim_{i\to\infty}} \left|\int_{E_k} f_{n_i}\cdot\varphi dx\right| + \left|\int_{E_k} f\cdot\varphi dx\right| \\ &\le \overline{\lim_{i\to\infty}} \left|\int_{E_k} f_{n_i}\cdot\varphi dx\right| + \int_{E_k} |f\cdot\varphi|dx \\ &\le \overline{\lim_{i\to\infty}} \left|\int_{E_k} f_{n_i}\cdot\varphi dx\right| + \|\varphi\|_{L^\infty(\Omega;R^N)} \int_{E_k} |f|dx.\end{aligned}$$

Since Φ is an $\mathcal{N}$-function, we obtain that for any $M > 0$, there exists $s_M > 0$, such that $\frac{\Phi(s)}{s} \ge M$ for $s > s_M$.

Then

$$
\begin{aligned}
&0 \le \left|\int_{E_k} f_{n_i}\cdot\varphi dx\right| \le \int_{E_k} |f_{n_i}\cdot\varphi| dx\\
&\le \|\varphi\|_{L^\infty(\Omega;R^N)} \int_{E_k} |f_{n_i}| dx\\
&= \|\varphi\|_{L^\infty(\Omega;R^N)} \left(\int_{E_k\cap\{|f_{n_i}|>s_M\}} |f_{n_i}| dx + \int_{E_k\cap\{|f_{n_i}|\le s_M\}} |f_{n_i}| dx\right)\\
&\le \|\varphi\|_{L^\infty(\Omega;R^N)} \left(\int_{E_k\cap\{|f_{n_i}|>s_M\}} |f_{n_i}| dx + s_M\mu(E_k)\right)\\
&\le \|\varphi\|_{L^\infty(\Omega;R^N)} \left(\tfrac{1}{M}\int_{E_k\cap\{|f_{n_i}|>s_M\}} \Phi(|f_{n_i}|) dx + s_M\mu(E_k)\right)\\
&\le \|\varphi\|_{L^\infty(\Omega;R^N)} \left(\tfrac{1}{M}\int_{E_k} \Phi(|f_{n_i}|) dx + s_M\mu(E_k)\right)\\
&\le \|\varphi\|_{L^\infty(\Omega;R^N)} \left(\tfrac{C}{M} + s_M\mu(E_k)\right).
\end{aligned}
$$

Therefore, for every k and M, we conclude that

$$
\overline{\lim_{i\to\infty}} \left|\int_\Omega (f_{n_i}-f)\cdot\varphi dx\right| \le \|\varphi\|_{L^\infty(\Omega;R^N)} \left(\frac{C}{M} + s_M\mu(E_k) + \int_{E_k}|f|dx\right).
$$

Using $f\in L^1(\Omega;\mathbb{R}^N)$ and $\mu(E_k)\to 0$ as $k\to\infty$, passing to limits as $k\to\infty$, we have

$$
\overline{\lim_{i\to\infty}} \left|\int_\Omega (f_{n_i}-f)\cdot\varphi dx\right| \le \|\varphi\|_{L^\infty(\Omega;\mathbb{R}^N)} \frac{C}{M}.
$$

Then passing to limits as $k\to\infty$, we obtain

$$
\overline{\lim_{i\to\infty}} \left|\int_\Omega (f_{n_i}-f)\cdot\varphi dx\right| = 0.
$$

This shows that $f_{n_i} \rightharpoonup f$ weakly in $L^1(\Omega;\mathbb{R}^N)$ as $i\to\infty$.

