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WEAK COMPACTNESS OF THE SET OF ε-EXTENSIONS

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Abstract. Let W be a subspace of a normed space X, $\varepsilon > 0$ be given and let $f \in W^*$. Every extension of f (in the sense of Hahn-Banach Theorem) is a ε-extension of f. We determine when the set of all ε-extensions of every $f \in W^*$ is weakly compact. Finally by using of ε-extensions, we characterize weak*-closed ε-weakly Chebyshev subspaces of X^*.

1. Introduction

Let X be a (complex or real) normed space, $\varepsilon > 0$ be given and let W be a subspace of X. A point $y_0 \in W$ is said to be a ε-approximation for $x \in X$ if $\|x - y_0\| \leq d(x, W) + \varepsilon$. For $x \in X$, put

$$P_{W,\varepsilon}(x) = \{y \in W : \|x - y\| \leq d(x, W) + \varepsilon\}.$$

It is clear that $P_{W,\varepsilon}(x)$ is a non-empty, bounded and convex subset of X. Also, $P_{W,\varepsilon}(x)$ is closed for all $x \in X$, if W is closed.

Recently, the author has defined ε-quasi Chebyshev and ε-weakly Chebyshev subspaces of a normed space (see [8,9]). A subspace W of a normed space X is called ε-quasi Chebyshev (ε-Weakly Chebyshev) if $P_{W,\varepsilon}(x)$ is compact (weakly compact) for all $x \in X$. Also,

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some types of proximinality are investigated (see [2]-[7]).

Let W be a subspace of a normed space X, $\varepsilon > 0$ be given and let $f \in W^*$. A linear functional $g \in X^*$ is called ε-extension of f if $g|_W = f$ and $\|g\| - \varepsilon \leq \|f\| \leq \|g\|$. We denote the set of all ε-extensions of f by $E_\varepsilon(f)$. Note that $E_\varepsilon(f)$ is a non-empty, closed and convex subset of X^*.

Recall that if W is a subspace of a normed space X, then $W^\perp = \{f \in X^* : f(w) = 0 \text{ for all } w \in W\}$. Also if M is a subspace of X^*, then ${}^\perp M = \{x \in X : f(x) = 0 \text{ for all } f \in M\}$ and $\|f\|_{M^\perp} = sup\{|\varphi(f)| : \|\varphi\| \leq 1, \varphi \in M^\perp\}$, for all $f \in X^*$. Finally, $\hat{x}$ stands for the canonical image of $x \in X$ in the second dual X^{**}.

We conclude this section by a list of known lemmas needed in the proof of the main results.

Lemma 1.1. [8; Lemma 2.1]. *Let W be a subspace of a normed linear space X $x \in X$, $y_0 \in W$ and $\varepsilon > 0$ be given. Then, $y_0 \in P_{W,\varepsilon}(x)$ if and only if $\|x - y_0\| \leq \|x - y_0\|_{W^\perp} + \varepsilon$, where $\|x - y_0\|_{W^\perp} = sup\{|f(x - y_0)| : \|f\| \leq 1, f \in W^\perp\}$.*

Lemma 1.2. [8; Lemma 2.2]. *Let W be a closed subspace of a normed linear space X, $x \in X$ and $\varepsilon > 0$ be given. Then, $M \subseteq P_{W,\varepsilon}(x)$ if and only if there exists $f \in X^*$ such that $\|f\| = 1$, $f|_W = 0$ and $f(x - y) \geq \|x - y\| - \varepsilon$ for all $y \in M$.*

Lemma 1.3. [8; Corollary 2.12(a)]. *All closed subspaces of a normed space X are ε-quasi Chebyshev in X if and only if X is finite dimensional.*

Lemma 1.4. [9; Corollary 2.8(a)]. *All closed subspaces of a normed space X are ε-weakly Chebyshev in X if and only if X is reflexive.*

Lemma 1.5. [3; Lemma 1.4]. *Let X be a normed space and let W be weak*-closed subspace of X^*. Then, $\|f\|_{W^\perp} = \|f|_{{}^\perp W}\|$ for all $f \in X^*$.*

2. Weak compactness of the set of ε-extensions

Now, we are ready to state and prove our results on ε-extensions.

