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Behavior of R-groups for p-adic inner forms of quasi-split special unitary groups

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BEHAVIOR OF R-GROUPS FOR p-ADIC INNER FORMS OF QUASI-SPLIT SPECIAL UNITARY GROUPS

K. CHOIY AND D. GOLDBERG*

For Freydoon, with deep gratitude for all he has done for us, and for the field.

ABSTRACT. We study R-groups for p-adic inner forms of quasi-split special unitary groups. We prove Arthur's conjecture, the isomorphism between the Knapp-Stein R-group and the Langlands-Arthur R-group, for quasi-split special unitary groups and their inner forms. Furthermore, we investigate the invariance of the Knapp-Stein R-group within L-packets and between inner forms. This work is applied to transferring known results in the second-named author's earlier work for quasi-split special unitary groups to their non-quasi-split inner forms.

Keywords: Invariance of Knapp-Stein R-groups, tempered spectrum, Arthur's conjecture on R-groups, p-adic inner forms of special unitary groups.

MSC(2010): Primary: 22E50; Secondary: 11F70, 22E35.

1. Introduction

In the representation theory of p-adic groups, in particular, in the framework of the local Langlands correspondence for a connected reductive algebraic group $\mathbf{G}$ over a p-adic field F of characteristic zero, it is of great importance to study the reducibility of parabolically induced representations. This study yields information on constructing tempered L-packets of $\mathbf{G}$ from discrete L-packets of its F-Levi subgroups. The determination of reducibility has been developing over decades via several approaches, for example, by means of harmonic analysis from investigation of poles or zeros of residues of intertwining operators, Plancherel measures, and local L-functions, [31, 32, 35]. The method we address here is in terms of the Knapp-Stein R-group, which provides a combinatorial description of the tempered dual of $\mathbf{G}(F)$ as well as its elliptic tempered spectrum. Further, as conjectured by Arthur, the isomorphism of the Knapp-Stein and Langlands-Arthur R-groups, via the endoscopic R-group,

plays a significant role in the comparison of trace formulas and the endoscopic classification of automorphic representations [3, 20, 25].

While there has been a great deal of progress on the theory of Knapp-Stein R-groups for F-quasi-split groups $\mathbf{G}$, little is known for non F-quasi-split groups $\mathbf{G}'$. In [10], we investigated the behavior of R-groups between F-inner forms SL'_n of SL_n, and determined the Knapp-Stein R-groups for SL'_n from those of the split SL_n. We also proved the Knapp-Stein R-group for SL'_n embeds as a subgroup of the R-group for SL_n, and we characterized the quotient. Another approach to this case was carried out in [8] and an example was discovered for which the Knapp-Stein R-group for SL'_n is strictly smaller than that of SL_n. In [11], we further showed the invariance of R-groups between F-inner forms of quasi-split classical groups SO_{2n+1}, Sp_{2n}, SO_{2n}, or SO^*_{2n}, and transferred all known, relevant facts developed by the second-named author in the quasi-split classical groups to their non-quasi-split inner forms. Non-quasi-split inner forms of Sp_{4n} and SO_{4n} are also treated in [16].

To study the Knapp-Stein R-group for a non-quasi-split group $\mathbf{G}'$ in general, as one may notice from our previous works [10, 11], it is natural that we investigate its behavior between $\mathbf{G}$ and $\mathbf{G}'$ and transfer the developed theories regarding the Knapp-Stein R-group from $\mathbf{G}$ to $\mathbf{G}'$. This is due to the notion of Langlands functoriality. Since $\mathbf{G}'$ is induced from a new Galois action twisted by a Galois 1-cocycle in the inner automorphisms of $\mathbf{G}$, their L-groups are isomorphic over the Galois group and the set of L-parameters for $\mathbf{G}'$ is contained in that for $\mathbf{G}$. In this paper, combining this strategy and the restriction method, we study the Knapp-Stein R-group for non-quasi-split F-inner forms of quasi-split special unitary groups.

More precisely, we fix a quadratic extension E of a p-adic field F of characteristic zero. Let $\mathbf{G}_n = \mathrm{SU}_n$ be a quasi-split special unitary group over F with respect to E/F and let $\mathbf{G}'_n$ be its non-quasi-split inner form over F. A simple consequence from the Satake classification or a computation of Galois cohomology reduces our study to the case when n is even. In fact, there is a unique non-quasi-split F-inner form $\mathbf{G}'_n$, up to F-isomorphism (see Section 3). For the rest of the introduction, we assume that n is even, unless otherwise stated. Let $\mathbf{M}'$ be an F-Levi subgroup of $\mathbf{G}'_n$, which is an F-inner form of an F-Levi subgroup $\mathbf{M}$ of $\mathbf{G}_n$. Then, $\mathbf{M} = \boldsymbol{\mathcal{M}} \cap \mathbf{G}_n$ and $\mathbf{M}' = \boldsymbol{\mathcal{M}}' \cap \mathbf{G}'_n$, where $\boldsymbol{\mathcal{M}}$ is an F-Levi subgroup of a quasi-split unitary group $\boldsymbol{\mathcal{G}}_n = \mathrm{U}_n$ over F with respect to E/F and $\boldsymbol{\mathcal{M}}'$ is an F-Levi subgroup of a non-quasi-split F-inner form $\boldsymbol{\mathcal{G}}'_n$ of $\boldsymbol{\mathcal{G}}_n$. We shall use G for the group $\mathbf{G}(F)$ of F-points of any connected reductive algebraic group $\mathbf{G}$ over F.

Given an elliptic tempered L-parameter $\phi \in \Phi_{\mathrm{disc}}(M)$, by [24, Théorèm 8.1], we have a lifting $\varphi \in \Phi_{\mathrm{disc}}(\mathcal{M})$ commuting with the natural projection $\widehat{\mathcal{M}} \twoheadrightarrow \widehat{M}$, where $\widehat{\mathcal{M}}$ and $\widehat{M}$ respectively denote the connected components of

the L-groups of $\boldsymbol{\mathcal{M}}$ and $\mathbf{M}$ (see Section 2 for the details). Restricting the L-packet $\Pi_\varphi(\mathcal{M})$ constructed by Rogawski [28] and Mok [25] (see Section 5.1), we construct an L-packet $\Pi_\phi(M)$ as the set of isomorphism classes of irreducible constituents in the restriction from $\mathcal{M}$ to M. All the arguments used for $\mathbf{M}$ can be applied to $\mathbf{M}'$, as in Kaletha-Minguez-Shin-White [20] and the details are described in Section 6.1. For any $\sigma \in \Pi_\phi(M)$, and $\sigma' \in \Pi_\phi(M')$, we prove

$$R_\sigma \simeq R_{\phi,\sigma} \text{ and } R_{\sigma'} \simeq R_{\phi,\sigma'}.$$

In each of the above isomorphisms, the left side is the Knapp-Stein R-group, and the right side is the Langlands-Arthur R-group, defined in Section 2.2. This is known as Arthur's conjecture, predicted in [1], for $\mathbf{G}$ and $\mathbf{G}'$ (See Theorem 6.1). In the course of the proofs, we apply some known results about R-groups for $\boldsymbol{\mathcal{G}}_n$ and $\boldsymbol{\mathcal{G}}'_n$ in [14, 20, 25], which are recalled in Section 5. We also investigate and utilize some relationships between identity components of the centralizers of the images of φ and ϕ in $\widehat{G}$ and $\widehat{\mathcal{G}}$ (see Section 3.3 and Lemma 6.2).

We further study the invariance of the Knapp-Stein R-groups for $\mathbf{G}_n$ and $\mathbf{G}'_n$. Namely, given σ_1, $\sigma_2 \in \Pi_\phi(M)$, and σ'_1, $\sigma'_2 \in \Pi_\phi(M')$, we prove $R_{\sigma_1} \simeq R_{\sigma_2}$, and $R_{\sigma'_1} \simeq R_{\sigma'_2}$ (Theorem 6.6). Moreover, given $\sigma \in \Pi_\phi(M)$ and $\sigma' \in \Pi_\phi(M')$, we prove $R_\sigma \simeq R_{\sigma'}$ (Theorem 6.8). The crucial idea is to study the stabilizers $W(\sigma)$ and $W(\sigma')$. Theorem 6.3 shows

(1.1) $W(\sigma) \simeq \{w \in W_M : {}^w\Sigma \simeq \Sigma\lambda \text{ for some } \lambda \in (\mathcal{M}/M)^D\},$

where W_M is the Weyl group of $\mathbf{M}$ in $\mathbf{G}$, $\Sigma \in \Pi_\varphi(\mathcal{M})$ is a lifting of σ, and $(\mathcal{M}/M)^D$ is the group of continuous characters on $\mathcal{M}$ which are trivial on M. The same is true for $W(\sigma')$. The isomorphism (1.1) stems from the group structure of M and M' and the description of irreducible representations of M and M', which are discovered in [14, Section 2] and explained in Section 6.3.

As an application, a combination of Theorem 6.8 and the second-named author's earlier result [15, Theorem 3.7] shows that the Knapp-Stein R-group, $R_{\sigma'}$, for non-quasi-split inner forms $\mathbf{G}'$ can be expressed in terms of a subgroup in $R_{\sigma'}$ and $\mathbb{Z}_2^d$ for some integer d. It follows that $R_{\sigma'}$ is in general non-abelian (see Remark 6.9).

In Section 2, we recall basic notation and background. In Section 3, we provide the detailed group structure of $\boldsymbol{\mathcal{G}}_n$, $\mathbf{G}_n$, $\boldsymbol{\mathcal{G}}'_n$, $\mathbf{G}'_n$, and their F-Levi subgroups, and study relations between their L-groups. In Section 4, we describe Weyl group actions on Levi subgroups and their representations. In Section 5, we revisit the theory of R–groups for $\boldsymbol{\mathcal{G}}_n$ and $\boldsymbol{\mathcal{G}}'_n$ based on [14, 20, 25]. In Section 6, we prove the Arthur's conjecture for $\mathbf{G}_n$ and $\mathbf{G}'_n$ and the invariance of their R-groups.