Since Φ is an $\mathcal{N}$-function, by Lemma 2.8, we know that $\Phi(|\xi|)$ is a convex function with respect to $\xi\in\mathbb{R}^N$. Therefore we get

$$
\Phi(|f|) \le \Phi(|f_{n_i}|) + \nabla_\xi\Phi(|f|)\cdot(f-f_{n_i}), \tag{2.7}
$$

where

$$
\begin{aligned}
\nabla_\xi\Phi(|f|) &= \left(\tfrac{\Phi'(|\xi|)\xi_1}{|\xi|}, \tfrac{\Phi'(|\xi|)\xi_2}{|\xi|}, \ldots, \tfrac{\Phi'(|\xi|)\xi_N}{|\xi|}\right)\Bigg|_{\xi=f}\\
&= \Big(a(|\xi|)\xi_1, a(|\xi|)\xi_2, \ldots, a(|\xi|)\xi_N\Big)\Bigg|_{\xi=f}.
\end{aligned}
$$

Integrating the above inequality (2.7) over the set $\Omega \cap \{|f| \le M\}$, we have

$$\begin{aligned}
&\int_{\Omega\cap\{|f|\le M\}} \Phi(|f|)dx \\
&\le \int_{\Omega\cap\{|f|\le M\}} \Phi(|f_{n_i}|)dx + \int_{\Omega\cap\{|f|\le M\}} \nabla_\xi \Phi(|f|)\cdot(f-f_{n_i})dx \\
&\le \int_\Omega \Phi(|f_{n_i}|)dx + \int_\Omega \nabla_\xi \Phi(|f|)\chi_{\{|f|\le M\}}\cdot(f-f_{n_i})dx.
\end{aligned}$$

As $\nabla_\xi \Phi(|f|)\chi_{\{|f(x)|\le M\}} \in L^\infty(\Omega; R^N)$, passing to limits as $i \to \infty$, we obtain

$$\int_{\Omega\cap\{|f|\le M\}} \Phi(|f|)dx \le \underline{\lim}_{n\to\infty} \int_\Omega \Phi(|f_{n_i}|)dx \le C.$$

And passing to limits as $M \to \infty$, we conclude that

$$\int_\Omega \Phi(|f|)dx \le C.$$

This completes the proof of the Lemma. $\square$

3. **Existence and uniqueness**

To prove Theorem 1.2, as preparation we first study the existence and uniqueness of weak solutions of the following auxiliary elliptic problems. For $h > 0$ and $u_0 \in L^2(\Omega)$, we consider

$$\left\{\begin{array}{lll} \frac{u-u_0}{h} - \operatorname{div}(a(|\nabla u|)\nabla u) = 0 & \text{in} & \Omega, \\ \frac{\partial u}{\partial \mathbf{n}}(x) = 0 & \text{on} & \partial\Omega. \end{array}\right. \tag{3.1}$$

Definition 3.1. A function $u \in L^2(\Omega) \cap W^{1,1}(\Omega)$ with $\int_\Omega \Phi(|\nabla u|)dx < \infty$ is called a weak solution of problem (3.1) if the following conditions are satisfied:

(i)
$$\int_\Omega u dx = \int_\Omega u_0 dx.$$

(ii) For any $\varphi \in C^1(\overline{\Omega})$, we have
$$\int_\Omega \frac{u-u_0}{h}\varphi dx + \int_\Omega a(|\nabla u|)\nabla u \cdot \nabla\varphi dx = 0.$$

Now we state the following theorem.

Theorem 3.2. *Assume that $u_0 \in L^2(\Omega)$, then there exists a unique weak solution for problem (3.1).*

Proof. We consider the variational problem

$$\min\{J(v) \mid v \in V\},$$

where

$$V = \{v \in L^2(\Omega) \cap W^{1,1}(\Omega) \Big| \int_\Omega \Phi(|\nabla v|)dx < \infty, \ \int_\Omega v dx = \int_\Omega u_0 dx\},$$

and the functional J is

$$J(v) = \frac{1}{2h} \int_\Omega (v - u_0)^2 dx + \int_\Omega \Phi(|\nabla v|)dx.$$

We will establish that $J(v)$ has a minimizer u_1 in V.