Lemma 2.1. *Let W be a subspace of a normed space X and $f, g \in X^*$. Then*

$$g \in P_{W^\perp,\epsilon}(f) \quad \Longleftrightarrow \quad \widehat{f-g} \in E_\epsilon(\hat{f}|_{W^{\perp\perp}}).$$

Proof. Let $g \in P_{W^\perp,\varepsilon}(f)$. Then, by Lemma 1.2, there exists $\Lambda \in X^{**}$ such that $\|\Lambda\| = 1$, $\Lambda|_{W^\perp} = 0$ and $\Lambda(f-g) \geq \|f-g\| - \varepsilon$. It follows that, $\|f-g\| - \varepsilon \leq \|(\widehat{f-g})|_{W^{\perp\perp}}\| \leq \|f-g\|$. Thus, $\|\widehat{f-g}\| - \varepsilon \leq \|\hat{f}|_{W^{\perp\perp}}\| \leq \|\widehat{f-g}\|$. Therefore, $\widehat{f-g}$ is a ε-extension of $\hat{f}|_{W^{\perp\perp}}$. Now, let $\widehat{f-g}$ be a ε-extension of $\hat{f}|_{W^{\perp\perp}}$. Then, $\hat{g}|_{W^{\perp\perp}} = 0$ and hence $g \in W^\perp$. On the other hand, $\|f-g\| = \|\widehat{f-g}\| \leq \|\hat{f}|_{W^{\perp\perp}}\| + \varepsilon = \|(\widehat{f-g})|_{W^{\perp\perp}}\| + \varepsilon = \|f-g\|_{W^{\perp\perp}} + \varepsilon$. Therefore, by Lemma 1.1, $g \in P_{W^\perp,\varepsilon}(f)$. $\square$

Lemma 2.2. *Let W be a subspace of a normed space X, $f \in W^*$ and let $\tilde{f} \in X^*$ be an extension of f. Then*

$$E_\epsilon(\hat{\tilde{f}}|_{W^{\perp\perp}}) = \{\hat{g} : g \in E_\epsilon(f)\}.$$

Proof. Let $g \in X^*$ be such that $\hat{g}$ is a ε-extension of $(\hat{\tilde{f}})|_{W^{\perp\perp}}$. Then, $\hat{g}(\varphi) = (\hat{\tilde{f}})(\varphi)$, for all $\varphi \in W^{\perp\perp}$. Thus,

$g(x) = \hat{x}(g) = \hat{g}(\hat{x}) = (\hat{\tilde{f}})(\hat{x}) = \hat{x}(\tilde{f}) = \tilde{f}(x) = f(x)$, for all $x \in W$. Since

$$\|g\| - \varepsilon = \|\hat{g}\| - \varepsilon \leq \|(\hat{\tilde{f}})|_{W^{\perp\perp}}\| = \|\tilde{f}\| = \|f\| \leq \|\hat{g}\| = \|g\|,$$

g is a ε-extension of f. Now, let $g \in X^*$ be a ε-extension of f. Then, $g(x) = f(x) = \tilde{f}(x)$, for all $x \in W$. Since $\hat{g}$ and $(\hat{\tilde{f}})$ are continuous with respect to the weak-topology on X^{**} and W is dense in $W^{\perp\perp}$ for the weak-topology on X^{**}, we have $\hat{g}(\varphi) = (\hat{\tilde{f}})(\varphi)$, for all $\varphi \in W^{\perp\perp}$. Hence, $\hat{g}|_{W^{\perp\perp}} = (\hat{\tilde{f}})|_{W^{\perp\perp}}$ and

$$\|\hat{g}\| - \varepsilon = \|g\| - \varepsilon \leq \|f\| = \|\tilde{f}\| = \|(\hat{\tilde{f}})|_{W^{\perp\perp}}\| \leq \|g\| = \|\hat{g}\|.$$

Therefore, $\hat{g}$ is a ε-extension of $(\hat{\tilde{f}})|_{W^{\perp\perp}}$. $\square$

Theorem 2.3. *Let W be a subspace of a normed space X and let $\varepsilon > 0$ be given. Then the following are equivalent:*

(a) For each $f \in W^*$, $E_\varepsilon(f)$ is weakly compact.

(b) For each $f \in X^*$, the set of all $g \in X^*$ such that $\hat{g}$ is a ε-extension of $\hat{f}|_{W^{\perp\perp}}$ is weakly compact.

(c) $W^\perp$ is a ε-weakly Chebyshev subspace of X^*.

