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DISTINGUISHING NUMBER AND DISTINGUISHING INDEX OF NATURAL AND FRACTIONAL POWERS OF GRAPHS

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Abstract. The distinguishing number (resp. index) $D(G)$ $(D'(G))$ of a graph G is the least integer d such that G has an vertex labeling (resp. edge labeling) with d labels that is preserved only by a trivial automorphism. For any $n \in \mathbb{N}$, the n-subdivision of G is a simple graph $G^{\frac{1}{n}}$ which is constructed by replacing each edge of G with a path of length n. The m^{th} power of G, is a graph with same set of vertices of G and an edge between two vertices if and only if there is a path of length at most m between them in G. The fractional power of G, is the m^{th} power of the n-subdivision of G, i.e., $(G^{\frac{1}{n}})^m$ or n-subdivision of m-th power of G, i.e., $(G^m)^{\frac{1}{n}}$. In this paper we study the distinguishing number and the distinguishing index of the natural and the fractional powers of G. We show that the natural powers more than one of a graph are distinguished by at most three edge labels. We also show that for a connected graph G of order $n \geqslant 3$ with maximum degree $\Delta(G)$, and for $k \geqslant 2$, $D(G^{\frac{1}{k}}) \leqslant \lceil \sqrt[k]{\Delta(G)} \rceil$. Finally we prove that for $m \geqslant 2$, the fractional power of G, i.e., $(G^{\frac{1}{k}})^m$ and $(G^m)^{\frac{1}{k}}$ are distinguished by at most three edge labels.

Keywords: Distinguishing index, distinguishing number, fractional power.
MSC(2010): Primary: 05C15; Secondary: 05E18.

1. Introduction

Let $G = (V, E)$ be a simple finite graph with n vertices. We use the standard graph notation ([8]). An automorphism of G is a permutation σ on the vertex set of G with the property that, for any vertices u and v, $u\sigma \sim v\sigma$ if and only if $u \sim v$ (note that $v\sigma$ denotes the image of the vertex v under the permutation σ). The set of all automorphisms of G, with the operation of composition of permutations, is a permutation group on V and is denoted by $\mathrm{Aut}(G)$.

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A labeling of G, $\phi : V \to \{1, 2, \dots, r\}$, is r-distinguishing, if no non-trivial automorphism of G preserves all of the vertex labels. In other words, ϕ is r-distinguishing if for every non-trivial $\sigma \in \mathrm{Aut}(G)$, there exists an x in V such that $\phi(x) \neq \phi(x\sigma)$. The distinguishing number of a graph G has been defined by Albertson and Collins [2] and is the minimum number r such that G has a labeling that is r-distinguishing. Similar to this definition, Kalinowski and Pilśniak [13] have defined the distinguishing index $D'(G)$ of G which is the least integer d such that G has an edge colouring with d colours that is preserved only by a trivial automorphism. The introduction of these two parameters was a great success; by now more than one hundred papers were written motivated by these two papers! (See, for example [3, 15, 10]).

If x and y are two vertices of G, then the distance $d(x, y)$ between x and y, is defined as the length of a minimum path connecting x and y. The eccentricity of a vertex x is $ecc(x) = \max\{d(x, u) : u \in V(G)\}$ and the radius r and the diameter d of G are defined as the minimum and the maximum eccentricity among vertices of G, respectively. A vertex u of G is called the central vertex if $ecc(u) = r$. The set of all central vertices of G is denoted by $Z(G)$ and is called the center of G. For $k \in \mathbb{N}$, the k-power of G, denoted by G^k, is defined on the vertex set $V(G)$ by adding edges joining any two distinct vertices x and y with distance at most k [1, 16]. In other words, $E(G^k) = \{xy : 1 \leqslant d_G(x, y) \leqslant k\}$. Also the k-subdivision of G, denoted by $G^{\frac{1}{k}}$, is constructed by replacing each edge v_iv_j of G with a path of length k, say $P_{v_iv_j}$. These k-paths are called superedges, any new vertex is an internal vertex, and is denoted by $w_l^{\{v_i,v_j\}}$ if it belongs to the superedge $P_{v_iv_j}$, $i < j$ with distance l from the vertex v_i, where $l \in \{1, 2, \dots, k-1\}$. Note that for $k = 1$, we have $G^{1/1} = G^1 = G$, and if the graph G has v vertices and e edges, then the graph $G^{\frac{1}{k}}$ has $v + (k-1)e$ vertices and ke edges. The fractional power of G, denoted by $G^{\frac{m}{n}}$ is m^{th} power of the n-subdivision of G or n-subdivision of m-th power of G ([11]). Note that the graphs $(G^{\frac{1}{n}})^m$ and $(G^m)^{\frac{1}{n}}$ are different graphs. The fractional power of a graph has been introduced by Iradmusa in [11]. He has investigated the chromatic number and the clique number of the fractional power of graphs. Also, he has studied the domination number and the independent domination number of the fractional powers of graphs ([12]). In the study of the distinguishing number and the distinguishing index of graphs, this may raise the question: What happens to the distinguishing number and the distinguishing index, when we consider the natural power and the fractional power of a graph?