Let $u_{0,\Omega} = \frac{1}{\mu(\Omega)} \int_\Omega u_0 dx$. Note that $u_{0,\Omega} \in V$ and

$$0 \le \inf_{v \in V} J(v) \le J(u_{0,\Omega}) = \frac{1}{2h} \int_\Omega (u_0 - u_{0,\Omega})^2 dx.$$

Then we can find a minimizing sequence $\{v_m\} \subset V$ such that $J(v_m) \le J(u_{0,\Omega})+$ 1 and

$$\lim_{m\to\infty} J(v_m) = \inf_{v\in V} J(v).$$

Since

$$\int_\Omega \Phi(|\nabla v_m|)dx \le J(v_m) \le J(u_{0,\Omega}) + 1,$$

$$\begin{aligned}
&\int_\Omega v_m^2 dx = \int_\Omega (v_m - u_{0,\Omega})^2 dx + u_{0,\Omega}^2 \mu(\Omega) \\
&\le 2 \int_\Omega [(v_m - u_{0,\Omega})^2 + (u_0 - u_{0,\Omega})^2] dx + u_{0,\Omega}^2 \mu(\Omega) \\
&\le 4h[J(v_m) + J(u_{0,\Omega})] + u_{0,\Omega}^2 \mu(\Omega) \\
&\le 4h[2J(u_{0,\Omega}) + 1] + u_{0,\Omega}^2 \mu(\Omega),
\end{aligned}$$

it follows that

$$\int_\Omega v_m^2 dx + \int_\Omega \Phi(|\nabla v_m|)dx \le C(h, \Omega, u_0).$$

By Lemma 2.10 and the weak compactness of bounded sets in reflexive Banach spaces, we can find a subsequence $\{v_{m_i}\}$ of $\{v_m\}$ and a function $u_1 \in L^2(\Omega) \cap W^{1,1}(\Omega)$ such that

$$v_{m_i} \rightharpoonup u_1 \text{ weakly in } L^2(\Omega),$$
$$\nabla v_{m_i} \rightharpoonup \nabla u_1 \text{ weakly in } L^1(\Omega; \mathbb{R}^N).$$

Thus, we have

$$\int_\Omega u_1 dx = \lim_{i\to\infty} \int_\Omega v_{m_i} dx = \int_\Omega u_0 dx, \tag{3.2}$$

$$\int_\Omega (u_1 - u_0)^2 dx \le \varliminf_{i\to\infty} \int_\Omega (v_{m_i} - u_0)^2 dx,$$

and

$$\int_\Omega \Phi(|\nabla u_1|)dx \le \varliminf_{i\to\infty} \int_\Omega \Phi(|\nabla v_{m_i}|)dx,$$

which follows that $J(u_1) \le \varliminf_{i\to\infty} J(v_{m_i}) = \inf_{v\in V} J(v)$.

This implies that $u_1 \in V$ is a minimizer of the functional $J(u)$ in V, i.e.,

$$J(u_1) = \inf_{v\in V} J(v).$$

Now for every $\varphi \in C^1(\overline{\Omega})$ and every $t \in \mathbb{R}$, we have $u_1 + t(\varphi - \varphi_\Omega) \in V$ and then $j(0) \le j(t)$, where

$$j(t) = J\big(u_1 + t(\varphi - \varphi_\Omega)\big),$$

and φ_Ω is the integral mean of φ over Ω. Therefore we have $j^{'}(0) = 0$, i.e.,

$$\int_\Omega \frac{u_1 - u_0}{h}(\varphi - \varphi_\Omega)dx + \int_\Omega a(|\nabla u_1|)\nabla u_1 \cdot \nabla\varphi dx = 0.$$

In view of (3.2), we obtain that

$$\int_\Omega \frac{u_1 - u_0}{h}\varphi dx + \int_\Omega a(|\nabla u_1|)\nabla u_1 \cdot \nabla\varphi dx = 0.$$

This implies u_1 is a weak solution of problem (3.1).

Suppose that there exists another weak solution v of problem (3.1). Then, for every $\varphi \in C^1(\overline{\Omega})$, we have

$$\int_\Omega \frac{v - u_0}{h}\varphi dx + \int_\Omega a(|\nabla v|)\nabla v \cdot \nabla\varphi dx = 0.$$

Thus for every $\varphi \in C^1(\overline{\Omega})$, we have

$$\int_\Omega \frac{v - u_1}{h}\varphi dx + \int_\Omega \big[a(|\nabla v|)\nabla v - a(|\nabla u_1|)\nabla u_1\big] \cdot \nabla\varphi dx = 0. \tag{3.3}$$