2. **Preliminaries**

2.1. **Basic notation and background.** Let p be a prime. We let F be a p-adic field of characteristic 0, i.e., a finite extension of $\mathbb{Q}_p$. Fix an algebraic closure $\bar{F}$ of F. Given a connected reductive algebraic group $\mathbf{G}$ defined over F, we write $\mathbf{G}(F)$ for the group of F-points. Unless otherwise stated, we shall use G for $\mathbf{G}(F)$ and likewise for other algebraic groups defined over F.

Fix a minimal F-parabolic subgroup $\mathbf{P}_0$ of $\mathbf{G}$ with Levi decomposition $\mathbf{P}_0 = \mathbf{M}_0 \mathbf{N}_0$, where $\mathbf{M}_0$ denotes a Levi factor and $\mathbf{N}_0$ denotes the unipotent radical. We denote by $\mathbf{A}_0$ the split component of $\mathbf{M}_0$, that is, the maximal F-split torus in the center of $\mathbf{M}_0$, and by Δ the set of simple roots of $\mathbf{A}_0$ in $\mathbf{N}_0$. We say an F-parabolic subgroup $\mathbf{P}$ of $\mathbf{G}$ is standard if it contains $\mathbf{P}_0$.

Given an F-parabolic subgroup $\mathbf{P}$ with Levi decomposition $\mathbf{P} = \mathbf{MN}$, there exist a subset $\Theta \subseteq \Delta$ such that $\mathbf{M}$ equals $\mathbf{M}_\Theta$, the Levi subgroup generated by Θ. Note that $\mathbf{M} \supseteq \mathbf{M}_0$ and $\mathbf{N} \subseteq \mathbf{N}_0$. We write $\mathbf{A}_{\mathbf{M}_\Theta}$ for the split component $\mathbf{A}_\mathbf{M}$ of $\mathbf{M} = \mathbf{M}_\Theta$. It follows that $\mathbf{A}_\mathbf{M}$ equals the identity component $(\cap_{\alpha \in \Theta} \ker \alpha)^\circ$ in $\mathbf{A}_0$, so that $\mathbf{M} = Z_\mathbf{G}(\mathbf{A}_\mathbf{M})$, where $Z_\mathbf{G}(\mathbf{A}_\mathbf{M})$ is the centralizer of $\mathbf{A}_\mathbf{M}$ in $\mathbf{G}$. We refer the reader to [7, Proposition 20.4] and [36, Section 15.1] for the details.

We let $\Phi(P, A_M)$ denote the set of reduced roots of $\mathbf{P}$ with respect to $\mathbf{A}_\mathbf{M}$. Denote by $W_M = W(\mathbf{G}, \mathbf{A}_\mathbf{M}) := N_\mathbf{G}(\mathbf{A}_\mathbf{M})/Z_\mathbf{G}(\mathbf{A}_\mathbf{M})$ the Weyl group of $\mathbf{A}_\mathbf{M}$ in $\mathbf{G}$, where $N_\mathbf{G}(\mathbf{A}_\mathbf{M})$ is the normalizer of $\mathbf{A}_\mathbf{M}$ in $\mathbf{G}$. For simplicity, we write $\mathbf{A}_0 = \mathbf{A}_{\mathbf{M}_0}$.

For any topological group H, we write $Z(H)$ for the center of H. Denote by H° the identity component of H and by $\pi_0(H)$ the group H/H° of connected components of H. We write H^D for the group, $\mathrm{Hom}(H, \mathbb{C}^\times)$, of all continuous characters and by $\mathbb{1}$ the trivial character. We say a character is unitary if its image is in the unit circle $S^1 \subset \mathbb{C}^\times$. For any Galois module J, we denote by $H^i(F, J) := H^i(\mathrm{Gal}(\bar{F}/F), J(\bar{F}))$ the Galois cohomology of J for $i \in \mathbb{N}$.

Let $\mathbf{G}$ be a connected reductive algebraic group over F. We denote by $\mathrm{Irr}(G)$ the set of isomorphism classes of irreducible admissible complex representations of G. If there is no confusion, we do not make a distinction between each isomorphism class and its representative. For any $\sigma \in \mathrm{Irr}(M)$, we write $i_{G,M}(\sigma)$ for the normalized (twisted by $\delta_P^{1/2}$) induced representation, where δ_P denotes the modulus character of P. We denote by $\sigma^\vee$ the contragredient of σ.

Denote by $\Pi_{\mathrm{disc}}(G)$ and $\Pi_{\mathrm{temp}}(G)$ the subsets of $\mathrm{Irr}(G)$ which respectively consist of discrete series and tempered representations, where, a discrete series representation is an irreducible, admissible, unitary representation whose matrix coefficients are square-integrable modulo the center of G, that is, in $L^2(G/Z(G))$, and a tempered representation is an irreducible, admissible, unitary representation whose matrix coefficients are in $L^{2+\epsilon}(G/Z(G))$ for all $\epsilon > 0$. It is clear that $\Pi_{\mathrm{disc}}(G) \subset \Pi_{\mathrm{temp}}(G)$.

We denote by W_F the Weil group of F and by Γ the absolute Galois group $\mathrm{Gal}(\bar{F}/F)$. By fixing Γ-invariant splitting data, the L-group of G is defined as a semi-direct product $^L G := \widehat{G} \rtimes W_F$ (see [6, Section 2]). Following [6, Section 8.2], an L-parameter for G is an admissible homomorphism

$$\phi : W_F \times SL_2(\mathbb{C}) \to {}^L G.$$

Two L-parameters are said to be equivalent if they are conjugate by $\widehat{G}$. We denote by $\Phi(G)$ the set of equivalence classes of L-parameters for G.

We denote by $C_\phi(\widehat{G})$ the centralizer of the image of ϕ in $\widehat{G}$. The center of $^L G$ is the Γ-invariant group $Z(\widehat{G})^\Gamma$. Note that $C_\phi \supset Z(\widehat{G})^\Gamma$. We say that ϕ is elliptic if the quotient group $C_\phi(\widehat{G})/Z(\widehat{G})^\Gamma$ is finite, and ϕ is tempered if $\phi(W_F)$ is bounded. We denote by $\Phi_{\mathrm{ell}}(G)$ and $\Phi_{\mathrm{temp}}(G)$ the subset of $\Phi(G)$ which respectively consist of elliptic and tempered L-parameters of G. We set $\Phi_{\mathrm{disc}}(G) = \Phi_{\mathrm{ell}}(G) \cap \Phi_{\mathrm{temp}}(G)$.

The local Langlands conjecture for G predicts that there is a surjective finite-to-one map from $\mathrm{Irr}(G)$ to $\Phi(G)$. Given $\phi \in \Phi(G)$, we write $\Pi_\phi(G)$ for the L-packet attached to ϕ, and then the local Langlands conjecture implies that

$$\mathrm{Irr}(G) = \bigsqcup_{\phi \in \Phi(G)} \Pi_\phi(G).$$

It is expected that $\Phi_{\mathrm{disc}}(G)$ and $\Phi_{\mathrm{temp}}(G)$ respectively parameterize $\Pi_{\mathrm{disc}}(G)$ and $\Pi_{\mathrm{temp}}(G)$.

Given two connected reductive algebraic groups $\mathbf{G}$ and $\mathbf{G}'$ over F, $\mathbf{G}'$ is said to be an F-*inner form* of $\mathbf{G}$ with respect to an $\bar{F}$-isomorphism $\varphi : \mathbf{G}' \xrightarrow{\sim} \mathbf{G}$ if $\varphi \circ \tau(\varphi)^{-1}$ is an inner automorphism ($g \mapsto xgx^{-1}$) defined over $\bar{F}$ for all $\tau \in \mathrm{Gal}(\bar{F}/F)$ (see [6, 2.4(3)] or [23, p. 280]). If there is no confusion, we often omit the references to F and φ. It is well known from [23, p. 280] that there is a bijection between $H^1(F, \mathbf{G}_{\mathrm{ad}})$ and the set of isomorphism classes of F-inner forms of $\mathbf{G}$, where $\mathbf{G}_{\mathrm{ad}} := \mathbf{G}/Z(\mathbf{G})$. We note that, when $\mathbf{G}$ and $\mathbf{G}'$ are inner forms of each other, we have $^L G \simeq {}^L G'$ [6, Section 2.4(3)].

2.2. R-**groups.** We review the definitions of Knapp-Stein, Langlands-Arthur, and endoscopic R–groups. Let $\mathbf{G}$ be a connected reductive algebraic group over F, and let $\mathbf{M}$ be an F-Levi subgroup of $\mathbf{G}$. Given $\sigma \in \mathrm{Irr}(M)$ and $w \in W_M$, we write $^w\sigma$ for the representation given by $^w\sigma(x) = \sigma(w^{-1}xw)$. Note that, for the purpose of studying R–groups, we do not distinguish the representative of w, since the isomorphism class of $^w\sigma$ is independent of the choices of representatives in G of $w \in W_M$. Assume that σ lies in $\Pi_{\mathrm{disc}}(M)$, we define the stabilizer of σ in W_M

$$W(\sigma) := \{w \in W_M : {}^w\sigma \simeq \sigma\}.$$

Write Δ'_σ for $\{\alpha \in \Phi(P, A_M) : \mu_\alpha(\sigma) = 0\}$, where $\mu_\alpha(\sigma)$ is the rank one Plancherel measure for σ attached to α [13, p. 1108]. The *Knapp-Stein R-group* is defined by

$$R_\sigma := \{w \in W(\sigma) : w\alpha > 0, \ \forall \alpha \in \Delta'_\sigma\}.$$

We denote by W°_σ the subgroup of $W(\sigma)$, generated by the reflections in the roots of Δ'_σ. Then, for any $\sigma \in \Pi_{\mathrm{disc}}(M)$, we have

$$W(\sigma) = R_\sigma \ltimes W^\circ_\sigma,$$

which yields another description of the Knapp-Stein R-group

$$R_\sigma \simeq W(\sigma)/W^\circ_\sigma.$$

We refer to [21, 33, 34] for the details.

