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BOUNDS FOR THE DIMENSION OF THE c-NILPOTENT MULTIPLIER OF A PAIR OF LIE ALGEBRAS

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Abstract. In 2009, Salemkar et al. extended the notion of the Schur multiplier of a Lie algebra to the c-nilpotent multiplier. In this paper, we study the c-nilpotent multiplier of a pair of Lie algebras and give some inequalities for the dimension of the c-nilpotent multiplier of a pair of Lie algebras.
Keywords: Pair of Lie algebras, Schur multiplier, c-nilpotent multiplier.
MSC(2010): Primary: 17B99; Secondary: 16W25.

1. Introduction

The notion of Schur multiplier arises from works of Schur on the projective representation in 1904 [18]. Let G be a group with a free presentation $1 \to R \to F \to G \to 1$. The abelian group

$$\mathcal{M}(G) = (R \cap F^2)/[F, R]$$

is said to be the Schur multiplier of G (See [7,8,11,12] for more information). Analogous to the Schur multiplier of a group, the Schur multiplier of a Lie algebra L, can be defined as $\mathcal{M}(L) = (R \cap F^2)/[R, F]$, where $L \cong F/R$ and F is a free Lie algebra (See [3,4,6,13] for more details).

In 2009, Salemkar and colleagues [17] generalized the concept of the Schur multiplier to the c-nilpotent multiplier as follows. For a given Lie algebra L, the c-nilpotent multiplier of L, $c \geq 1$, is

$$\mathcal{M}^{(c)}(L) = (R \cap \gamma_{c+1}(F))/\gamma_{c+1}(R, F),$$

where $\gamma_{c+1}(F)$ is the $(c+1)$-st term of the lower central series of F, $\gamma_1(R, F) = R$, $\gamma_{c+1}(R, F) = [\gamma_c(R, F), F]$ and $L \cong F/R$ for a free Lie algebra F. This is analogous to the definition of the Bear-invariant of a group with respect to the variety of nilpotent groups of class at most c (See [2]). The Lie algebra $\mathcal{M}^{(1)}(L)$ is the Schur multiplier of L.

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In [15], Saeedi et al. defined the Schur multiplier of a pair of Lie algebras (Also, see [5] for more details). Let (N, L) be a pair of Lie algebras, in which N is an ideal in L, if N has a complement in L, then for each free presentation $0 \to R \to F \to L \to 0$ of L, $\mathcal{M}(N, L)$ is isomorphic to the factor Lie algebra $(R \cap [S, F])/[R, F]$, in which S is an ideal in F such that $N \cong S/R$ (See [1, 14] for more information). Using the above concept, we define the c-nilpotent multiplier of a pair (N, L) as $\mathcal{M}^{(c)}(N, L) = (R \cap [S,_c F])/[R,_c F]$. In particular, if $N = L$, then $\mathcal{M}^{(c)}(L, L)$ is the c-nilpotent multiplier of L (See [16, 17]). In this paper, we generalize some results of Rismanchian and Araskhan [14].

2. Preliminaries

In this section, we study some notions and results, which are needed for the next section.

All Lie algebras are considered over a fixed field Λ and $[,]$ denotes the Lie bracket. Recall from [9] that Kassel and Loday investigated the notion of Lie crossed module of pairs of Lie algebras (N, L) to be a Lie homomorphism $\sigma : M \to L$ together with an action of L on M, which is denoted by ${}^{l}m$ for all $l \in L, m \in M$ satisfying the following conditions:

(i) $\sigma({}^{l}m) = [l, \sigma(m)]$, for all $l \in L, m \in M$
(ii) ${}^{\sigma(m)}m' = [m, m']$, for all $m, m' \in M$
(iii) $\sigma(M) = N$.

Also, see [10] for more information. The inclusion map $i : N \to L$ is a crossed module of the pair (N, L). In this case, $[N, L]$ and $Z(N, L)$ denote the commutator subalgebra and centralizer of L in N, respectively. Using the above notions, we define the subalgebras $Z_c(N, L)$ and $[N,_c L]$, for all $c \geq 1$, as follows:

$$Z_c(N, L) = \{n \in N \mid [n, l_1, \ldots, l_c] = 0, \forall l_1, , l_c \in L\},$$

$$[N,_c L] = \langle [n, l_1, \ldots, l_c] \mid n \in N, l_1, \ldots, l_c \in L \rangle,$$

where $[n, l_1, \ldots, l_c] = [\ldots[n, l_1], l_2], \ldots, l_c]$. Moreover, a pair (N, L) is called nilpotent of class k, if $[N,_k L] = 0$ and $[N,_{k-1} L] \neq 0$, for some positive integer k.