In this paper, we answer to this question. As usual, we denote the complete graph, the path and the cycle of order n by K_n, P_n and C_n, respectively. Also $K_{1,n}$ is the star graph with $n + 1$ vertices.

In the next section, we state some results on the distinguishing number and the distinguishing index of the natural powers of a graph. We show that the natural powers more than one of a graph distinguished by three edge labels. In Sections 3 and 4, we study the distinguishing number and the distinguishing index of the fractional powers of graphs, respectively.

2. The distinguishing number and the distinguishing index of the natural powers of a graph

In this section, we consider the natural powers of a graph and study their distinguishing number and distinguishing index. We begin with the following lemma which follows from the definition of the power of a graph.

Lemma 2.1. *Let G be a connected graph of order n and diameter d.*

(i) *For every natural number $t \geqslant d$, $G^t = K_n$.*

(ii) ([9, Theorem 1]) *Let $k = mn$, where m and n are positive integers. Then $G^k = (G^m)^n$.*

(iii) [4, Lemma 2.1]) *Let u and v be two vertices of a graph G. Then $d_{G^k}(u,v) = \lceil \frac{d_G(u,v)}{k} \rceil$.*

Proposition 2.2. *If G is a connected graph, then for $k \geqslant 2$, $\mathrm{Aut}(G)$, is a subgroup of $\mathrm{Aut}(G^k)$.*

Proof. Since $\mathrm{Aut}(G)$ is a group, it is suffices to show that $\mathrm{Aut}(G) \subseteq \mathrm{Aut}(G^k)$. Let f be an automorphism of G. It is clear that v_i and v_j are adjacent in G^k if and only if $d_G(v_i, v_j) \leqslant k$ and this is true if and only if $d_G(f(v_i), f(v_j)) \leqslant k$ and so $f(v_i)$ and $f(v_j)$ are adjacent in G^k. So $f \in \mathrm{Aut}(G^k)$, and we have the result. □

Now we state a simple and useful proposition.

Proposition 2.3 ([5])**.** *If G_1 and G_2 are two graphs with the same vertex set, and $Aut(G_1)$ is a subgroup of $Aut(G_2)$, then $D(G_1) \leqslant D(G_2)$.*

By Propositions 2.2 and 2.3 we have the following results which are comparison between the distinguishing number of a graph and the distinguishing number of its natural powers:

Corollary 2.4. *For a connected graph G and every $k \geqslant 2$, $D(G) \leqslant D(G^k)$.*

Theorem 2.5. *Let G be a connected graph of order n with diameter d and radius r. If $Z(G) = \{x_1, \ldots, x_t\}$, $t \geqslant 1$, is the center of G, then for $0 \leqslant i \leqslant d-r$ we have*

$$D(G^{r+i}) \geqslant \Big|\{x \in V(G) |\ 0 \leqslant d_G(x_j, x) \leqslant i \text{ for some } j = 1, \ldots, t\}\Big|.$$

Proof. We first prove the case $i = 0$. If $i = 0$, then we have

$$\Big\{x \in V(G) |\ d_G(x_j, x) = 0 \text{ for some } j = 1, \ldots, t\Big\} = \{x_1, \ldots, x_t\}.$$

By the definition of a central vertex and a power graph, the vertices $x_1, \ldots, x_t$ are the vertices of G^r such that $deg_{G^r} x_j = n-1$, where $1 \leqslant j \leqslant t$. So the maps that fix noncentral vertices and act on central vertices as an permutation of $\mathbb{S}_t$, are automorphisms of G^r, because they preserve the adjacency relation in G^r. So we need at least t labels, to have a vertex distinguishing labeling that is not preserved by the mentioned automorphisms. Therefore $D(G^r) \geqslant t$.

The proof for $i > 0$ is similar. Indeed, the elements of the set $\{x \in V(G) | \, 0 \leqslant d_G(x_j, x) \leqslant i \text{ for some } j = 1, \ldots, t\}$ are the vertices of $V(G)$ such that their degree is $n-1$ in G^{r+i}. □

A graph G is Hamiltonian connected, if and only if every two distinct vertices of G are joined by a Hamiltonian path in G (see [6, 17]). The following theorem implies that the cube of every connected graph is Hamiltonian connected (see also [7]).