Given an L-parameter $\phi \in \Phi(M)$, we also consider ϕ as an L-parameter for G via the inclusion $\widehat{M} \hookrightarrow \widehat{G}$. Fix a maximal torus T_ϕ in $C_\phi(\widehat{G})^\circ$. We set

$$W^\circ_\phi := N_{C_\phi(\widehat{G})^\circ}(T_\phi)/Z_{C_\phi(\widehat{G})^\circ}(T_\phi), \quad W_\phi := N_{C_\phi(\widehat{G})}(T_\phi)/Z_{C_\phi(\widehat{G})}(T_\phi).$$

The *endoscopic R-group* R_ϕ is defined as follows

$$R_\phi := W_\phi/W^\circ_\phi.$$

We identify W_ϕ with a subgroup of W_M (see [1, p. 45]). For $\sigma \in \Pi_\phi(M)$, we set

$$W^\circ_{\phi,\sigma} := W^\circ_\phi \cap W(\sigma), \quad W_{\phi,\sigma} := W_\phi \cap W(\sigma).$$

The *Langlands-Arthur R-group* $R_{\phi,\sigma}$ is defined as follows

$$R_{\phi,\sigma} := W_{\phi,\sigma}/W^\circ_{\phi,\sigma}.$$

3. Structure theory of Levi subgroups

We discuss the group structure of U_n, SU_n, their F-inner forms, and their F-Levi subgroups. We mainly refer to [14, 15, 20, 25, 29]. Fix a quadratic extension E/F with $\bar{\ }$ the non-trivial Galois element. For any positive integer n, we let

$$J_n := \begin{pmatrix} & & & & 0 & & & 1 \\ & & & & & & -1 & \\ & & & & & 1 & & \\ & & & \cdot\cdot\cdot & & & & \\ & (-1)^{n-1} & & & & & & 0 \end{pmatrix} \in \mathrm{Mat}_{n\times n}(\mathbb{Z}).$$

We denote by $\mathrm{Res}_{E/F}$ the Weil restriction of scalars of E/F (see [38, Chapter 1] and [27, 2.1.2]). For $g = (g_{ij}) \in \mathrm{Res}_{E/F} \mathrm{GL}_n$, we let $\bar{g} = (\bar{g}_{ij})$ and set $\varepsilon_n(g) = J_n {}^t\bar{g}^{-1} J_n^{-1}$, where $g \mapsto {}^tg$ is the transpose. If there is no confusion, we often use ε instead of ε_n. We note that ε defined in [15, Section 1] is identical to that defined here, since the matrix u_n used in *loc. cit.* is equal to $(-1)^n J_n$.

3.1. U_n **and its inner forms.** Let $\boldsymbol{\mathcal{G}} = \boldsymbol{\mathcal{G}}_n$ denote the quasi-split unitary group U_n with respect to E/F and J_n. Thus,

$$\boldsymbol{\mathcal{G}} = \{g \in \mathrm{Res}_{E/F}\mathrm{GL}_n : gJ_n\,{}^t\bar{g} = J_n\}$$

(cf., [15, Section 1] and [25, Chapter 1]). We denote by $\boldsymbol{\mathcal{M}}$ an F-Levi subgroup of $\boldsymbol{\mathcal{G}}$. Then, $\boldsymbol{\mathcal{M}}$ is of the form

(3.1) $$\mathrm{Res}_{E/F}\mathrm{GL}_{n_1} \times \cdots \times \mathrm{Res}_{E/F}\mathrm{GL}_{n_k} \times \boldsymbol{\mathcal{G}}_m,$$

where $\sum_{i=1}^{k} 2n_i + m = n$ with $n_i \geq 0$ and $m \geq 0$. By convention, we note that, $\boldsymbol{\mathcal{G}}_0 = 1$ for n even, $\boldsymbol{\mathcal{G}}_1 = U_1$ for n odd, and $\mathrm{GL}_0 = 1$. The group of F-points, $A_{\boldsymbol{\mathcal{M}}}$, of the split component $\mathbf{A}_{\boldsymbol{\mathcal{M}}}$ is of the form

(3.2) $$\{\mathrm{diag}(x_1 I_{n_1}, \cdots, x_k I_{n_k}, I_m, x_k^{-1} I_{n_k}, \cdots, x_1^{-1} I_{n_1}) : x_i \in F^{\times}\}.$$

We let $\boldsymbol{\mathcal{G}}' = \boldsymbol{\mathcal{G}}'_n$ denote an F-inner form of $\boldsymbol{\mathcal{G}}$. By the Satake classification in [29, Section 3.3], for n **odd**, there is *no non-quasi-split* F-inner form of $\boldsymbol{\mathcal{G}}_n$. On the other hand, for n **even**, there is a *unique* non-quasi-split F-inner form $\boldsymbol{\mathcal{G}}'_n$, up to F-isomorphism. The Γ-diagram of the connected, simply-connected, semi-simple type of such $\boldsymbol{\mathcal{G}}'$ is

(3.3)
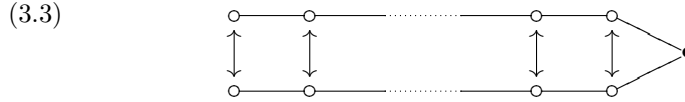
(see the table in [29, p. 119]). In the diagram above, the arrow indicates the non-trivial Galois action. Furthermore, the black vertex indicates a root in the set of simple roots of a fixed minimal F-Levi subgroup $\boldsymbol{\mathcal{M}}'_0$ of $\boldsymbol{\mathcal{G}}'$. So, we remove only a subset (denoted by ϑ) of pairs ($\mathrm{Gal}(E/F)$-orbits) of white vertices to obtain an F-Levi subgroup $\boldsymbol{\mathcal{M}}'$ (see [29, Section 2.2] and [6, Section I.3]). As discussed in Section 2.1, the F-Levi subgroup $\boldsymbol{\mathcal{M}}'$, corresponding to $\Theta = \Delta \setminus \vartheta$, is the centralizer in $\boldsymbol{\mathcal{G}}'$ of the split component $\mathbf{A}_{\boldsymbol{\mathcal{M}}'} = (\cap_{\alpha \in \Theta} \ker \alpha)^{\circ}$. Then, $\boldsymbol{\mathcal{M}}'$ is of the form

(3.4) $$\mathrm{Res}_{E/F}\mathrm{GL}_{n'_1} \times \cdots \times \mathrm{Res}_{E/F}\mathrm{GL}_{n'_{k'}} \times \boldsymbol{\mathcal{G}}'_{m'},$$

where $\sum_{i=1}^{k'} 2n'_i + m' = n$ with $n'_i \geq 0$ and $m' \geq 2$. Notice here that m' is always even. Considering the forms (3.1) and (3.4), it is obvious that, if $n_i = n'_i$ for all i, $k = k'$, and $m = m'$, then $\boldsymbol{\mathcal{M}}'$ is an F-inner form of $\boldsymbol{\mathcal{M}}$. In this case, furthermore, two split components $\mathbf{A}_{\boldsymbol{\mathcal{M}}}$ and $\mathbf{A}_{\boldsymbol{\mathcal{M}}'}$ are isomorphic over F.

Remark 3.1. Since there is a bijection between $H^1(F, (\boldsymbol{\mathcal{G}}_n)_{\mathrm{ad}})$ and the set of isomorphism classes of F-inner forms of $\boldsymbol{\mathcal{G}}_n$ (see Section 2.1), using the fact ([17, Lemma 1.2.1(ii) and p. 657]) that

$$H^1(F, (\boldsymbol{\mathcal{G}}_n)_{\mathrm{ad}}) = \begin{cases} 1, & \text{if } n \text{ is odd,} \\ \mathbb{Z}/2\mathbb{Z}, & \text{if } n \text{ is even,} \end{cases}$$

it follows that, when n is odd, there is no non-quasi-split F-inner form of $\mathcal{G}_n$, and when n is even, there is a unique non-quasi-split F-inner form, $\mathcal{G}'_n$, of $\mathcal{G}_n$, up to F-isomorphism, as we discussed above.

3.2. SU_n and its inner forms. We let $\mathbf{G} = \mathbf{G}_n$ denote the quasi-split special unitary group SU_n with respect to E/F and J_n. Thus,

$$\mathbf{G} = \mathcal{G} \cap \mathrm{Res}_{E/F}\mathrm{SL}_n = \{g \in \mathrm{Res}_{E/F}\mathrm{SL}_n : gJ_n{}^t\bar{g} = J_n\}.$$

We denote by $\mathbf{M}$ an F-Levi subgroup of $\mathbf{G}$. Then, $\mathbf{M}$ is of the form $\mathcal{M} \cap \mathbf{G}$ and $\mathbf{A_M}$ is of the form $\mathbf{A}_{\mathcal{M}} \cap \mathbf{G}$.

Due to (3.2), the F-points $A_\mathbf{M}$ of the split component $\mathbf{A_M}$ thus equals

$$\{\mathrm{diag}(x_1 I_{n_1}, \cdots, x_k I_{n_k}, I_m, x_k^{-1}I_{n_k}, \cdots, x_1^{-1}I_{n_1}) : x_i \in F^\times\}.$$

Especially, when $m \geq 2$, from (3.1) and [15, Lemma 2.8] we have a useful isomorphism

(3.5) $$M \simeq (\mathrm{GL}_{n_1}(E) \times \cdots \times \mathrm{GL}_{n_k}(E)) \rtimes G_m,$$

where $\sum_{i=1}^k 2n_i + m = n$ with $n_i \geq 0$. More precisely, for (g,h) with $g \in \mathrm{GL}_{n_1}(E) \times \cdots \times \mathrm{GL}_{n_k}(E)$ and $h \in G_m$, the isomorphism (3.5) from $(\mathrm{GL}_{n_1}(E) \times \cdots \times \mathrm{GL}_{n_k}(E)) \rtimes G_m$ to M is given by

(3.6) $$(g,h) \longmapsto \begin{pmatrix} g & & \\ & \alpha_m(g)^{-1}h & \\ & & \varepsilon(g) \end{pmatrix} \in \mathcal{M} \cap \mathrm{SU}_n(F) = M,$$

where $\alpha_m(g) = \alpha_m(\det g)$ and for $a \in E^\times$,

(3.7) $$\alpha_m(a) := \begin{pmatrix} a & & \\ & I_{m-2} & \\ & & \bar{a}^{-1} \end{pmatrix} \in \mathrm{GL}_m(E).$$