The following Lemmas are useful for the proof of our main results.

Lemma 2.1 (See [11, Lemma 2.2]). *Let L be a finite dimensional Lie algebra with an ideal N. Let $0 \to R \to F \to L \to 0$ be a free presentation of L and $N \cong \dfrac{S}{R}$ for some ideal S of the free Lie algebra F such that $K = \dfrac{L}{N} \cong \dfrac{F}{S}$. Then, there exists the following epimorphism*

$$\otimes^{c+1}(N, K) \longrightarrow \frac{[S,_c F]}{[R,_c F] + [S,_{c+1} F] + \sum_{i=2}^{c+1} \gamma_{c+1}(S, F)_i},$$

where for all $2 \leq i \leq c$, $\gamma_{c+1}(S,F)_i = [D_1, \ldots, D_{c+1}]$ *such that* $D_1 = D_i = S$, $D_j = F$, *for all* $j \neq 1, i$ *and* $\otimes^{c+1}(N,K) = N \otimes \underbrace{K \otimes \cdots \otimes K}_{c-times}$ *is the abelian tensor product.*

Lemma 2.2 (See [14, Lemma 2.5])**.** *Let* H *and* N *be ideals of Lie algebra* L *and* $N = N_0 \supseteq N_1 \supseteq \cdots$, *a chain of ideals of* N *such that* $[N_i, L] \subseteq N_{i+1}$ *for all* $i = 1, 2, \ldots$. *Then*

$$[N_i, [H,_j L]] \subseteq N_{i+j+1}$$

for all i, j.

Proposition 2.3. *Let* L *be a Lie algebra and* K *be an ideal in* L *contained in* N*; then the following sequences are exact*

(a)

$$0 \longrightarrow \mathcal{M}^{(c)}(K,L) \longrightarrow \mathcal{M}^{(c)}(N,L) \xrightarrow{\alpha}$$
$$\mathcal{M}^{(c)}(\frac{N}{K}, \frac{L}{K}) \longrightarrow \frac{K \cap [N,_c L]}{[K,_c L]} \longrightarrow 0;$$

(b)

$$\mathcal{M}^{(c)}(N,L) \longrightarrow \mathcal{M}^{(c)}(\frac{N}{K}, \frac{L}{K}) \longrightarrow K$$
$$\longrightarrow \frac{L}{[N,_c L]} \longrightarrow \frac{L}{[N,_c L] + K} \longrightarrow 0.$$

Proof. Let $0 \longrightarrow R \longrightarrow F \longrightarrow L \longrightarrow 0$ be a free presentation of L and let S and T be ideals in F such that $K \cong \frac{F}{T}$ and $N \cong \frac{F}{S}$. By definition we obtain

(i) $\mathcal{M}^{(c)}(K,L) = \frac{R \cap [T + R,_c F]}{[R,_c F]}$;

(ii) $\mathcal{M}^{(c)}(N,L) = \frac{R \cap [S + R,_c F]}{[R,_c F]}$;

(iii) $\mathcal{M}^{(c)}(\frac{N}{K}, \frac{L}{K}) = \frac{(T+R) \cap [S + R,_c F]}{[T + R,_c F]}$;

(iv) $\frac{K \cap [N,_c L]}{[K,_c L]} = \frac{((T+R) \cap [S + R,_c F]) + R}{[T + R,_c F] + R}$.

(a) Clearly the following sequence, with obvious natural homomorphism is exact

$$\begin{aligned} 0 \quad &\longrightarrow \frac{R \cap [T + R,_c F]}{[R,_c F]} \longrightarrow \frac{R \cap [(S+R),_c F]}{[R,_c F]} \\ &\longrightarrow \frac{(T+R) \cap [(S+R),_c F]}{[T + R,_c F]} \\ &\longrightarrow \frac{((T+R) \cap [(S+R),_c F]) + R}{[(T+R),_c F] + R} \longrightarrow 0. \end{aligned}$$

(b) The inclusion maps

$$R\cap[(S+R),_c F]\longrightarrow (T+R)\cap[(S+R),_c F]$$
$$\longrightarrow (T+R)\longrightarrow F\longrightarrow F;$$

induce the following exact sequence of homomorphisms

$$\frac{R\cap[(S+R),_c F]}{[R,_c F]}\longrightarrow\frac{(T+R)\cap[(S+R),_c F]}{[(T+R),_c F]}\longrightarrow\frac{T+R}{R}$$
$$\longrightarrow\frac{F}{[(S+R),_c F]+R}\longrightarrow\frac{F}{[(S+R),_c F]+T+R}\longrightarrow 0;$$

which gives the result.

□

The following corollary is an immediate consequence of Proposition 2.3.