Theorem 2.6 ([19])**.** *If G is a connected finite graph, then G^3 is Hamiltonian connected.*

We recall that a traceable graph is a graph that possesses a Hamiltonian path.

Theorem 2.7 ([18])**.** *If G is a traceable graph of order $n \geqslant 7$, then $D'(G) \leqslant 2$.*

The assumption $n \geqslant 7$ is substantial in this theorem, because for example $D'(K_{3,3}) = 3$. The following corollary shows that the natural powers more than two of a graph of order at least seven can be distinguished by two edge labels.

Corollary 2.8. *If G is a connected finite graph of order $n \geqslant 7$, then for any $i \geqslant 3$, $D'(G^i) \leqslant 2$.*

Proof. It can follows from Theorem 2.6 that G^i, $i \geqslant 3$, is Hamiltonian connected, and so it is a traceable graph. Since the order of graph is $n \geqslant 7$, so by Theorem 2.7 we have $D'(G^i) \leqslant 2$. □

Remark 2.9. If G is a connected finite graph of order $1 \leqslant n \leqslant 5$ which is not a path graph, then $d \leq 3$. So for $i \geqslant 3$ we have $D'(G^i) = D'(K_n) = 3$. For $n = 6$, by considering all graphs of order $n = 6$, observe that the diameter of G is less than or equal to three except eight cases (see Figure 1). For these eight graphs, the diameter of G is 4 (G is not a path graph), and so $D'(G^i) = D'(K_6) = 2$ for $i \geqslant 4$. It can be easily computed that $D'(G^3) \leqslant 3$ for these eight graphs. Also it can be shown that $D'(P_n^3) \leqslant 3$, for all $1 \leqslant n \leqslant 6$.

To see what happens for the distinguishing index of G^2, we prove the following theorem.

Theorem 2.10. *If G is a connected graph, then $D'(G^2) \leqslant 3$.*

Proof. We present an edge distinguishing labeling of G^2 with three labels in the following steps:

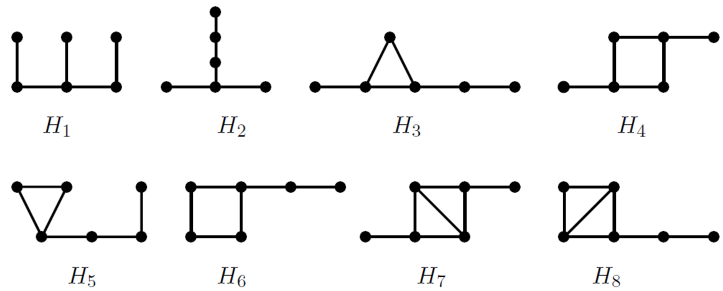

FIGURE 1. The graphs of order $n = 6$ and diameter $d = 4$.

Step 1) The induced graph of G^2 with the vertices of $N_G[x]$, i.e., $G^2[N_G[x]]$, is a complete graph of order $deg_G x + 1$. Hence we can label all its edges with three labels $0, 1$, and 2 distinguishingly and we do not use label 0 any more. So with respect to the edge labeling of $G^2[N_G[x]]$, we can conclude that if f is an automorphism of G^2 preserving the distinguishing edge labeling of G^2, then the restriction of f to $N_G[x]$ is the identity.

Step 2) If $N_G[N_G[x]] \subseteq N_G[x]$, then we labeled all edges of G^2. Otherwise, let $v \in N_G[x]$ such that $N_G[v] \nsubseteq N_G[x]$. So the induced subgraph $G^2[N_G[v]]$ is a complete graph of order $deg_G v + 1$. Since the vertex v of G is fixed under each automorphism of G^2 preserving the labeling, so we can label the remaining edges of $G^2[N_G[v]]$ which did not label in the Step 1, with two labels 1 and 2 such that for any two different vertices a and b of complete graph $G^2[N_G[v]]$, except the vertex v, there exists a label, say 1, such that the number of label 1 used for the labeling of incident edges to a and b is different, and hence $G^2[N_G[v]]$ has a distinguishing labeling.

Step 3) If there exists element of $N_G[x]$, say v, such that $N_G[v] \nsubseteq N_G[x]$, then we do the same work as Step 2 for v. So the induced subgraph $G^2[N_G[v]]$ (for every $v \in N_G[x]$ with $N_G[v] \nsubseteq N_G[x]$), and hence the elements of $N_G[N_G[x]]$ are fixed under each nontrivial automorphism of G^2 preserving the distinguishing edge labeling of G^2, by Steps 1 and 2.