By convention, we note that $\mathbf{G}_0 = \mathbf{G}_1 = 1$. Given (g,h) with $g \in \mathrm{GL}_{n_1}(E) \times \cdots \times \mathrm{GL}_{n_k}(E)$ and $h \in G_m$, the action of g on h is given by

$$g \circ h = \alpha_m(g) \cdot h \cdot \alpha_m(g)^{-1}.$$

Next, we let $\mathbf{G}' = \mathbf{G}'_n$ denote an F–inner form of $\mathbf{G}$. As discussed in Section 3.1, for n **odd**, there is *no non-quasi-split* F-inner form of $\mathbf{G}_n$. On the other hand, for n **even**, there is a *unique* non-quasi-split F-inner form $\mathbf{G}'_n$, up to F-isomorphism, whose Γ-diagram is (3.3). Any F-Levi subgroup $\mathbf{M}'$ of $\mathbf{G}'$ is of the form $\mathcal{M}' \cap \mathbf{G}'$. Thus, from (3.4) and (3.5), we have an isomorphism

(3.8) $$M' \simeq (\mathrm{GL}_{n'_1}(E) \times \cdots \times \mathrm{GL}_{n'_{k'}}(E)) \rtimes G'_{m'},$$

where $\sum_{i=1}^{k'} 2n'_i + m' = n$ with $n'_i \geq 0$ and $m' \geq 2$. Notice as before that m' is always even. To explain (3.8) more precisely, for (g',h') with $g' \in \mathrm{GL}_{n'_1}(E) \times$

$\cdots \times \mathrm{GL}_{n'_k}(E)$ and $h' \in G'_{m'}$, the isomorphism (3.8) from $(\mathrm{GL}_{n'_1}(E) \times \cdots \times \mathrm{GL}_{n'_{k'}}(F)) \rtimes G'_{m'}$ to M' is given by

$$(g', h') \longmapsto \begin{pmatrix} g' & & \\ & \alpha_{m'}(g')^{-1}h' & \\ & & \varepsilon(g') \end{pmatrix} \in \mathcal{M}' \cap G'_n = M',$$

where $\alpha_{m'}(g') = \alpha_{m'}(\det g')$ and for $a \in E^\times$,

$$\alpha_{m'}(a) := \begin{pmatrix} a & & \\ & I_{m'-2} & \\ & & \bar{a}^{-1} \end{pmatrix} \in \mathrm{GL}_{m'}(E).$$

By convention, we note that, $\mathbf{G}'_0 = 1$. Given $(g', h') \in M'$ with $g' \in \mathrm{GL}_{n'_1}(E) \times \cdots \times \mathrm{GL}_{n'_k}(E)$ and $h' \in G'_{m'}$, the action of g' on h' is given by

$$g' \circ h' = \alpha_{m'}(g') \cdot h' \cdot \alpha_{m'}(g')^{-1}.$$

Like the unitary case in Section 3.1, it is obvious from (3.5) and (3.8) that, if $n_i = n'_i$ for all i, $k = k'$, and $m = m'$, then $\mathbf{M}'$ is an F-inner form of $\mathbf{M}$. In this case, there is an F–isomorphism $\mathbf{A_M} \simeq \mathbf{A_{M'}}$ between two split components.

We have following exact sequences of algebraic groups

$$1 \longrightarrow \mathbf{G} \longrightarrow \mathcal{G} \xrightarrow{\det} \mathrm{U}_1 \longrightarrow 1,$$

and

$$1 \longrightarrow \mathbf{G}' \longrightarrow \mathcal{G}' \xrightarrow{\det} \mathrm{U}_1 \longrightarrow 1.$$

Applying Galois cohomology, since $H^1(F, \mathbf{G}) = H^1(F, \mathbf{G}') = 1$ due to [27, Theorem 6.4], we have following exact sequences of their F-points

$$1 \longrightarrow G \longrightarrow \mathcal{G} \xrightarrow{\det} E^1 \longrightarrow 1,$$

and

$$1 \longrightarrow G' \longrightarrow \mathcal{G}' \xrightarrow{\det} E^1 \longrightarrow 1,$$

where $E^1 = \{x \in E : N_{E/K}(x) = x\bar{x} = 1\}$. All above exact sequences are true for F-Levi subgroups. In particular, $H^1(F, \mathbf{M}) = H^1(F, \mathbf{M}') = 1$ since $H^1(F, \mathbf{M}) \hookrightarrow H^1(F, \mathbf{G}) = 1$ and $H^1(F, \mathbf{M}') \hookrightarrow H^1(F, \mathbf{G}') = 1$ (see [26, p. 95, footnote], [23, p. 270], and [10, Remark 2.5]). We further have

$$\mathcal{G}_{\mathrm{der}} = \mathbf{G}_{\mathrm{der}} = \mathbf{G}, \ \mathcal{G}'_{\mathrm{der}} = \mathbf{G}'_{\mathrm{der}} = \mathbf{G}',$$

$$\mathcal{M}_{\mathrm{der}} = \mathbf{M}_{\mathrm{der}} \subset \mathbf{M}, \ \mathcal{M}'_{\mathrm{der}} = \mathbf{M}'_{\mathrm{der}} \subset \mathbf{M}'.$$

3.3. **L-groups.** We describe L-groups of $\boldsymbol{\mathcal{G}} = \mathrm{U}_n, \mathbf{G} = \mathrm{SU}_n$, their inner forms $\boldsymbol{\mathcal{G}'}, \mathbf{G}'$, and the L–groups of their F-Levi subgroups $\boldsymbol{\mathcal{M}}, \boldsymbol{\mathcal{M}'}, \mathbf{M}$, and $\mathbf{M}'$. Furthermore, we explain a relationship between L-groups of $\boldsymbol{\mathcal{M}}$ and $\mathbf{M}$, and investigate a connection between two Γ-split components $\mathbf{A}_{\widehat{\mathcal{M}}}$ and $\mathbf{A}_{\widehat{M}}$ of $\widehat{\mathcal{M}}$ and $\widehat{M}$, defined as follows

$$\mathbf{A}_{\widehat{\mathcal{M}}} := (Z(\widehat{\mathcal{M}})^{\Gamma})^{\circ} \quad \text{and} \quad \mathbf{A}_{\widehat{M}} := (Z(\widehat{M})^{\Gamma})^{\circ}.$$

These will be used in Section 6. Based on [25], we set

$$^L\boldsymbol{\mathcal{G}} = {}^L\boldsymbol{\mathcal{G}'} = \mathrm{GL}_n(\mathbb{C}) \rtimes W_F,$$

where W_E acts trivially on $\mathrm{GL}_n(\mathbb{C})$, and the action of $w_c \in W_F \setminus W_E$ on $\hat{g} \in \mathrm{GL}_n(\mathbb{C})$ is given by

(3.9) $$w_c(\hat{g}) = J_n^{-1t}\hat{g}^{-1}J_n.$$

For $\mathbf{G}$ and $\mathbf{G}'$, the L-group is

$$^LG = {}^LG' = \mathrm{PGL}_n(\mathbb{C}) \rtimes W_F,$$

where W_E acts trivially on $\mathrm{PGL}_n(\mathbb{C})$, and the action of $w_c \in W_F \setminus W_E$ on $\hat{g} \in \mathrm{PGL}_n(\mathbb{C})$ is given by (3.9).

Let $\boldsymbol{\mathcal{M}}$, $\boldsymbol{\mathcal{M}'}$, $\mathbf{M}$, and $\mathbf{M}'$ be F–Levi subgroups of $\boldsymbol{\mathcal{G}}, \boldsymbol{\mathcal{G}'}, \mathbf{G}$, and $\mathbf{G}'$, respectively, such that $\boldsymbol{\mathcal{M}'}$ is an F–inner form of $\boldsymbol{\mathcal{M}}$, and $\mathbf{M}'$ is an F–inner form of $\mathbf{M}$. We have

$$\widehat{\mathcal{M}} = \widehat{\mathcal{M}'} \subset \widehat{\mathcal{G}} = \widehat{\mathcal{G}'}, \ {}^L\mathcal{M} = {}^L\mathcal{M}' = \widehat{\mathcal{M}} \rtimes W_F,$$

$$\widehat{M} = \widehat{M'} \subset \widehat{G} = \widehat{G'}, \ {}^LM = {}^LM' = \widehat{M} \rtimes W_F.$$

Considering (3.1) and (3.4), we set

$$\boldsymbol{\mathcal{M}} = \mathrm{Res}_{E/F}\mathrm{GL}_{n_1} \times \cdots \times \mathrm{Res}_{E/F}\mathrm{GL}_{n_k} \times \boldsymbol{\mathcal{G}}_m,$$

$$\boldsymbol{\mathcal{M}'} = \mathrm{Res}_{E/F}\mathrm{GL}_{n_1} \times \cdots \times \mathrm{Res}_{E/F}\mathrm{GL}_{n_k} \times \boldsymbol{\mathcal{G}'}_m,$$

where $\boldsymbol{\mathcal{G}'}_m$ is an F-inner form of $\boldsymbol{\mathcal{G}}_m$, $\sum_{i=1}^{k} 2n_i + m = n$ with $n_i \geq 0$, and $m \geq 0$. Then, we have

$$\widehat{\boldsymbol{\mathcal{M}}} = \widehat{\boldsymbol{\mathcal{M}'}} = (\mathrm{GL}_{n_1}(\mathbb{C}) \times \mathrm{GL}_{n_1}(\mathbb{C}))$$
$$\times \cdots \times (\mathrm{GL}_{n_k}(\mathbb{C}) \times \mathrm{GL}_{n_k}(\mathbb{C})) \times \mathrm{GL}_m(\mathbb{C}).$$

W_E acts trivially on $\widehat{\boldsymbol{\mathcal{M}}}$, and the action of $w_c \in W_F \setminus W_E$ on

$$((\hat{g}_{11}, \hat{g}_{12}), (\hat{g}_{21}, \hat{g}_{22}), \cdots, (\hat{g}_{k1}, \hat{g}_{k2}), \hat{h}) \in \widehat{\boldsymbol{\mathcal{M}}}$$

is given by

(3.10) $$w_c((\hat{g}_{11}, \hat{g}_{12}), (\hat{g}_{21}, \hat{g}_{22}), \cdots, (\hat{g}_{k1}, \hat{g}_{k2}), \hat{h})$$
$$= ((\hat{g}_{12}, \hat{g}_{11}), (\hat{g}_{22}, \hat{g}_{21}), \cdots, (\hat{g}_{k2}, \hat{g}_{k1}), J_m^{-1t}\hat{h}^{-1}J_m).$$