Recalling (2.1), (2.2), (2.3), (2.4), (1.5) and Lemma 2.6, we conclude that

$$\begin{aligned} |a(|\nabla v|)\nabla v \cdot \nabla u_1| &\le |a(|\nabla v|)||\nabla v||\nabla u_1| \\ &\le \Phi(|\nabla u_1|) + \Psi(a(|\nabla v|)|\nabla v|) \\ &\le \Phi(|\nabla u_1|) + |\nabla v|\phi(|\nabla v|) - \Phi(|\nabla v|) \\ &\le \Phi(|\nabla u_1|) + \Phi(2|\nabla v|) - \Phi(|\nabla v|) \\ &\le \Phi(|\nabla u_1|) + K\Phi(|\nabla v|) - \Phi(|\nabla v|) \\ &= \Phi(|\nabla u_1|) + (K-1)\Phi(|\nabla v|) \in L^1(\Omega). \end{aligned}$$

Making use of the approximation argument, we conclude that $v - u_1$ can be a test function in (3.3). Hence

$$\int_\Omega \frac{(v - u_1)^2}{h}dx + \int_\Omega \big[a(|\nabla v|)\nabla v - a(|\nabla u_1|)\nabla u_1\big] \cdot (\nabla v - \nabla u_1)dx = 0.$$

By Lemma 2.7, we have

$$\int_\Omega (v-u_1)^2 dx = 0.$$

This implies $v = u_1$ a.e. in Ω. Thus we complete the proof of the Theorem. □

Now we begin to prove Theorem 1.2.

Proof. First we prove the uniqueness of weak solutions. Suppose there exist two weak solutions u and v of problem (1.1). Then $u-v$ satisfies the following problem

$$\begin{cases} (u-v)_t - \operatorname{div}[a(|\nabla u|)\nabla u - a(|\nabla v|)\nabla v] = 0 & \text{in} \quad \Omega\times(0,T], \\ \frac{\partial(u-v)}{\partial \mathbf{n}} = 0 & \text{on} \quad \partial\Omega\times(0,T], \\ u(x,0)-v(x,0) = 0 & \text{on} \quad \Omega. \end{cases}$$

By approximation [18] or [6] and using Proposition 2.4, we can choose $u-v$ as a test function in the above problem. Then we know, for every $t \in (0,T)$,

$$\frac{1}{2}\int_\Omega (u-v)^2(t)dx + \int_0^t \int_\Omega (a(|\nabla u|)\nabla u - a(|\nabla v|)\nabla v)\cdot(\nabla u - \nabla v)dxd\tau = 0.$$

By Lemma 2.7, the two term on the left-hand side are nonnegative, we have $u = v$ a.e. in Q. Therefore we obtain the uniqueness of weak solutions.

Then we prove the existence of weak solutions. Let n be a positive integer. Denote $h = \frac{T}{n}$. In order to construct an approximation solution sequence $\{u_h\}$ for problem (1.1), we consider the following elliptic problems

$$\begin{cases} \frac{u_k - u_{k-1}}{h} - \operatorname{div}(a(|\nabla u_k|)\nabla u_k) = 0 & \text{in} \quad \Omega, \\ \frac{\partial u_k}{\partial \mathbf{n}} = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{3.4}$$

for $k = 1, 2, \ldots, n$. As $k = 1$, it follows from Theorem 3.2 that there is a unique $u_1 \in V$ satisfying (3.4). Following the same procedures, we can find weak solutions $u_k \in V$ of (3.4) for $k = 2, 3, \ldots, n$. Moreover, for every $\varphi \in C^1(\overline{\Omega})$, we have

$$\int_\Omega \frac{u_k - u_{k-1}}{h}\varphi dx + \int_\Omega a(|\nabla u_k|)\nabla u_k \cdot \nabla\varphi dx = 0, \tag{3.5}$$

Now for every $h = \frac{T}{n}$, we define

$$
(3.6)\qquad u_h(x,t)=\begin{cases} u_0(x), & t=0,\\ u_1(x), & 0<t\le h,\\ \dots\dots, & \dots\dots,\\ u_j(x), & (j-1)<t\le jh,\\ \dots\dots, & \dots\dots,\\ u_n(x), & (n-1)h<t\le nh=T. \end{cases}
$$