Step 4) If $N_G[N_G[N_G[x]]] \subseteq N_G[N_G[x]]$, then we labeled all edges of G^2. Otherwise, we repeat the same method in Steps 2 and 3 for the vertices of $N_G[N_G[x]]$, say y, such that $N_G[y] \nsubseteq N_G[N_G[x]]$. Similarly the elements of $N_G[N_G[N_G[x]]]$ are fixed under each nontrivial automorphism of G^2 preserving the distinguishing edge labeling of G^2.

After finite steps we get $\underbrace{N_G[\ldots[N_G[x]]\ldots]}_{l-times} \subseteq \underbrace{N_G[\ldots[N_G[x]]\ldots]}_{(l-1)-times}$, where $l \geqslant 2$ is an integer number. In this case we labeled all edges of G^2. We checked that the elements of $\underbrace{N_G[\dots[N_G[x]]\dots]}_{(i)-times}$ are fixed under each nontrivial automorphism of G^2 preserving the distinguishing edge labeling of G^2 in the prior steps for $1 \leqslant i \leqslant l-1$. Thus the identity is the only automorphism preserving the labeling. Since we used three labels, so $D'(G^2) \leqslant 3$. □

Now, we are ready to state the following result which obtain from Corollary 2.8, Remark 2.9 and Theorem 2.10. This result implies that all natural powers greater than or equal two of a graph G are distinguished by three edge labels.

Corollary 2.11. *If G is a connected finite graph, then $D'(G^m) \leqslant 3$ for $m \geqslant 2$.*

It seems that $D'(G^2) \leqslant 2$ for all connected graphs with diameter $d > 2$. For instance, by Theorem 2.7 we have $D'(C_n^k) = 2$ where $n \geqslant 7$ and $k \geqslant 2$. However, until now all attempts to show this result failed, and it remains as open problem. We close this section with the following conjecture:

Conjecture 2.12. If G is a connected graph with diameter $d > 2$, then $D'(G^2) \leqslant 2$.

3. Distinguishing number of the fractional power of graphs

In this section, we study the distinguishing number of the fractional powers of graphs. It can easily be verified that for $n \geqslant 2$ and $k \geqslant 2$, $D(P_n^{\frac{1}{k}}) = 2$. Also for $n \geqslant 3$ and $k \geqslant 2$, $D(C_n^{\frac{1}{k}}) = 2$. We can show by Theorem 3.5 that $D(K_n^{\frac{1}{2}}) = 2$ for $n \geqslant 3$ and easily show that $D(K_n^{\frac{1}{k}}) = 2$ for $n \geqslant 3$ and $k \geqslant 2$. We state and prove the following lemma to obtain more results on the distinguishing number of the fractional power of graphs.

Lemma 3.1. *Let G be a connected graph of order $n \geqslant 3$ which is not a cycle. If $f \in \mathrm{Aut}(G^{\frac{1}{k}})$, then the restriction of f to the set of vertices of G is $V(G)$, i.e., $f|_{V(G)} \in \mathrm{Aut}(G)$.*

Proof. Let $\{v_1, \dots, v_n\}$ be the vertex set of G. By contradiction, suppose that there exists $t \in \{1, \dots, k-1\}$ such that $w_t^{\{v_i,v_j\}}$ is an internal vertex of $G^{\frac{1}{k}}$ such that $f(w_t^{\{v_i,v_j\}}) = v_s$. Since $deg_{G^{\frac{1}{k}}}(w_t^{\{v_i,v_j\}}) = 2$, so $deg(v_s) = 2$ (note that $deg_G(v_i) = deg_{G^{\frac{1}{k}}}(v_i)$ for all $i \in \{1 \dots, n\}$). Let v_{s_1} be the adjacent vertex to v_s in G and $w_1^{\{v_s,v_{s_1}\}}$ be the adjacent vertex to v_s in $G^{\frac{1}{k}}$. Without loss of generality, we can assume that

$$f(w_{t-1}^{\{v_i,v_j\}}) = w_1^{\{v_s,v_{s_1}\}}, f(w_{t-2}^{\{v_i,v_j\}}) = w_2^{\{v_s,v_{s_1}\}}, \dots, f(w_1^{\{v_i,v_j\}}) = w_{t-1}^{\{v_s,v_{s_1}\}},$$
$$f(v_i) = w_t^{\{v_s,v_{s_1}\}}.$$

Therefore $deg_G(v_i) = 2$. Continuing this process, we see that any vertex of G has degree two, and so G is a cycle, which is a contradiction. □