Next, we note that, since $\boldsymbol{\mathcal{M}}_{\mathrm{der}}$ and $\mathbf{M}_{\mathrm{der}}$ are simply connected, we have

$$\widehat{\mathcal{M}_{\mathrm{der}}} = (\widehat{\mathcal{M}})_{\mathrm{ad}} = \widehat{\mathcal{M}}/Z(\widehat{\mathcal{M}}), \ \widehat{M_{\mathrm{der}}} = (\widehat{M})_{\mathrm{ad}} = \widehat{M}/Z(\widehat{M}).$$

Further, from [22, (1.8.1) p. 616], the exact sequence of algebraic groups

$$1 \longrightarrow \mathbf{M} \longrightarrow \boldsymbol{\mathcal{M}} \longrightarrow \mathrm{U}_1 \longrightarrow 1$$

yields an exact sequence

(3.11) $\qquad 1 \longrightarrow \widehat{\mathrm{U}_1} = \mathbb{C}^\times \longrightarrow Z(\widehat{\mathcal{M}}) \simeq (\mathbb{C}^\times)^{2k+1} \longrightarrow Z(\widehat{M}) \longrightarrow 1.$

We thus have the following commutative diagram of L-groups (cf., [10, Remark 2.4])

$$
\begin{array}{ccccccc}
& & 1 & & 1 & & \\
& & \downarrow & & \downarrow & & \\
1 \longrightarrow & & \widehat{\mathrm{U}_1} = \mathbb{C}^\times & \xrightarrow{\simeq} & \ker & \longrightarrow & 1 \\
& & \downarrow & & \downarrow & & \downarrow \\
1 \longrightarrow & Z(\widehat{\mathcal{M}}) = (\mathbb{C}^\times)^{2k+1} & \longrightarrow & \widehat{\mathcal{M}} & \longrightarrow & \widehat{\mathcal{M}_{\mathrm{der}}} & \longrightarrow 1 \\
& & \downarrow & & \downarrow & & \| \\
1 \longrightarrow & Z(\widehat{M}) = (\mathbb{C}^\times)^{2k+1}/\mathbb{C}^\times & \longrightarrow & \widehat{M} & \longrightarrow & \widehat{M_{\mathrm{der}}} & \longrightarrow 1 \\
& & \downarrow & & \downarrow & & \downarrow \\
& & 1 & & 1 & & 1.
\end{array}
$$

The middle vertical exact sequence becomes

(3.12) $\qquad 1 \longrightarrow \widehat{\mathrm{U}_1} = \mathbb{C}^\times \longrightarrow \widehat{\mathcal{M}} = \widehat{\mathcal{M}'} \longrightarrow \widehat{M} = \widehat{M'} \longrightarrow 1.$

We also have

(3.13) $\qquad 1 \longrightarrow \widehat{\mathrm{U}_1} = \mathbb{C}^\times \longrightarrow \widehat{\mathcal{G}} = \widehat{\mathcal{G}'} \longrightarrow \widehat{G} = \widehat{G'} \longrightarrow 1.$

Note that $\widehat{\mathrm{U}_1}$ in (3.12) and (3.13) equals $Z(\widehat{\mathcal{G}}) = Z(\widehat{\mathcal{G}'})$. Furthermore, $\widehat{\mathrm{U}_1}$ is diagonally embedded into $\widehat{\mathcal{G}}$ in (3.13) and the action of W_F on $\widehat{\mathrm{U}_1} = \mathbb{C}^\times$ is obtained from (3.9). The action of W_F can be also obtained from (3.10), since it is diagonally embedded into $Z(\widehat{\mathcal{M}})$ in (3.11) as well. It thus follows that the subgroups of $\Gamma = \mathrm{Gal}(\bar{F}/F)$-invariants satisfy

(3.14) $\qquad\qquad Z(\widehat{\mathcal{G}})^\Gamma = Z(\widehat{\mathcal{G}'})^\Gamma = \widehat{\mathrm{U}_1}^\Gamma = \{\pm 1\}.$

Moreover, using the action of W_F on $\widehat{\mathcal{M}} = \widehat{\mathcal{M}'}$ in (3.10) and the surjective map $\widehat{\mathcal{M}} = \widehat{\mathcal{M}'} \twoheadrightarrow \widehat{M} = \widehat{M'}$ in (3.12), the action of W_F on $\widehat{M} = \widehat{M'}$ can be obtained.

From (3.10), we have

$$\mathbf{A}_{\widehat{\mathcal{M}}} := (Z(\widehat{\mathcal{M}})^\Gamma)^\circ \simeq ((\mathbb{C}^\times)^k \times \{\pm 1\})^\circ = (\mathbb{C}^\times)^k.$$

We note that
$$\mathbf{A}_{\widehat{\mathcal{M}}}/Z(\widehat{\mathcal{G}})^\Gamma \simeq (\mathbf{A}_{\widehat{\mathcal{M}}} \cdot Z(\widehat{\mathcal{G}}))/Z(\widehat{\mathcal{G}}) \subset \widehat{G}.$$
Then, we have the following lemmas.

Lemma 3.2. *The quotient* $\mathbf{A}_{\widehat{\mathcal{M}}}/Z(\widehat{\mathcal{G}})^\Gamma$ *is connected.*

Proof. Since $\mathbf{A}_{\widehat{\mathcal{M}}} = (Z(\widehat{\mathcal{M}})^\Gamma)^\circ$ by definition and since $Z(\widehat{\mathcal{G}}) \subset \mathbf{A}_{\widehat{\mathcal{M}}}$, we have
$$\mathbf{A}_{\widehat{\mathcal{M}}}/Z(\widehat{\mathcal{G}})^\Gamma = (Z(\widehat{\mathcal{M}})^\Gamma)^\circ/(Z(\widehat{\mathcal{G}})^\Gamma \cap (Z(\widehat{\mathcal{M}})^\Gamma)^\circ)$$
which is isomorphic to $Z(\widehat{\mathcal{M}})^\Gamma/Z(\widehat{\mathcal{G}})^\Gamma$ by [2, Lemma 1.1]. We note that $Z(\widehat{\mathcal{M}})^\Gamma/$ $Z(\widehat{\mathcal{G}})^\Gamma$ is connected due to the proof of [2, Lemma 1.1]. $\qquad\square$

From (3.11) and (3.14), we have the following exact sequence
$$1 \longrightarrow \mathbf{A}_{\widehat{\mathcal{M}}}/Z(\widehat{\mathcal{G}})^\Gamma \longrightarrow Z(\widehat{M})^\Gamma \longrightarrow H^1(F, Z(\widehat{\mathcal{G}})).$$
The following lemma proves that the embedding $\mathbf{A}_{\widehat{\mathcal{M}}}/Z(\widehat{\mathcal{G}})^\Gamma \hookrightarrow Z(\widehat{M})^\Gamma$ is in fact an equality.

Lemma 3.3. *With the notation above, we have*
$$\mathbf{A}_{\widehat{M}} = Z(\widehat{M})^\Gamma \ \text{and} \ \mathbf{A}_{\widehat{M}} = \mathbf{A}_{\widehat{\mathcal{M}}}/Z(\widehat{\mathcal{G}})^\Gamma.$$
Proof. We note from [2, Lemma 1.1] that
$$Z(\widehat{M})^\Gamma = Z(\widehat{G})^\Gamma \cdot (Z(\widehat{M})^\Gamma)^\circ.$$
Since $Z(\widehat{G}) = 1$ and $A_{\widehat{M}} = (Z(\widehat{M})^\Gamma)^\circ$, the first equality is verified (hence, $Z(\widehat{M})^\Gamma$ is connected). We note that
$$\mathbf{A}_{\widehat{\mathcal{M}}}/Z(\widehat{\mathcal{G}})^\Gamma \subset Z(\widehat{M})^\Gamma = \mathbf{A}_{\widehat{M}}$$
and $\mathbf{A}_{\widehat{\mathcal{M}}}/Z(\widehat{\mathcal{G}})^\Gamma$ is connected by Lemma 3.2. Thus, $\mathbf{A}_{\widehat{\mathcal{M}}}/Z(\widehat{\mathcal{G}})^\Gamma$ is a torus in $Z(\widehat{M})^\Gamma$ having the same dimension with $\mathbf{A}_{\widehat{M}}$ (cf., Lemma 6.2 in Section 6.1). Since $Z(\widehat{M})^\Gamma = \mathbf{A}_{\widehat{M}}$, it follows that $\mathbf{A}_{\widehat{\mathcal{M}}}/Z(\widehat{\mathcal{G}})^\Gamma = \mathbf{A}_{\widehat{M}}$. $\qquad\square$

4. Weyl group actions

For an F-Levi subgroup $\mathbf{M}$ of a connected reductive algebraic group $\mathbf{G}$, we recall from 2.1 that the Weyl groups are $W_M = W(\mathbf{G}, \mathbf{A_M}) := N_\mathbf{G}(\mathbf{A_M})/Z_\mathbf{G}(\mathbf{A_M})$, $W_{M'} = W(\mathbf{G'}, \mathbf{A_{M'}}) := N_{\mathbf{G'}}(\mathbf{A_{M'}})/Z_{\mathbf{G'}}(\mathbf{A_{M'}})$, and $W_{\widehat{M}} = W(\widehat{G}, A_{\widehat{M}}) := N_{\widehat{G}}(A_{\widehat{M}})/Z_{\widehat{G}}(A_{\widehat{M}})$. Through the duality
$$s_\alpha \mapsto s_{\alpha^\vee}$$
between simple reflections for $\alpha \in \Delta$, we have
$$W_M \simeq W_{\widehat{M}} \quad W_{\widehat{M'}} \simeq W_{M'}.$$

We thus identify

(4.1) $$W_M = W_{\widehat{M}} = W_{\widehat{M'}} = W_{M'}.$$

4.1. **On Levi subgroups.** For simplicity, in Sections 4.1 and 4.2, we will write $\mathcal{G}$ for *both* quasi-split unitary groups U_n and its non-quasi-split F–inner forms $\mathcal{G'}$, and $\mathbf{G}$ for *both* quasi-split special unitary groups SU_n and its non-quasi-split F–inner forms $\mathbf{G'}$.