Choosing u_k as a test function in (3.5), and using $u_k u_{k-1}\le \frac{u_k^2+u_{k-1}^2}{2}$, we have

$$
(3.7)\qquad \frac{1}{2}\int_\Omega u_k^2 dx + h\int_\Omega a(|\nabla u_k|)|\nabla u_k|^2 dx \le \frac{1}{2}\int_\Omega u_{k-1}^2 dx.
$$

For each $t\in(0,T]$, there exists some $j\in\{1,2,\dots,n\}$ such that $t\in((j-1)h, jh]$. Adding the inequality (3.7) from $k=1$ to $k=j$, we get

$$
\frac{1}{2}\int_\Omega u_j^2 dx + h\sum_{k=1}^{j}\int_\Omega a(|\nabla u_k|)|\nabla u_k|^2 dx \le \frac{1}{2}\int_\Omega u_0^2 dx.
$$

By the definition of $u_h(x,t)$, we obtain

$$
\frac{1}{2}\int_\Omega u_h^2(x,t) dx + \int_0^{jh}\int_\Omega a(|\nabla u_h|)|\nabla u_h|^2 dx d\tau \le \frac{1}{2}\int_\Omega u_0^2 dx.
$$

In particular, we get

$$
(3.8)\qquad \frac{1}{2}\int_\Omega u_h^2(x,t) dx + \int_0^{t}\int_\Omega a(|\nabla u_h|)|\nabla u_h|^2 dx d\tau \le \frac{1}{2}\int_\Omega u_0^2 dx.
$$

Therefore, after taking the supermum over $[0,T]$, we have

$$
\sup_{0\le t\le T}\int_\Omega u_h^2(x,t) dx + \int_0^{T}\int_\Omega a(|\nabla u_h|)|\nabla u_h|^2 dx dt \le \frac{3}{2}\int_\Omega u_0^2 dx.
$$

Recalling (2.3), we have

$$
\sup_{0\le t\le T}\int_\Omega u_h^2(x,t) dx + \int_0^{T}\int_\Omega \Phi(|\nabla u_h|) dx dt \le \frac{3}{2}\int_\Omega u_0^2 dx.
$$

We conclude that

$$
\sup_{0\le t\le T}\int_\Omega u_h^2(x,t) dx \le C = C(u_0,\Omega),
$$

and

$$
(3.9)\qquad \int_0^{T}\int_\Omega \Phi(|\nabla u_h|) dx dt \le C = C(u_0,\Omega).
$$

Thank to Lemma 2.10, we may choose a subsequence (for simplicity, we also denote it by the original sequence) such that

$$u_h \rightharpoonup u, \text{ weakly -}* \text{ in } L^\infty\left(0,T;L^2(\Omega)\right),$$

$$u_h \rightharpoonup u, \text{ weakly in } L^1\left(0,T;W^{1,1}(\Omega)\right).$$

These yield that

$$\sup_{0\le t\le T}\int_\Omega u^2(x,t)dx \le C \textit{ and } \int_0^T\int_\Omega \Phi(|\nabla u|)dxd\tau \le C.$$

Denote

$$\xi_h = a(|\nabla u_h|)\nabla u_h.$$

Using (2.2), (2.3), (1.5) and Lemma 2.6, we have

$$\begin{aligned}(3.10)\qquad \Psi(a(|\nabla u_h|)|\nabla u_h|) &= |\nabla u_h| a(|\nabla u_h|)|\nabla u_h| - \Phi(|\nabla u_h|)\\ &\le \Phi(2|\nabla u_h|) - \Phi(|\nabla u_h|) \le K\Phi(|\nabla u_h|) - \Phi(|\nabla u_h|)\\ &= (K-1)\Phi(|\nabla u_h|).\end{aligned}$$

It follows from (3.9) and (3.10) that

$$\int_0^T\int_\Omega \Psi(|\xi_h|)dxdt \le \int_0^T\int_\Omega (K-1)\Phi(|\nabla u_h|)dxdt \le C.$$