Proof. (a)$\Rightarrow$(b). For each $f \in X^*$, $f_1 \in W^*$, where $f_1 = f|_W$. Let $\{g_n\}_{n\geq 1}$ be an arbitrary sequence in X^* such that $\hat{g_n}$ is a ε-extension of $\hat{f}|_{W^{\perp\perp}}$ $(n = 1, 2, \dots)$. Then, by Lemma 2.2, $\{g_n\}_{n\geq 1}$ is a sequence in the set of all ε-extensions of f_1. Therefore, $\{g_n\}_{n\geq 1}$ has a weakly convergent subsequence, and hence the set of all $g \in X^*$ such that $\hat{g}$ is a ε-extension of $\hat{f}|_{W^{\perp\perp}}$ is weakly compact.

(b)$\Rightarrow$(a). Let $f \in W^*$ be given. Then, f has an extension $\tilde{f} \in X^*$, by Hahn-Banach Theorem. Now, (a) is an immediate consequence of Lemma 2.2.

(b)$\Rightarrow$(c). Let $f \in X^*$ be given and let $\{g_n\}_{n\geq 1}$ be a sequence in $P_{W^\perp,\varepsilon}(f)$. Then, by Lemma 2.1, $\{f - g_n\}_{n\geq 1}$ is a sequence in the set of all $g \in X^*$ such that $\hat{g}$ is a ε-extension of $\hat{f}|_{W^{\perp\perp}}$. Thus, there exists a weakly convergent subsequence $\{f - g_{n_k}\}_{k\geq 1}$ of $\{f - g_n\}_{n\geq 1}$. Therefore, $\{g_{n_k}\}_{k\geq 1}$ is a weakly convergent subsequence of $\{g_n\}_{n\geq 1}$. Hence, $W^\perp$ is a ε-weakly Chebyshev subspace of X^*.

(c)$\Rightarrow$(b). Let $f \in X^*$ be given and let $\{g_n\}_{n\geq 1}$ be a sequence in X^* such that $\hat{g_n}$ is an extension of $\hat{f}|_{W^{\perp\perp}}$, for all $n \geq 1$. Then, by Lemma 2.1, $\{f - g_n\}_{n\geq 1}$ is a sequence in $P_{W^\perp,\varepsilon}(f)$. Thus, there exists a weakly convergent subsequence $\{f - g_{n_k}\}_{k\geq 1}$ of $\{f - g_n\}_{n\geq 1}$. Therefore, $\{g_n\}_{n\geq 1}$ has a weakly convergent subsequence. $\square$

The proof of the following Theorem is similar to that of Theorem 2.3.

Theorem 2.4. *Let W be a subspace of a normed space X and let $\varepsilon > 0$ be given. Then the following are equivalent:*

(a) For each $f \in W^*$, the set $E_\varepsilon(f)$ is compact.

(b) For each $f \in X^*$, the set of all $g \in X^*$ such that $\hat{g}$ is a ε-extension of $\hat{f}|_{W^{\perp\perp}}$, is compact.

(c) $W^\perp$ is a ε-quasi Chebyshev subspace of X^*.

The following corollary is a consequence of Theorems 2.3, 2.4, Lemma 1.3 and Lemma 1.4.

Corollary 2.5. *Let X be a normed space.*

(a) For each subspace W of X and for each $f \in W^*$, the set of all ε-extensions of f is weakly compact if and only if X is reflexive.

(b) For each subspace W of X and $f \in W^*$, the set of all ε-extensions of f is compact if and only if X is finite dimensional.

3. ε-weakly Chebyshev subspaces in dual spaces

In this section by using of ε-extensions, we shall characterize weak*-closed ε-weakly Chebyshev subspaces in dual spaces.

A subspace W of a normed space X is said to have the property $(\varepsilon - W)$ if for every $f \in W^*$ the set $E_\varepsilon(f)$ is weakly compact.

A subspace M of X^* is said to have the property $(\varepsilon - W^*)$ if the set

$D^M_{x,\varepsilon} = \{y \in X : \; f(y) = f(x) \; for \; all \; f \in M \; \& \; \|y\| \leq \|x\|_M + \varepsilon\}$

is weakly compact for all $x \in X$, where

$$\|x\|_M = sup\{|f(x)| : \|f\| \leq 1, \quad f \in M\}.$$

Note that $D^M_{x,\varepsilon}$ is a non-empty, closed, bounded and convex subset of X for all $x \in X$.