Let $\boldsymbol{\mathcal{M}}$ and $\mathbf{M}$ be F-Levi subgroups of $\mathcal{G}$ and $\mathbf{G}$, respectively. We recall from Section 3 that

$$\mathcal{M} \simeq \mathrm{GL}_{n_1}(E) \times \cdots \times \mathrm{GL}_{n_k}(E) \times \mathcal{G}_m,$$

$$M \simeq (\mathrm{GL}_{n_1}(E) \times \cdots \times \mathrm{GL}_{n_k}(E)) \rtimes G_m,$$

where $\sum_{i=1}^{k} 2n_i + m = n$ with $n_i \geq 0$ and $m \geq 0$. Notice here that $m \geq 2$ and is always even for non-quasi-split inner forms. We describe the action of Weyl group on Levi subgroups $\boldsymbol{\mathcal{M}}$ and $\mathbf{M}$ as well as irreducible representations of $\mathcal{M}$ and M, based on the results in [14, 15]. We denote by S_k the symmetric group in k letters. From [14, 15], we have

(4.2) $$W_{\mathcal{M}} = W_M \subset S_k \ltimes \mathbb{Z}_2^k.$$

More precisely, $W_{\mathcal{M}} \simeq S \ltimes \mathcal{C}$, where $S = \langle (ij) | n_i = n_j \rangle$, and $\mathcal{C} = \mathbb{Z}_2^k$. For $g \in \mathcal{M}$, write

$$g = (g_1, \ldots g_i, \ldots, g_j, \ldots, g_k, h) \in \mathrm{GL}_{n_1}(E) \times \mathrm{GL}_{n_2}(E) \times \cdots \times \mathrm{GL}_{n_k}(E) \times \mathcal{G}_m.$$

The permutation (ij) acts on $g \in \mathcal{M}$ by

(4.3) $$(ij) : (g_1, \ldots g_i, \ldots, g_j, \ldots, g_k, h) \mapsto (g_1, \ldots, g_j, \ldots, g_i, \ldots, g_k, h).$$

The finite 2-group $\mathbb{Z}_2^k$ is generated by "block sign changes" C_i which acts on $g \in \mathcal{M}$ by

(4.4) $$C_i : (g_1, \ldots, g_i, \ldots, g_k, h) \mapsto (g_1, \ldots, \varepsilon(g_i), \ldots, g_k, h).$$

4.2. **On representations of Levi subgroups.** Set Σ to be $\Sigma_1 \otimes \Sigma_2 \otimes \cdots \otimes \Sigma_k \otimes \Upsilon$. From (4.3) and (4.4), we have

$$(ij)\Sigma = \Sigma_1 \otimes \cdots \otimes \Sigma_j \otimes \cdots \otimes \Sigma_i \otimes \cdots \otimes \Sigma_k \otimes \Upsilon;$$

$$C_i\Sigma = \Sigma_1 \otimes \cdots \otimes \varepsilon(\Sigma_i) \otimes \cdots \otimes \Sigma_k \otimes \Upsilon,$$

where $\varepsilon(\Sigma_i)(g_i) = \Sigma_i(\varepsilon(g_i))$, and these describe the action of $W_{\mathcal{M}}$ on Σ.

Now we turn to the case of $\mathbf{M}$. Note that, since $G_0 = G_1 = 1$, W_M acts on M and an irreducible representation of M in the same manner for $m = 0, 1$. In particular, when $m = 0$ (thus, n is even), from the proof of [15, Lemma 2.4], M is of the form

$$M = \{ \begin{pmatrix} g & \\ & \varepsilon(g) \end{pmatrix} : \det(g) \det(\varepsilon(g)) = 1,$$

$$g \in \mathrm{GL}_{n_1}(E) \times \mathrm{GL}_{n_2}(E) \times \cdots \times \mathrm{GL}_{n_k}(E) \},$$

which implies $M \simeq \{g \in \mathrm{GL}_{n_1}(E) \times \mathrm{GL}_{n_2}(E) \times \cdots \times \mathrm{GL}_{n_k}(E) : \det g \in F^\times\}$. When $m = 1$ (thus, n is odd), since $\mathcal{M} \simeq \mathrm{GL}_{n_1}(E) \times \mathrm{GL}_{n_2}(E) \times \cdots \times \mathrm{GL}_{n_k}(E) \times \mathrm{U}_1(F)$, we have

$$M \simeq \mathrm{GL}_{n_1}(E) \times \mathrm{GL}_{n_2}(E) \times \cdots \times \mathrm{GL}_{n_k}(E).$$

Note $\mathrm{U}_1(F) = E^1 = \{x \in E : x\bar{x} = 1\}$. Further, for any $\Sigma \simeq \Sigma_1 \otimes \Sigma_2 \otimes \cdots \otimes \Sigma_k \otimes \Omega \in \mathrm{Irr}(\mathcal{M})$, we have

$$\mathrm{Res}^{\mathcal{M}_n}_{M_n}(\Sigma) \simeq \Sigma_1 \otimes \Sigma_2 \otimes \cdots \otimes \Sigma_k \otimes \Omega\Omega^\varepsilon,$$

which is always irreducible (see the proof of [15, Lemma 2.5]). Let us move to the case of $m \geq 2$. For $g \in M$, from (3.8), we write

$$g = (g_0, h) = (g_1, \ldots g_i, \ldots, g_j, \ldots, g_k, h)$$
$$\in (\mathrm{GL}_{n_1}(E) \times \mathrm{GL}_{n_2}(E) \times \cdots \times \mathrm{GL}_{n_k}(E)) \rtimes G_m.$$

Denote wgw^{-1} by (wg_0, h) for $w \in W_{\mathcal{M}}$. Following arguments in [15, p. 353], the permutation (ij) acts on $g \in M$

$$(4.5) \qquad (ij) : (g_1, \ldots g_i, \ldots, g_j, \ldots, g_k, h)$$
$$\mapsto (g_1, \ldots, g_j, \ldots, g_i, \ldots, g_k, \alpha_m(g_0)^{-1}\alpha_m((ij)g_0)h)$$
$$= (g_1, \ldots, g_j, \ldots, g_i, \ldots, g_k, h),$$

since $\alpha_m(g_0)^{-1}\alpha_m((ij)g_0) = I_m$. That is, the permutation (ij) acts trivially on G_m.

The finite 2-group $\mathbb{Z}_2^k$ is generated by "block sign changes" C_i which acts on $g \in M$
(4.6)
$$C_i : (g_1, \ldots, g_i, \ldots, g_k, h) \mapsto (g_1, \ldots, \varepsilon(g_i), \ldots, g_k, \alpha_m(g_0)^{-1}\alpha_m(C_ig_0)h).$$

Note from the definition of α_m in (3.7) that

$$\alpha_m(g_0)^{-1}\alpha_m(C_ig_0)$$
$$:= \begin{pmatrix} (\det(g_i)\det(\bar{g}_i))^{-1} & & \\ & I_{m-2} & \\ & & \det(g_i)\det(\bar{g}_i) \end{pmatrix} \in \mathrm{SU}_m(F).$$

Given $\sigma \in \mathrm{Irr}(M)$ whose component of the tensor product coming from G_m is τ, we fix a lift $\Sigma \in \mathrm{Irr}(\mathcal{M})$. Set $\Sigma \in \mathrm{Irr}(\mathcal{M})$ to be $\Sigma_1 \otimes \Sigma_2 \otimes \cdots \otimes \Sigma_k \otimes \Upsilon$, and Υ to be a lift in $\mathrm{Irr}(\mathcal{G}_m)$ with $\tau \hookrightarrow \mathrm{Res}^{\mathcal{G}_m}_{G_m}(\Upsilon)$. Denote by V_σ the $\mathbb{C}$-vector space which σ acts on. It follows from [12, Lemma 2.1(c)] and [15, p. 353] that, for $(g_0, h) = (g_1, \ldots g_i, \ldots, g_j, \ldots, g_k, h) \in (\mathrm{GL}_{n_1}(E) \times \mathrm{GL}_{n_2}(E) \times \cdots \times \mathrm{GL}_{n_k}(E)) \rtimes G_m$, and $(v_1, v_2, \cdots, v_k, v_0) \in V_\sigma$, we have

$$\sigma(g_0, h)(v_1, v_2, \cdots, v_k, v_0)$$
$$= \Sigma_1(g_1)(v_1) \otimes \Sigma_2(g_2)(v_2) \otimes \cdots \otimes \Sigma_k(g_k)(v_k) \otimes \Upsilon(\alpha_m(g_0)^{-1})\tau(h)(v_0).$$

Further, V_σ has the following decomposition

$$V_{\sigma_1} \otimes V_{\sigma_2} \otimes \cdots \otimes V_{\sigma_k} \otimes V_\tau,$$

as $\mathbb{C}$-vector spaces. We then note that

$$\sigma_i = \Sigma_i \text{ and } \quad V_{\sigma_i} = V_{\Sigma_i}$$

for each $i = 1, \ldots, k$. Here, $V_{\sigma_i}, V_{\Sigma_i}$ for $i = 1, \ldots, k$, and V_τ are the corresponding representation vector spaces over $\mathbb{C}$.

From (4.5) and [15, p. 353], we thus have

$$
\begin{aligned}
(ij)\sigma(g_0, h) =& \sigma_1(g_1) \otimes \cdots \otimes \sigma_j(g_i) \otimes \cdots \otimes \sigma_i(g_j) \cdots \otimes \sigma_k(g_k) \\
& \otimes \Upsilon(\alpha_m((ij)g_0)^{-1})\tau(\alpha_m(g_0)^{-1}\alpha_m((ij)g_0)h) \\
=& \sigma_1(g_1) \otimes \cdots \otimes \sigma_i(g_j) \otimes \cdots \otimes \sigma_j(g_i) \cdots \otimes \sigma_k(g_k) \\
& \otimes \Upsilon(\alpha_m(g_0)^{-1})\tau(h).
\end{aligned}
$$

Likewise, using (4.6), we have

$$
\begin{aligned}
C_i\sigma(g_0, h) =& \sigma_1(g_1) \otimes \cdots \otimes \varepsilon(\sigma_i)(g_i) \otimes \cdots \otimes \sigma_k(g_k) \\
& \otimes \Upsilon(\alpha_m(C_i g_0)^{-1})\tau(\alpha_m(g_0)^{-1}\alpha_m(C_i g_0)h) \\
=& \sigma_1(g_1) \otimes \cdots \otimes \sigma_i(\varepsilon(g_i)) \otimes \cdots \otimes \sigma_k(g_k) \\
& \otimes \Upsilon(\alpha_m(g_0)^{-1})\tau(h).
\end{aligned}
$$

Therefore, W_M acts non-trivially only on $\sigma_1, \sigma_2, \ldots, \sigma_k$, but trivially on τ.