Observation 3.2. Let G be a connected graph of order $n \geqslant 3$ which is not a cycle. Let $i < j$ and v_i and v_j be two adjacent vertices of G. Suppose that f is an automorphism of $G^{\frac{1}{k}}$ such that $f(v_i) = v_{i'}$ and $f(v_j) = v_{j'}$. We have two following cases:

(i) If $i' < j'$, then $f(w_t^{\{v_i,v_j\}}) = w_t^{\{v_{i'},v_{j'}\}}$, where $1 \leqslant t \leqslant k-1$.
(ii) If $i' > j'$, then $f(w_t^{\{v_i,v_j\}}) = w_{k-t}^{\{v_{j'},v_{i'}\}}$, where $1 \leqslant t \leqslant k-1$.

Corollary 3.3. *Let G be a connected graph of order $n \geqslant 3$ which is not a cycle. Then for every natural number k,*

(i) $|\mathrm{Aut}(G^{\frac{1}{k}})| = |\mathrm{Aut}(G)|$.
(ii) $D(G^{\frac{1}{k}}) \leqslant D(G)$.

Proof. (i) It follows from Observation 3.2.

(ii) By Observation 3.2, if we label the vertices of the graph G with $D(G)$ labels in a distinguishing way and assign the internal vertices the label 1, then we have a distinguishing labeling. Therefore $D(G^{\frac{1}{k}}) \leqslant D(G)$. □

Here we state the following definition:

Definition 3.4 ([14])**.** The total distinguishing number $D''(G)$ of a graph G is the least number d such that G has a total colouring with d colours that is preserved only by the identity automorphism of G.

Theorem 3.5. *If G is a connected graph of order $n \geqslant 3$, then for every natural number k, $D(G^{\frac{1}{2k}}) = D''(G^{\frac{1}{k}})$.*

Proof. Note that $G^{\frac{1}{2k}}$ is constructed by replacing each edge v_iv_j of $G^{\frac{1}{k}}$ with a path of length 2, say $P_{v_iv_j}$. If we consider the label of the edges in total labeling of $G^{\frac{1}{k}}$ as the label of internal vertices of $G^{\frac{1}{2k}}$, then the result follows. □

Now we want to obtain a better upper bound for $D(G^{\frac{1}{k}})$. For this purpose, let $S_k^G(x)$, $k \geqslant 0$ denote a sphere of radius k with a center x, i.e., the set of all vertices of G at distance k from x. We know that every finite tree T has either a central vertex or a central edge which is fixed by every automorphism of T. The following theorem gives an upper bound for the distinguishing number of $T^{\frac{1}{k}}$.

Theorem 3.6. *If T is a tree of order $n \geqslant 3$ with maximum degree $\Delta(T)$, then $D(T^{\frac{1}{k}}) \leqslant \lceil \sqrt[k]{\Delta(T)} \rceil$.*

Proof. The basic idea of proof follows directly from the proof of [14, Theorem 2.1]. If T has a central vertex v_0, then the label of v_0 can be arbitrary. Having $\lceil \sqrt[k]{\Delta(T)} \rceil$ labels, we have at least $\Delta(T)$ different k-arys $(c_1, \ldots, c_k)$ of labels, as the colouring need not be proper. Every $k-1$ added vertices on the edge incident to v_0, in T, and its end vertex of this edge in T, obtain a distinct k-ary of labels $(c_1, \ldots, c_k)$ in the $\bigcup_{i=1}^{k} S_i^{T^{\frac{1}{k}}}(v_0)$. Hence, all vertices of distance at most k to v_0 are fixed by every automorphism of $T^{\frac{1}{k}}$ preserving this colouring. Next, we label the internal vertices on the edges going to subsequent spheres of T by k-arys of labels in the same way as for the $\bigcup_{i=1}^{k} S_i^{T^{\frac{1}{k}}}(v_0)$. By induction on the distance from v_0, all vertices of $T^{\frac{1}{k}}$ are fixed.

Suppose that T has a central edge e_0. Let T_1 and T_2 be subtrees obtained by deleting the edge e_0. If we put distinct labels on the end vertices of e_0, and assign arbitrary label to the internal vertices on e_0 in $T^{\frac{1}{k}}$, then these verices are fixed by every automorphism. Next, for $i = 1, 2$, we label the tree T_i using the same method as in the previous case. $\square$

To see that the bound in Theorem 3.6 is sharp, we show the following theorem:

Theorem 3.7. *For $m \geqslant 3$ and $k \geqslant 2$, $D(K_{1,m}^{\frac{1}{k}}) = \lceil \sqrt[k]{m} \rceil$.*