Theorem 3.1. *Let X be a normed space, M be a subspace of X^* and let $\varepsilon > 0$ be given. Then the following are equivalent:*

(a) M is a ε-weakly Chebyshev subspace of X^*.

(b) $M^\perp$ has the property $(\varepsilon - W^*)$.

(c) ${}^\perp M$ has the property $(\varepsilon - W)$.

Proof. (a)$\Rightarrow$(b). If $D_{f,\varepsilon}^{M^\perp}$ is not weakly compact for some $f \in X^*$, then there exists a sequence $\{g_n\}_{n\geq 1}$ in $D_{f,\varepsilon}^{M^\perp}$ without a weakly convergent subsequence. Since M is weak*-closed, $g_1 - g_n \in M$ for all $n \geq 1$ and

$$\|g_1 - (g_1 - g_n)\| = \|g_n\| \leq \|f\|_{M^\perp} + \varepsilon = \|g_1 - (g_1 - g_n)\|_{M^\perp} + \varepsilon$$

for all $n \geq 1$. Therefore, $g_1 - g_n \in P_{M,\varepsilon}(g_1)$ for all $n \geq 1$. It follows that M is not ε-weakly Chebyshev subspace of X^*. Hence, (a) implies (b).

(b)$\Rightarrow$(c). suppose that ${}^\perp M$ does not have the property $(\varepsilon - W)$. Then, there exists $f_0 \in ({}^\perp M)^*$ and a sequence $\{g_n\}_{n\geq 1}$ in $E_{f_0,\varepsilon}$ without a convergent subsequence. Since, $M^\perp = ({}^\perp M)^{\perp\perp}$, ${}^\perp M$ is dense in $M^\perp$ for the weak topology on X^{**} ([1; Propositions 1.10.15 and 2.6.6]). Next, define $F_0 : X^{**} \longrightarrow \Phi$ by $F_0(\varphi) = \varphi(f_0)$ and $F_n : X^{**} \longrightarrow \Phi$ by $F_n(\varphi) = \varphi(g_n)$ for all $n \geq 1$, where Φ is the real or complex field. Then, $F_0, F_1, F_2, \ldots$ are continuous with respect to the weak-topology on X^{**}. Therefore, $\varphi(g_n) = \varphi(f_0)$ for all $n \geq 1$ and $\varphi \in M^\perp$. By Lemma 1.5, $g_n \in D_{g_1,\varepsilon}^{M^\perp}$ for all $n \geq 1$, because $\|g_1\|_{M^\perp} = \|f_0\| \geq \|g_n\| - \varepsilon$ for all $n \geq 1$. Thus, $M^\perp$ does not have the property $(\varepsilon - W^*)$. Hence, (b) implies (c).

(c)$\Rightarrow$(a). suppose that $P_{M,\varepsilon}(f_0)$ is not weakly compact for some $f_0 \in X^*$. Then, there exists a sequence $\{g_n\}_{n\geq 1}$ in $P_{M,\varepsilon}(f_0)$ without a weakly convergent subsequence . By Lemma 1.1, $\|f_0 - g_n\| \leq \|f_0 - g_n\|_{M^\perp} + \varepsilon = \|f_0\|_{W^\perp} + \varepsilon$ for all $n \geq 1$. It follows that $f_0 - g_n \in D_{f_0,\varepsilon}^{M^\perp}$ for all $n \geq 1$. If we define the linear functional $\Lambda_0 \in ({}^\perp M)^*$ by $\Lambda_0 = f_0|_{{}^\perp M}$, then by Lemma 1.5, $\Lambda_0(x) = f_0(x) - g_n(x)$ and $\|\Lambda_0\| + \varepsilon = \|f_0\|_{M^\perp} + \varepsilon \geq \|f_0 - g_n\|$ for all $n \geq 1$ and all $x \in^\perp M$. It follows that $f_0 - g_n \in E_{\Lambda_0,\varepsilon}$ for all $n \geq 1$. Therefore, ${}^\perp M$ does not have the property $(\varepsilon - W)$. $\square$

Corollary 3.2. *Let X be a normed space and $\varepsilon > 0$ be given. Then the following are equivalent:*

(a) All weak*-closed subspaces of a X^* are ε-weakly Chebyshev.

(b) All subspaces of X have the property $(\varepsilon - W)$.

(c) X is reflexive.

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