5. Revisiting R–groups for U_n and their inner forms

Based on some known results in [4, 14, 20, 25] regarding R-groups for U_n and its F-inner form, we discuss Arthur's conjecture for U_n and its inner forms, behavior of R-groups within L-packets of U_n and its inner forms, and behavior of R-groups between U_n and its inner forms.

5.1. **L-packets for U_n and its inner forms.** Let $\mathcal{G} = \mathcal{G}_n$ denote the quasi-split unitary group U_n with respect to E/F and J_n, and $\mathcal{M}$ an F-Levi subgroup of $\mathcal{G}$. For our purpose of studying R-groups, we focus on $\Phi_{\text{temp}}(\mathcal{G})$. In [25, Theorem 2.5.1.(b)], Mok generalized Rogawski's results [28] in the case of unitary groups in three variables as follows. There is a surjective finite-to-one map

$$\Pi_{\text{temp}}(\mathcal{G}) \longrightarrow \Phi_{\text{temp}}(\mathcal{G}),$$

and for $\varphi \in \Phi_{\text{temp}}(\mathcal{G})$, the tempered L-packet $\Pi_\varphi(\mathcal{G})$ is constructed. The same is true for an F-inner form $\mathcal{G}'$ of $\mathcal{G}$ by Kaletha-Minguez-Shin-White [20, Section 1.6.1].

Let $\mathcal{M}$ and $\mathcal{M}'$ be F-Levi subgroups of $\mathcal{G}$ and $\mathcal{G}'$, respectively, which are F-inner forms of each other. Then, from (3.1) and (3.4), we recall

$$\mathcal{M} \simeq \text{Res}_{E/F}\text{GL}_{n_1} \times \cdots \times \text{Res}_{E/F}\text{GL}_{n_k} \times \mathcal{G}_m,$$

$$\mathcal{M}' \simeq \mathrm{Res}_{E/F}\mathrm{GL}_{n_1} \times \cdots \times \mathrm{Res}_{E/F}\mathrm{GL}_{n_k} \times \boldsymbol{\mathcal{G}}'_m,$$

where $\boldsymbol{\mathcal{G}}'_m$ is an F-inner form of $\boldsymbol{\mathcal{G}}_m$, $\sum_{i=1}^{k} 2n_i + m = n$ with $n_i \geq 0$, and $m \geq 0$. Let $\varphi \in \Phi_{\mathrm{disc}}(\mathcal{M})$ be given. Then, φ is $\boldsymbol{\mathcal{M}}$-relevant (see [6, Section 8.2]) and lies in $\Phi_{\mathrm{disc}}(\mathcal{M}')$ as well. We note that φ is of the form $\varphi_1 \oplus \varphi_2 \oplus \cdots \oplus \varphi_k \oplus \varphi_-$, where $\varphi_i \in \Phi_{\mathrm{disc}}(\mathrm{GL}_{n_i}(E))$ and $\varphi_- \in \Phi_{\mathrm{disc}}(\mathcal{G}_m) = \Phi_{\mathrm{disc}}(\mathcal{G}'_m)$. For each φ_i, due to [18, 19, 30], we construct L-packets $\Pi_{\varphi_i}(\mathrm{GL}_{n_i}(E))$ consisting of discrete series representations of $\mathrm{GL}_{n_i}(E)$. Note that $\Pi_{\varphi_i}(\mathrm{GL}_{n_i}(E))$ is a singleton. For φ_-, due to [20, 25, 28], we construct L-packets $\Pi_{\varphi_-}(G_m)$ and $\Pi_{\varphi_-}(G'_m)$ consisting of discrete series representations of G_m and G'_m, respectively. By taking the tensor product of members in packets for each φ_i and φ_-, we thus construct L-packets $\Pi_\varphi(\mathcal{M})$ of $\mathcal{M}$ and $\Pi_\varphi(\mathcal{M}')$ of $\mathcal{M}'$, associated to the elliptic tempered L-parameter φ.

5.2. Invariance of R-groups between U_n and its inner forms.

Theorem 5.1. *(Goldberg, [14, Theorem 3.4]) Let $\boldsymbol{\mathcal{M}}$ be an F-Levi subgroup of $\boldsymbol{\mathcal{G}}$. Given $\Sigma \simeq \Sigma_1 \otimes \Sigma_2 \otimes \cdots \otimes \Sigma_k \otimes \Upsilon \in \Pi_{\mathrm{disc}}(\mathcal{M})$, we have*

$$R_\Sigma \simeq \mathbb{Z}_2^d,$$

where d is the number of inequivalent Σ_i such that the induced representation $i_{\mathcal{G}_{n_i+m}, \ \mathrm{GL}_{n_i}(E) \times \mathcal{G}_m}(\Sigma \otimes \Upsilon)$ is reducible. $\square$

Theorem 5.2. *Let $\varphi \in \Phi_{\mathrm{disc}}(\mathcal{M})$ be given. Under the identity (4.1), for any $\Sigma \in \Pi_\varphi(\mathcal{M})$ and $\Sigma' \in \Pi_\varphi(\mathcal{M}')$, we have*

$$W(\Sigma) = W_\varphi = W(\Sigma').$$

and

(5.1) $$W_\Sigma^\circ = W_\varphi^\circ = W_{\Sigma'}^\circ.$$

Therefore, we have

$$R_\Sigma \simeq R_{\varphi,\Sigma} \simeq R_\varphi \simeq R_{\varphi,\Sigma'} \simeq R_{\Sigma'}.$$

Proof. For quasi-split cases, the arguments, $W(\Sigma) = W_\varphi$, $W_\Sigma^\circ = W_\varphi^\circ$, and $R_\Sigma \simeq R_{\varphi,\Sigma} \simeq R_\varphi$ immediately follow from [4, Lemma 11], its proof, and [5, Lemma 4.1]. Another proof for them can be found in [25, Section 7.6]. For non-quasi-split cases, $R_{\Sigma'} \simeq R_{\varphi,\Sigma'} \simeq R_\varphi$ is proved in [20, Lemma 4.6.5]. The equality $W(\Sigma') = W_\varphi$ is verified in the second paragraph after the proof of [20, Lemma 4.6.3]. The argument $W_{\Sigma'}^\circ = W_\varphi^\circ$ is a consequence of (4.6.3) and Lemma 4.6.5 in [20]. This completes the proof of Theorem 5.2. $\square$

We have following corollaries to Theorems 5.1 and 5.2.

Corollary 5.3. *Let $\varphi \in \Phi_{\mathrm{disc}}(\mathcal{M})$ be given. For $\Sigma \in \Pi_\varphi(\mathcal{M})$ and $\Sigma' \in \Pi_\varphi(\mathcal{M}')$, we have $i_{\mathcal{G},\mathcal{M}}(\Sigma)$ is irreducible if and only if $i_{\mathcal{G}',\mathcal{M}'}(\Sigma')$ is irreducible.* $\square$

Corollary 5.4. *Let $\boldsymbol{\mathcal{M}}'$ be an F-Levi subgroup of $\boldsymbol{\mathcal{G}}'$. Given $\Sigma' \simeq \Sigma'_1 \otimes \Sigma'_2 \otimes \cdots \otimes \Sigma'_k \otimes \Upsilon' \in \Pi_{\mathrm{disc}}(\mathcal{M}')$, we have*

$$R_{\Sigma'} \simeq \mathbb{Z}_2^d,$$

where d is the number of inequivalent Σ'_i such that the induced representation $i_{\mathcal{G}'_{n_i+m}, \, \mathrm{GL}_{n_i}(E) \times \mathcal{G}'_m}(\Sigma' \otimes \Upsilon')$ is reducible. $\square$

Corollary 5.5. *Suppose $w \in R_{\Sigma'}$ and $w = sc$, with $s \in S$ and $c \in \mathcal{C}$. Then $s = 1$.*

Proof. This is a consequence of [14, Lemma 3.2] and Theorem 5.2. $\square$

Let $\Sigma' \in \Pi_{\mathrm{disc}}(\mathcal{M}')$ be given. Denote by $\mathbb{C}[R_{\Sigma'}]_\eta$ the group algebra of $R_{\Sigma'}$ twisted by a 2-cocycle η, and by $C(\Sigma')$, known as the commuting algebra of $i_{\mathcal{G}',\mathcal{M}'}(\Sigma')$, the algebra $\mathrm{End}_{\mathcal{G}'}(i_{\mathcal{G}',\mathcal{M}'}(\Sigma'))$ of $\mathcal{G}'$-endomorphisms of $i_{\mathcal{G}',\mathcal{M}'}(\Sigma')$. Then, we have

$$C(\Sigma') \simeq \mathbb{C}[R_{\Sigma'}]_\eta$$

as group algebras (see [21, 33, 34]).

Proposition 5.6. *With the above notation, we have*

$$C(\Sigma') \simeq \mathbb{C}[R_{\Sigma'}].$$

Proof. The proof is similar to [14, Proposition 4.1], due to Theorem 5.2 and Corollaries 5.4 and 5.5. $\square$

As consequence of Proposition 5.6, we have the following corollary.