Proof. Denote the vertex of degree m of $K_{1,m}$ by v_0 and the remaining vertices by $v_1, \ldots, v_m$. Since v_0 is the only vertex of degree m, so the label of v_0 can be arbitrary. In an r-distinguishing labeling, each of the k-ary consisting of a $k-1$ internal vertices on the edge v_0v_i and the vertex v_i, $(1 \leqslant i \leqslant m)$ must have a different k-ary of labels. There are r^k possible k-ary of labels using r colors, hence $D(K_{1,m}^{\frac{1}{k}}) = \min\{r : r^k \geqslant m\} = \lceil \sqrt[k]{m} \rceil$. $\square$

Theorem 3.8. *If G is a connected graph of order $n \geqslant 3$ with maximum degree Δ, then for any $k \geqslant 2$, $D(G^{\frac{1}{k}}) \leqslant \lceil \sqrt[k]{\Delta(G)} \rceil$.*

Proof. The basic idea of proof follows directly from the proof of [14, Theorem 2.2]. Clearly, $\Delta \geqslant 2$ and we have at least two labels. If G is a tree, then the result is true by Theorem 3.6. Suppose that G contains a cycle. If G is just a cycle or a complete graph, then the claim follows from the information presented in the first of this section. Otherwise, we can always choose a vertex v_0 lying on a cycle such that the sphere $S_2^G(v_0)$ is nonempty. We label v_0 with the label 2 and consider a BFS tree T of G rooted at v_0. We first label the vertices of the tree $T^{\frac{1}{k}}$. For a given vertex v of G, let $M(v) = \{(w_1^{\{v,u\}}, \ldots, w_{k-1}^{\{v,u\}}, u) : vu \in E(G)\}$ (note that the vertices $w_1^{\{v,u\}}, \ldots, w_{k-1}^{\{v,u\}}$ are internal vertices on P_{vu}). Let $S_1^G(v_0) = \{v_1, \ldots, v_p\}$. Without loss of generality, we can assume that v_1 has a neighbour in $S_2^G(v_0)$. We label both k-ary $(w_1^{\{v_0,v_1\}}, \ldots, w_{k-1}^{\{v_0,v_1\}}, v_1)$ and $(w_1^{\{v_0,v_2\}}, \ldots, w_{k-1}^{\{v_0,v_2\}}, v_2)$ with a k-ary $(1, \ldots, 1)$. Then we label each k-ary of $M(v_0) \setminus \{(w_1^{\{v_0,v_1\}}, \ldots, w_{k-1}^{\{v_0,v_1\}}, v_1), (w_1^{\{v_0,v_2\}}, \ldots, w_{k-1}^{\{v_0,v_2\}}, v_2)\}$ with a distinct k-ary of labels different from $(1, \ldots, 1)$. Thus $(1, \ldots, 1)$ appears twice as a k-ary of labels in $M(v_0)$. We will then label the vertices of graph $G^{\frac{1}{k}}$ in such a way that v_0 will be the only vertex of $G^{\frac{1}{k}}$ labeled with the label 2 such that k-ary $(1, \ldots, 1)$ appears twice in the $M(v_0)$. Hence v_0 will be fixed by every automorphism of $G^{\frac{1}{k}}$ preserving the labeling. Therefore, all vertices in $S_1^G(v_0)$

and all internal vertices on $P_{v_0v_1},\ldots,P_{v_0v_p}$ will be also fixed, except, possibly $(w_1^{\{v_0,v_1\}},\ldots,w_{k-1}^{\{v_0,v_1\}},v_1)$ and $(w_1^{\{v_0,v_2\}},\ldots,w_{k-1}^{\{v_0,v_2\}},v_2)$. To distinguish them, we label the sets $\{(w_1^{\{v_1,u\}},\ldots,w_{k-1}^{\{v_1,u\}},u)\in M(v_1):u\in S_2^G(v_0)\}$ and $\{(w_1^{\{v_2,u\}},\ldots,w_{k-1}^{\{v_2,u\}},u)\in M(v_2):v_2u\in E(T),u\in S_2^G(v_0)\}$ with two distinct sets of k-ary of labels (this is possible, because each of these sets contains at most $\Delta-1$ elements, and we have Δ distinct k-ary of labels). Therefore every internal vertex on the superedges $P_{v_0v_1},\ldots,P_{v_0v_p}$ and P_{v_1u},P_{v_2u} where $u\in S_2^G(v_0)$ will be fixed by every automorphism preserving our labeling. For $i=3,\ldots,p$, we then label all elements of $\{(w_1^{\{v_i,u\}},\ldots,w_{k-1}^{\{v_i,u\}},u):v_iu\in E(T),u\in S_2^G(v_0)\}$ with distinct k-ary of labels different from the $(1,\ldots,1)$. Thus all other vertices in $S_2^G(v_0)$ and all internal vertices on $P_{v_0v_1},\ldots,P_{v_0v_p}$ and $P_{v_1u},\ldots,P_{v_pu}$ where $u\in S_2^G(v_0)\setminus S_1^G(v_0)$ will be also fixed.