Corollary 2.4. *Let (N,L) be a pair of finite dimensional Lie algebras and K be an ideal in L contained in N. Then*

$$\dim(\frac{K\cap[N,_c L]}{[K,_c L]})+\dim\mathcal{M}^{(c)}(N,L)$$
$$=\dim\mathcal{M}^{(c)}(\frac{N}{K},\frac{L}{K})+\dim\mathcal{M}^{(c)}(K,L).$$

3. Some inequalities on dimension of $\mathcal{M}^{(c)}(N,L)$

In this section, we give some inequalities for the dimension of the c-nilpotent multiplier of a pair of finite dimensional Lie algebras.

Theorem 3.1. *Let (N,L) be a pair of finite dimensional Lie algebras and K be a central subalgebra of L contained in N. Let $0\to R\to F\to L\to 0$ be a free presentation of L and $\frac{T}{R}\cong K$. Then*

$$\dim\frac{K\cap[N,_c L]}{[K,_c L]}+\dim\mathcal{M}^{(c)}(N,L)$$
$$\leq\dim\mathcal{M}^{(c)}(\frac{N}{K},\frac{L}{K})+\dim(\otimes^{c+1}(K,\frac{L}{K}))$$
$$+\dim\left(\frac{[R,_c F]+\sum_{i=2}^{c+1}\gamma_{c+1}(T,F)_i}{[R,_c F]}\right).$$

Proof. Since K is a central subalgebra of L, we have $[T,F]\leq R$. Then by Lemma 2.1,

$$\otimes^{c+1}(K,\frac{L}{K}))\longrightarrow\frac{[T,_c F]}{[R,_c F]+\sum_{i=2}^{c+1}\gamma_{c+1}(T,F)_i},$$

is an epimorphism. On the other hand, we have

$$\begin{aligned}&\dim \frac{(R\cap [T,_c F])/[R,_c F]}{([R,_c F]+\sum_{i=2}^{c+1}\gamma_{c+1}(T,F)_i)/[R,_c F]}\\&= \dim \frac{[T,_c F]}{[R,_c F]+\sum_{i=2}^{c+1}\gamma_{c+1}(T,F)_i}.\end{aligned}$$

Therefore,

$$\dim \frac{(R\cap [T,_c F])/[R,_c F]}{([R,_c F]+\sum_{i=2}^{c+1}\gamma_{c+1}(T,F)_i)/[R,_c F]} \le \dim(\otimes^{c+1}(K,\frac{L}{K})),$$

and so,

$$\begin{aligned}&\dim \mathcal{M}^{(c)}(K,L) \le \dim(\otimes^{c+1}(K,\frac{L}{K}))\\&+\dim\left(\frac{[R,_c F]+\sum_{i=2}^{c+1}\gamma_{c+1}(T,F)_i}{[R,_c F]}\right).\end{aligned}$$

Hence, the result holds by Corollary 2.4. □

In Theorems 3.2 and 3.3, we generalize [14, Corollary 2.7].

Theorem 3.2. *Let (N,L) be a pair of finite dimensional nilpotent Lie algebras of class t. Then*

(1) *If $t\ge c+1$, then*

$$\begin{aligned}&\dim[N,_{t-1} L]+\dim \mathcal{M}^{(c)}(N,L)\\&\le \dim \mathcal{M}^{(c)}\left(\frac{N}{[N,_{t-1} L]},\frac{L}{[N,_{t-1} L]}\right)\\&+\dim\left(\otimes^{c+1}([N,_{t-1} L],\frac{L}{Z_{t-1}(N,L)})\right).\end{aligned}$$

(2) *If $t<c+1$, then*

$$\begin{aligned}&\dim[N,_{c} L]+\dim \mathcal{M}^{(c)}(N,L)\\&\le \dim \mathcal{M}^{(c)}\left(\frac{N}{[N,_{t-1} L]},\frac{L}{[N,_{t-1} L]}\right)\\&+\dim\left(\otimes^{c+1}([N,_{t-1} L],\frac{L}{Z_{t-1}(N,L)})\right).\end{aligned}$$

Proof. Let $0\to R\to F\to L\to 0$ be a free presentation of L. Let $N\cong \frac{S}{R}$ and $Z_i(N,L)\cong \frac{T_i}{R}$, for all $0\le i\le t$. Consider the following chain

$$S=T_0\supseteq\cdots\supseteq T_k\supseteq\cdots\supseteq T_{t-1}\supseteq T_t=R\supseteq[R,F]\supseteq\cdots\supseteq[R,_c F].$$