Recalling (2.5), we have that

$$\int_0^T\int_\Omega |\xi_h|^q dxdt \le C,\ \ (q>1).$$

Thus we can draw another subsequence $\{\xi_h\}$ (we also denote it by the original sequence for simplicity) such that

$$(3.11)\qquad \xi_h \rightharpoonup \xi, \quad \textit{weakly in} \quad (L^q(Q))^N, \quad (q>1).$$

We conclude from Lemma 2.10 that

$$\int_0^T\int_\Omega \Psi(|\xi|)dxdt \le \varliminf_{h\to 0}\int_0^T\int_\Omega \Psi(|\xi_h|)dxdt \le C.$$

Recalling Inequality (2.1), we have

$$|\xi\cdot\nabla u| \le |\xi||\nabla u| \le \Psi(|\xi|) + \Phi(|\nabla u|).$$

This implies that $\xi\cdot\nabla u \in L^1(Q)$.

In the following, we prove that the function u is a weak solution of problem (1.1).

For every $\varphi\in C(\overline{Q})$ with $\varphi(\cdot,T)=0$, we take $\varphi(x,(k-1)h)$ as a test function in (3.5) for every $k\in\{1,2,\dots,n\}$ to have

$$\int_\Omega \frac{u_k-u_{k-1}}{h}\varphi(x,(k-1)h)dx + \int_\Omega a(|\nabla u_k|)\nabla u_k\cdot\nabla\varphi(x,(k-1)h)dx = 0.$$

Summing up all the equalities and recalling $\varphi(\cdot,T)=\varphi(\cdot,nh)=0$, we get

$$-\frac{1}{h}\int_\Omega u_0(x)\varphi(x,0)dx + \sum_{k=1}^n \int_\Omega u_k(x)\frac{\varphi(x,(k-1)h)-\varphi(x,kh)}{h}dx$$

$$+\sum_{k=1}^n \int_\Omega a(|\nabla u_k|)\nabla u_k \cdot \nabla\varphi(x,(k-1)h)dx = 0.$$

In view of the definition of $u_h(x,t)$ in (3.6), we obtain

$$\begin{aligned}&\int_0^T\int_\Omega u_h(x,t)\varphi_t(x,t)dxdt = \sum_{k=1}^n \int_{(k-1)h}^{kh}\int_\Omega u_h(x,t)\varphi_t(x,t)dxdt\\&= \sum_{k=1}^n \int_\Omega u_k(x)\left[\int_{(k-1)h}^{kh}\varphi_t(x,t)dt\right]dx\\&= \sum_{k=1}^n \int_\Omega u_k(x)\big[\varphi(x,kh)-\varphi(x,(k-1)h)\big]dx,\end{aligned}$$

and

$$\begin{aligned}&\int_0^T\int_\Omega a(|\nabla u_h|)\nabla u_h\cdot\nabla\varphi dxdt = \sum_{k=1}^n\int_{(k-1)h}^{kh}\int_\Omega a(|\nabla u_h|)\nabla u_h\cdot\nabla\varphi dxdt\\&= \sum_{k=1}^n\int_\Omega a(|\nabla u_k|)\nabla u_k\cdot\left[\int_{(k-1)h}^{kh}\nabla\varphi(x,t)dt\right]dx.\end{aligned}$$

Thus

$$\begin{aligned}&-\int_\Omega u_0(x)\varphi(x,0)dx - \int_0^T\int_\Omega u_h(x,t)\varphi_t(x,t)dxdt\\&+\int_0^T\int_\Omega a(|\nabla u_h|)\nabla u_h\cdot\nabla\varphi dxdt\\&= \sum_{k=1}^n\int_\Omega a(|\nabla u_k|)\nabla u_k\cdot\left[\int_{(k-1)h}^{kh}\nabla\varphi(x,t)d_t - h\nabla\varphi(x,(k-1)h)\right]dx.\end{aligned}$$

Letting $h\to 0$, we have

$$-\int_\Omega u_0(x)\varphi(x,0)dx + \int_0^T\int_\Omega[-u\varphi_t + \xi\cdot\nabla\varphi]dxdt = 0. \tag{3.12}$$