Then we proceed recursively with respect to the radius r of subsequent sphere $S_r^G(v_0)$ according to the ordering of the tree T. Suppose all vertices of $S_i^G(v_0)=\{u_1,\ldots,u_{l_i}\}$, $i=0,1,\ldots,r$ and all internal vertices on $P_{v_av_b}$, where $v_a,\in S_{i-1}^G(v_0)\setminus S_{i-2}^G(v_0)$ and $v_b\in S_i^G(v_0)\setminus S_{i-1}^G(v_0)$, are fixed by every automorphism preserving labels. For each subsequent vertex u_j, $j=1,\ldots,l_r$ we label every k-ary $(w_1^{\{u_j,u\}},\ldots,w_{k-1}^{\{u_j,u\}},u)$ where u is a descendent of u_j in T, with a distinct k-ary of labels except for $(1,\ldots,1)$. This is possible, because the number of k-ary to be labeled is not greater than the number of admissible k-ary of labels. Thus all neighbours of u_j in $S_{r+1}^G(v_0)$ and all internal vertices on the superedges P_{u_ju}, where u is a descendent of u_j in T, will be also fixed.

Finally, we label all remaining vertices in $V(G^{\frac{1}{k}})\setminus V(T^{\frac{1}{k}})$ with the label 2. It is easy to see that if v is a vertex labeled with the label 2 such that the k-ary $(1,\ldots,1)$ appears twice in $M(v)$, then $v=v_0$. Hence all vertices of $G^{\frac{1}{k}}$ are fixed by any automorphism of $G^{\frac{1}{k}}$ preserving this labeling. □

4. Distinguishing index of the fractional powers of graphs

In this section, we study the distinguishing index of the fractional powers of graphs. It can be easily verified that for $n\geqslant 2$ and $k\geqslant 2$, $D'(P_n^{\frac{1}{k}})=2$ and for $n\geqslant 3$ and $k\geqslant 2$, $D'(C_n^{\frac{1}{k}})=2$. We begin with the following theorem.

Theorem 4.1. *Let G be a connected graph of order $n\geqslant 3$ such that it is not a cycle. For any $k\geqslant 2$, $D(G^{\frac{1}{k+1}})\leqslant D'(G^{\frac{1}{k}})$.*

Proof. We define a distinguishing vertex labeling for $G^{\frac{1}{k+1}}$ with $D'(G^{\frac{1}{k}})$ labels. Suppose that $P_{v_iv_j}$ is a superedge that has been replaced with the edge v_iv_j in the structure of $G^{\frac{1}{k}}$. If we assign the label of edges $P_{v_iv_j}$ in $G^{\frac{1}{k}}$ to the internal vertices of the superedge that has been replaced with the edge v_iv_j in construction of $G^{\frac{1}{k+1}}$, and assign the remaining vertices the label 1, then by

Observation 3.2 we can conclude that the labeling is distinguishing. Since we used $D'(G^{\frac{1}{k}})$ labels, the result follows. $\square$

Theorem 4.2. *Let G be a connected graph of order $n \geqslant 3$ such that it is not a cycle. Then* $D'(G^{\frac{1}{2}}) \leqslant \lceil \frac{1+\sqrt{1+8D'(G)}}{2} \rceil$.

Proof. Using the label of edges, we partition the edge set of G, to $D'(G)$ classes, each class contains the edges with the same labels. Suppose that the elements of $[i]$-th class are $e_{i1}, \ldots, e_{is_i}$, where $1 \leqslant i \leqslant D'(G)$ and $\sum_{i=1}^{D'(G)} s_i = |E(G)|$. We know that each edge of G is replaced with a path of length 2 in $G^{\frac{1}{2}}$. Let the edge e_{ij} in G be replaced with two edges e_{ij}^1 and e_{ij}^2 in $G^{\frac{1}{2}}$. For $1 \leqslant i \leqslant D'(G)$, we assign the distinct pairs (c_{i1}, c_{i2}) of labels to all new edges e_{ij}^1 and e_{ij}^2, where $1 \leqslant j \leqslant s_i$ such that

(i) $c_{i1} \neq c_{i2}$ for $1 \leqslant i \leqslant D'(G)$,
(ii) $\{c_{i1}, c_{i2}\} \neq \{c_{i'1}, c_{i'2}\}$ for $1 \leqslant i, i' \leqslant D'(G)$ and $i \neq i'$.