Since $[T_k, F] \subseteq T_{k+1}$, we have $[T_i, [S,_{t-1} F]] \subseteq [R,_i F]$ by Lemma 2.2. This inclusion induces the following epimorphism

$$\begin{array}{ccc} \otimes^{c+1}\left(\dfrac{[S,_{t-1} F]+R}{R}, \dfrac{F}{T_{t-1}}\right) & \longrightarrow & \dfrac{[[S,_{t-1} F]+R,_c F]}{[R,_c F]} \\ (s+R)\otimes(x_1+T_{t-1})\otimes\cdots\otimes(x_c+T_{t-1}) & \longmapsto & [s,x_1,\ldots,x_c]+[R,_c F]. \end{array}$$

So, we have

$$\dim\left(\frac{[[S,_{t-1} F]+R,_c F]}{[R,_c F]}\right) \leq \dim\left(\otimes^{c+1}\left(\frac{[S,_{t-1} F]+R}{R}, \frac{F}{T_{t-1}}\right)\right). \tag{3.1}$$

On the other hand, considering $K = [N,_{t-1} L]$ in Corollary 2.4, if $t \geq c+1$, then

$$\dim[N,_{t-1} L] + \dim \mathcal{M}^{(c)}(N,L) = \dim \mathcal{M}^{(c)}\left(\frac{N}{[N,_{t-1} L]}, \frac{L}{[N,_{t-1} L]}\right)$$
$$+\dim\left(\frac{[[S,_{t-1} F],_c F]}{[R,_c F]}\right),$$

and if $t < c+1$, then

$$\dim[N,_c L] + \dim \mathcal{M}^{(c)}(N,L) = \dim \mathcal{M}^{(c)}\left(\frac{N}{[N,_{t-1} L]}, \frac{L}{[N,_{t-1} L]}\right)$$
$$+\dim\left(\frac{[[S,_{t-1} F]+R,_c F]}{[R,_c F]}\right).$$

Now the theorem follows by (3.1). □

Theorem 3.3. *Let (N,L) be a pair of finite dimensional nilpotent Lie algebras of class at most $t \geq 2$. Then*

$$\dim[N,_c L] + \dim \mathcal{M}^{(c)}(N,L) \leq \dim \mathcal{M}^{(c)}\left(\frac{N}{[N,L]}, \frac{L}{[N,L]}\right)$$
$$+\left(\sum_{i=1}^{t-1} \dim(\otimes^{c+1}([N,_i L], \frac{L}{[N,_i L]})\right).$$

Proof. Let F, S and R be as in Theorem 3.2. Considering $K = [N,L]$ in Corollary 2.4, we have

$$\dim[N,_c L] + \dim \mathcal{M}^{(c)}(N,L) = \dim \mathcal{M}^{(c)}\left(\frac{N}{[N,L]}, \frac{L}{[N,L]}\right)$$
$$+\dim \mathcal{M}^{(c)}([N,L],L) + \dim[N,_{c+1} L].$$

On the other hand,

$$\begin{aligned}
&\dim[N,_{c+1} L] + \dim \mathcal{M}^{(c)}([N,L],L) \\
= \quad & \dim\left(\frac{[S,_{c+1} F] + R}{R}\right) + \dim\left(\frac{(R \cap [S,F,_c F]) + [R,_c F]}{[R,_c F]}\right) \\
= \quad & \dim \frac{[[S,_t F] + R,_c F]}{[R,_c F]} + \sum_{i=1}^{t-1} \dim \frac{[[S,_i F] + R,_c F]}{[[S,_{i+1} F] + R,_c F]}.
\end{aligned}$$

By the assumption, $[N,_t L] = \dfrac{[S,_t F] + R}{R} = 0$, and hence we can write $[[S,_t F] + R,_c F] = [R,_c F]$. Therefore

$$\dim[N,_c L] + \dim \mathcal{M}^{(c)}(N,L) = \dim \mathcal{M}^{(c)}\left(\frac{N}{[N,L]}, \frac{L}{[N,L]}\right)$$
$$+ \sum_{i=1}^{t-1} \dim \frac{[[S,_i F] + R,_c F]}{[[S,_{i+1} F] + R,_c F]}.$$

On the other hand for all $1 \leq i \leq t-1$,

$$\sum_{j=2}^{c+1} \gamma_{c+1}([S,_i F] + R, F)_j + [[S,_i F] + R,_{c+1} F] + [R,_c F] \subseteq [[S,_{i+1} F] + R,_c F].$$

Considering this relation, Lemma 2.1 implies that

$$\dim \frac{[[S,_i F] + R,_c F]}{[[S,_{i+1} F] + R,_c F]} \leq \dim\left(\otimes^{c+1}([N,_i L], \frac{L}{[N,_i L]})\right),$$

and hence the proof is complete. □

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