Choosing $\varphi\in C_c^\infty(Q)$, we get

$$\int_0^T\int_\Omega u\varphi_t dxdt = \int_0^T\int_\Omega \xi\cdot\nabla\varphi dxdt. \tag{3.13}$$

By (3.11), we know that $\xi \in \left(L^2(Q)\right)^N$. In view of (3.13), we conclude that $u_t \in L^1(0,T;H^{-1}(\Omega))$. Since

$$u = \int_0^t u_t dt + u_0,$$

and $u_0 \in L^2(\Omega) \hookrightarrow H^{-1}(\Omega)$, it follows that $u \in C\left([0,T];H^{-1}(\Omega)\right)$. Here $H^{-1}(\Omega)$ is the dual space of $H_0^1(\Omega) = W_0^{1,2}(\Omega)$.

Denote

$$Av = a(|\nabla v|)\nabla v,$$

for $v \in L^1(Q)$ with $\int_0^T \int_\Omega \Phi(|\nabla v|)dxdt < \infty$.

Summing up the inequalities (3.7), we get

$$\frac{1}{2}\int_\Omega u_h^2(T)dx + \int_0^T \int_\Omega Au_h \cdot \nabla u_h dxdt \le \frac{1}{2}\int_\Omega u_0^2 dx. \tag{3.14}$$

Recalling Lemma 2.7, we have

$$\int_0^T \int_\Omega (Au_h - Av)\cdot(\nabla u_h - \nabla v)dxdt \ge 0.$$

Then it follows from (3.14) that

$$\begin{aligned}&\frac{1}{2}\int_\Omega u_h^2(T)dx + \int_0^T \int_\Omega (Au_h)\cdot \nabla v dxdt + \int_0^T \int_\Omega (Av)\cdot \nabla u_h dxdt\\ &- \int_0^T \int_\Omega (Av)\cdot \nabla v dxdt \le \frac{1}{2}\int_\Omega u_0^2 dx.\end{aligned}$$

Letting $h \to 0$, and noting

$$\int_\Omega u^2(T)dx \le \varliminf_{h\to 0}\int_\Omega u_h^2(T)dx.$$

We get

$$\begin{aligned}&\frac{1}{2}\int_\Omega u^2(T)dx + \int_0^T \int_\Omega \xi \cdot \nabla v dxdt + \int_0^T \int_\Omega Av \cdot \nabla u dxdt\\ &- \int_0^T \int_\Omega Av \cdot \nabla v dxdt \le \frac{1}{2}\int_\Omega u_0^2 dx.\end{aligned} \tag{3.15}$$

By an approximation, we may choose the test function $\varphi = u$ in (3.12) to have

$$\frac{1}{2}\int_\Omega u^2(T)dx + \int_0^T \int_\Omega \xi \cdot \nabla u dxdt = \frac{1}{2}\int_\Omega u_0^2 dx. \tag{3.16}$$

Combining (3.15) with (3.16), we obtain

$$\int_0^T \int_\Omega (\xi - Av)\cdot(\nabla v - \nabla u)dxdt \le 0.$$

Next we choose $v = u + \lambda\omega$ for any $\lambda > 0,\ \omega \in W^{1,p}(Q)(p > 1)$ in the above inequality to have

$$\int_0^T \int_\Omega \big(\xi - A(u+\lambda\omega)\big) \cdot \nabla\omega dxdt \le 0.$$

Passing to limits as $\lambda \to 0^+$ and using Lebesgue's Dominated Convergence Theorem, we obtain

$$\int_0^T \int_\Omega (\xi - Au) \cdot \zeta dxdt = 0,$$

for every $\zeta \in (L^p(Q))^N (p \ge 2)$ and conclude that $\xi = Au$ a.e. in Q.