By Observation 3.2, this labeling is distinguishing. Since there exist $\sum_{i=2}^{r}(i-1)$ possible ordered pairs of such labels using r labels, hence $D'(G^{\frac{1}{2}}) \leqslant \min\{r : \sum_{i=2}^{r}(i-1) \geqslant D'(G)\}$. By an easy computation, we see that

$$\min\{s : \sum_{i=2}^{r}(i-1) \geqslant D'(G)\} = \lceil \frac{1+\sqrt{1+8D'(G)}}{2} \rceil.$$

Therefore we have the result. $\square$

Let $k \geqslant 2$ and $(c_1, \ldots, c_k)$ be an k-ary of labels such that it is not symmetric, i.e., there exists i, $1 \leqslant i \leqslant k$ such that $c_i \neq c_{k-i}$. It can be computed that there are $r^k - r^{\lceil k/2 \rceil}$ of these kind of k-ary's using r labels. Let d'_k be the minimum number of labels that have been used in the construction of $D'(G)$ numbers of such k-ary. In fact, $d'_k = \min\{r : r^k - r^{\lceil k/2 \rceil} \geqslant D'(G)\}$. Here we state and prove the following theorem.

Theorem 4.3. *Let G be a connected graph of order $n \geqslant 3$ such that it is not a cycle. Then $D'(G^{\frac{1}{k}}) \leqslant d'_k$ for $k \geqslant 2$.*

Proof. Using the label of edges, we partition the edge set of G to $D'(G)$ classes, each class contains the edges with the same labels. Suppose that the elements of $[i]$-th class are $e_{i1}, \ldots, e_{is_i}$, where $1 \leqslant i \leqslant D'(G)$ and $\sum_{i=1}^{D'(G)} s_i = |E(G)|$. We know that each edge of G is replaced with a path of length k in $G^{\frac{1}{k}}$. Let the edge e_{ij} in G be replaced with the edges $e_{ij}^1, \ldots, e_{ij}^k$ in $G^{\frac{1}{k}}$. For $1 \leqslant i \leqslant D'(G)$, we assign the asymmetric distinct k-ary $(c_{i1}, \ldots, c_{ik})$ of labels to all new edges $e_{ij}^1, \ldots, e_{ij}^k$, where $1 \leqslant j \leqslant s_i$. By Observation 3.2 this labeling is distinguishing. Since the number of labels that have been used, is d'_k, so we have the result. $\square$

By Theorems 3.8, 4.1 and 4.3, we have two upper bounds for $D(G^{\frac{1}{k}})$, that are d'_{k-1} and $\lceil \sqrt[k]{\Delta(G)}\rceil$, and hence $D(G^{\frac{1}{k}}) \leqslant \min\{d'_{k-1}, \lceil \sqrt[k]{\Delta(G)}\rceil\}$.

Now by Corollary 2.11 and Theorem 4.3 we have the following result.

Corollary 4.4. *If G is a connected finite graph of order $n \geqslant 3$, then $D'(G^{\frac{m}{k}}) \leqslant 3$ for $m \geqslant 2$ and $k \geqslant 2$.*

Proof. We know that $G^{\frac{m}{k}}$ means $(G^{\frac{1}{k}})^m$ or $(G^m)^{\frac{1}{k}}$. In the case $(G^{\frac{1}{k}})^m$ the result follows directly from Corollary 2.11. For the case $(G^m)^{\frac{1}{k}}$, if G^m is a cycle, then $(G^m)^{\frac{1}{k}}$ is a cycle and so $D'(G^{\frac{m}{k}}) \leqslant 3$. If G^m is not a cycle, then we have $D'((G^m)^{\frac{1}{k}}) \leqslant d'_k$ by Theorem 4.3 and d'_k is the minimum number of labels that have been used in construction of $D'(G^m)$ numbers of asymmetric k-ary. Since $D'(G^m) \leqslant 3$, so $d'_k \leqslant 3$ and therefore $D'((G^m)^{\frac{1}{k}}) \leqslant 3$. □

The following result follows easily from Corollaries 2.4 and 3.3.

Corollary 4.5. *If G is a connected finite graph of order $n \geqslant 3$, then for $m \geqslant 1$ and $k \geqslant 1$ we have*

(i) $D(G^{\frac{1}{k}}) \leqslant D((G^{1/k})^m)$.

(ii) $D((G^m)^{\frac{1}{k}}) \leqslant D(G^m)$.

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