For every $0 < \delta < T$, we denote $v_\delta(x,t) = u(x, t+\delta)$. By the uniqueness of weak solutions, we conclude that v_δ is a weak solution for the following problem

$$\begin{cases} \frac{\partial v_\delta}{\partial t} - \operatorname{div}\big(a(|\nabla v_\delta|)\nabla v_\delta\big) = 0 & \text{in} \quad \Omega \times (0, T-\delta], \\ \frac{\partial v_\delta}{\partial \mathbf{n}} = 0 & \text{on} \quad \partial\Omega \times (0, T-\delta], \\ v_\delta(x,0) = u(x,\delta) & \text{in} \quad \Omega. \end{cases}$$

Then it follows that $\omega_\delta(x,t) = v_\delta(x,t) - u(x,t) = u(x,t+\delta) - u(x,t)$ satisfying

(3.17)

$$\begin{cases} \frac{\partial \omega_\delta}{\partial t} - \operatorname{div}\big(a(|\nabla v_\delta|)\nabla v_\delta - a(|\nabla u|)\nabla u\big) = 0 & \text{in} \quad \Omega \times (0, T-\delta], \\ \frac{\partial \omega_\delta}{\partial \mathbf{n}} = 0 & \text{on} \quad \partial\Omega \times (0, T-\delta], \\ \omega_\delta(x,0) = u(x,\delta) - u_0(x) & \text{in} \quad \Omega. \end{cases}$$

For each $t_0 \in [0, T-\delta]$, we choose a test function ω_δ for equations (3.17) over $[0, t_0]$ to have

$$\frac{1}{2}\int_\Omega \omega_\delta^2(x,t_0)dx + \int_0^{t_0} \int_\Omega \Big[a(|\nabla v_\delta|)\nabla v_\delta - a(|\nabla u|)\nabla u\Big] \cdot (\nabla v_\delta - \nabla u)dxdt \le \frac{1}{2}\int_\Omega \omega_\delta^2(x,0)dx.$$

Thank to Lemma 2.7, it yields

$$\int_\Omega |u(x,t_0+\delta) - u(x,t_0)|^2 dx \le \int_\Omega |u(x,\delta) - u_0(x)|^2 dx.$$

In order to prove that $u \in C\left([0,T]; L^2(\Omega)\right)$, we only need to prove

$$\overline{\lim_{\delta \to 0^+}} \int_\Omega |u(x,\delta) - u_0(x)|^2 dx = 0. \tag{3.18}$$

Suppose (3.18) is not true. Then there exit a positive number ε_0 and a sequence $\{\delta_i\}$ with $\delta_i \to 0$ as $i \to \infty$ such that

$$\lim_{\delta_i \to 0^+} \int_\Omega |u(x,\delta_i) - u_0(x)|^2 dx \ge \varepsilon_0. \tag{3.19}$$

By (3.8), we easily see that

$$\int_\Omega |u(x,\delta_i)|^2 dx \le \int_\Omega |u_0(x)|^2 dx. \tag{3.20}$$

Thus, we have from (3.19) that

$$\varliminf_{\delta_i\to 0^+} \left[\int_\Omega |u_0(x)|^2 dx - \int_\Omega u_0(x)u(x,\delta_i)dx\right] \ge \frac{\varepsilon_0}{2}. \tag{3.21}$$

Hence, it follows from (3.20) that $\{u(x,\delta_i)\}$ is a bounded sequence in $L^2(\Omega)$. Moreover, there exist a subsequence (for simplicity, we also denote it by the original sequence) and a $\widetilde{u_0} \in L^2(\Omega)$ such that

$$u(x,\delta_i) \rightharpoonup \widetilde{u_0}(x), \quad \textit{weakly in } L^2(\Omega).$$

Since $u \in C\left([0,T]; H^{-1}(\Omega)\right)$, it follows that

$$u(x,\delta_i) \to u_0(x), \quad \textit{in } H^{-1}(\Omega).$$

Thus we must have $\widetilde{u_0}(x) = u_0(x)$ and then

$$u(x,\delta_i) \rightharpoonup u_0(x), \quad \textit{weakly in } L^2(\Omega).$$

The above relation leads to a contradiction with (3.21). Therefore, we conclude that (3.18) is true and $u \in C\left([0,T]; L^2(\Omega)\right)$. Thus we complete the proof of the theorem.

□

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