Corollary 5.7. *Let $\Sigma' \in \Pi_{\mathrm{disc}}(\mathcal{M}')$ be given. Then, each constituent of $i_{\mathcal{G}',\mathcal{M}'}(\Sigma')$ appears with multiplicity one.* $\square$

Proposition 5.8. *Let $\Sigma' \in \Pi_{\mathrm{disc}}(\mathcal{M}')$ be given. Then, $i_{\mathcal{G}',\mathcal{M}'}(\Sigma')$ has an elliptic constituent if and only if all constituents of $i_{\mathcal{G}',\mathcal{M}'}(\Sigma')$ are elliptic if and only if*

$$R_{\Sigma'} \simeq \mathbb{Z}_2^k.$$

Proof. The proof is similar to [14, Theorem 4.3], due to Corollary 5.5 and Proposition 5.6. $\square$

Corollary 5.9. *Let $\varphi \in \Phi_{\mathrm{disc}}(\mathcal{M})$ be given. For any $\Sigma \in \Pi_\varphi(\mathcal{M})$ and $\Sigma' \in \Pi_\varphi(\mathcal{M}')$, there is an elliptic constituent in $i_{\mathcal{G},\mathcal{M}}(\Sigma)$ if and only if there is an elliptic constituent in $i_{\mathcal{G}',\mathcal{M}'}(\Sigma')$.*

Proof. This is a consequence of [14, Theorem 4.3] and Proposition 5.8. $\square$

6. Behavior of R–groups for SU_n and its inner forms

In this section, we prove Arthur's conjecture for both quasi-split special unitary groups SU_n and its inner forms. Furthermore, we study the behavior of R-groups within L-packets and between inner forms of SU_n. We will use the notation in Sections 3, 4, and 5.

6.1. **L-packets for SU_n and its inner forms.** We discuss tempered L-packets of $\mathbf{G} = \mathrm{SU}_n$ and its F-inner form $\mathbf{G}'$. It is natural to construct L-packets for $\mathbf{G}$ and $\mathbf{G}'$, by restricting L-packets for $\boldsymbol{\mathcal{G}} = \mathrm{U}_n$ and its F-inner form $\boldsymbol{\mathcal{G}}'$ which has been done by Rogawski [28], Mok [25], and Kaletha-Minguez-Shin-White [20] (see Section 5.1). Thus, given $\phi \in \Phi(G)$, from [24, Théorèm 8.1], there exists a lifting $\varphi \in \Phi(\mathcal{G})$ such that

$$\phi = pr \circ \varphi,$$

where pr is the projection $\widehat{\mathcal{G}} \twoheadrightarrow \widehat{G}$ in (3.13). Note that the homomorphism pr is compatible with Γ-actions on $\widehat{\mathcal{G}}$ and $\widehat{G}$ (see Section 3.3) and the lifting $\varphi \in \Phi(\mathcal{G})$ is chosen uniquely up to a 1-cocycle of W_F in $\widehat{(\mathcal{G}/G)}$ (see [24, Section 7] and [8, Theorem 3.5.1]).

For our purpose, we are interested in $\phi \in \Phi_{\mathrm{temp}}(G)$ and the lifting φ lies in $\Phi_{\mathrm{temp}}(\mathcal{G})$. Using the L-packet $\Pi_\varphi(\mathcal{G})$ for $\varphi \in \Phi_{\mathrm{temp}}(\mathcal{G})$ in Section 5.1, we construct an L-packet $\Pi_\phi(G)$ for $\phi \in \Phi_{\mathrm{temp}}(G)$ as the set of isomorphism classes of irreducible constituents in the restriction from $\mathcal{G}$ to G as follows:

$$\Pi_\phi(G) := \{\sigma \hookrightarrow \mathrm{Res}_G^{\mathcal{G}}(\Sigma), \ \Sigma \in \Pi_\varphi(\mathcal{G})\}/\simeq .$$

Likewise, given $\phi \in \Phi_{\mathrm{disc}}(M)$, we construct an L-packet $\Pi_\phi(M)$ for $\phi \in \Phi_{\mathrm{disc}}(M)$ as follows:

$$\Pi_\phi(M) := \{\sigma \hookrightarrow \mathrm{Res}_M^{\mathcal{M}}(\Sigma), \ \Sigma \in \Pi_\varphi(\mathcal{M})\}/\simeq,$$

where φ lies in $\Phi_{\mathrm{disc}}(\mathcal{M})$ such that $\phi = pr \circ \varphi$, with the projection $pr : \widehat{\mathcal{M}} \twoheadrightarrow \widehat{M}$ in (3.12).

All the above arguments apply verbatim to F-inner forms $\boldsymbol{\mathcal{G}}'$ and their F-Levi subgroups $\boldsymbol{\mathcal{M}}'$.

6.2. **Arthur's conjecture for SU_n and its inner forms.** The purpose of the section is to prove Arthur's conjecture, predicted in [1], for $\mathbf{G}$ and $\mathbf{G}'$.

Theorem 6.1. *Given $\phi \in \Phi_{\mathrm{disc}}(M)$, $\sigma \in \Pi_\phi(M)$, and $\sigma' \in \Pi_\phi(M')$, we have*

$$R_\sigma \simeq R_{\phi,\sigma} \ and \ R_{\sigma'} \simeq R_{\phi,\sigma'}.$$

The rest of the section is devoted to the proof of Theorem 6.1. Since all the following techniques apply to both $\mathbf{M}$ and $\mathbf{M}'$, we shall state the proof for $\mathbf{M}$.

Let $\phi \in \Phi_{\mathrm{disc}}(M)$ and $\sigma \in \Pi_\phi(M)$ be given. Identifying W_ϕ with a subgroup of W_M (see Section 2.2), we claim that

(6.1) $$W(\sigma) \subset W_\phi.$$

To see this, we let $w \in W_M$ be given such that $^w\sigma \simeq \sigma$. Let $\Sigma \in \mathrm{Irr}(\mathcal{M})$ be a lifting of σ such that $\sigma \hookrightarrow \mathrm{Res}_M^{\mathcal{M}}(\Sigma)$ (see [12, Lemma 2.3] and [37, Proposition 2.2]). Since $^w\sigma \simeq \sigma$, it follows from [12, Lemma 2.4] and [37, Corollary 2.5] that $^w\Sigma \simeq \Sigma\chi$ for some character χ of $\mathcal{M}/M$. Due to the construction of L-packets in Section 6.1, one can choose a lifting $\varphi \in \Phi_{\mathrm{disc}}(\mathcal{M})$ of ϕ such that

$\Sigma \in \Pi_\varphi(\mathcal{M})$ and $\phi = pr \circ \varphi$ (see [24, Théorèm 8.1]), where pr is the projection $\widehat{\mathcal{M}} \twoheadrightarrow \widehat{M}$ in (3.12). We then have

$$\Sigma\chi \in \Pi_{\varphi\chi}(\mathcal{M}) \text{ and } {}^w\Sigma \in \Pi_{{}^w\varphi}(\mathcal{M}),$$

where χ is considered as a 1-cocycle of W_F in $\widehat{(\mathcal{M}/M)}$ by the local class field theory (cf., [24, Section 7] and [8, Theorem 3.5.1]). Due to the disjointness in [25, Theorem 2.5.1(b)] for L-packets of U_n, it follows from the isomorphism ${}^w\Sigma \simeq \Sigma\chi$ that ${}^w\varphi \simeq \varphi\chi$. Through the projection $pr : \widehat{\mathcal{M}} \twoheadrightarrow \widehat{M}$, we have ${}^w\phi \simeq \phi$. Since the elements of W_ϕ stabilize $\mathbf{A}_{\widehat{M}}$ by definition (cf., [25, p. 60]), it thus follows that w lies in W_ϕ.

From the inclusion (6.1), we have

(6.2) $W(\sigma) = W_{\sigma,\phi} = W_\phi \cap W(\sigma).$

Denote by $\varphi \in \Phi_{\mathrm{disc}}(\mathcal{M})$ a lifting of ϕ as in Section 6.1. For any $\Sigma \in \Phi_\varphi(\mathcal{M})$, we have

$$W_\Sigma^\circ = W_\sigma^\circ,$$

since the Plancherel measure is compatible with restriction (see [9, Proposition 2.4], [15, Lemma 2.3], for example) and $\Phi(P, A_M) = \Phi(\mathcal{P}, A_\mathcal{M})$ (see Section 2.1). Using (5.1), we have

(6.3) $W_\sigma^\circ = W_\varphi^\circ.$

To prove Theorem 6.1, from (6.2) and (6.3), it is thus enough to show that

(6.4) $W_\varphi^\circ = W_\phi^\circ.$

Indeed, if so, then we have

$$W_{\sigma,\phi}^\circ \overset{\text{definition}}{=} W_\phi^\circ \cap W(\sigma) \overset{(6.4)}{=} W_\varphi^\circ \cap W(\sigma) \overset{(6.3)}{=} W_\sigma^\circ \cap W(\sigma) = W_\sigma^\circ.$$

In what follows, we prove (6.4). From [25, Chapter 3.4] we set a maximal torus T_φ in $C_\varphi(\widehat{\mathcal{G}})^\circ$ to be the identity component

$$\mathbf{A}_{\widehat{\mathcal{M}}} = (Z(\widehat{\mathcal{M}})^\Gamma)^\circ$$

of the Γ–invariants of the center $Z(\widehat{\mathcal{M}})$. So, we have $\widehat{\mathcal{M}} = Z_{\widehat{\mathcal{G}}}(T_\varphi)$. Likewise, we set $T_\phi = \mathbf{A}_{\widehat{M}} \subset C_\phi(\widehat{G})^\circ$, and $\widehat{M} = Z_{\widehat{G}}(T_\phi)$. Here, we put

$$\bar{C}_\varphi^\circ := C_\varphi(\widehat{\mathcal{G}})^\circ / Z(\widehat{\mathcal{G}})^\Gamma, \ \bar{T}_\varphi := T_\varphi / Z(\widehat{\mathcal{G}})^\Gamma.$$

Note that $\bar{C}_\varphi^\circ \subset C_\phi(\widehat{G})^\circ \subset \widehat{G}$ and $\bar{T}_\varphi \subset T_\phi \subset \widehat{M}$. Using an equality in [25, p. 64], we have

$$W_\varphi^\circ \overset{\text{definition}}{=} N_{C_\varphi(\widehat{\mathcal{G}})^\circ}(T_\varphi)/T_\varphi = N_{\bar{C}_\varphi^\circ}(\bar{T}_\varphi)/\bar{T}_\varphi.$$

Thus, it suffices to show that

$$N_{\bar{C}_\varphi^\circ}(\bar{T}_\varphi) = N_{C_\phi(\widehat{G})^\circ}(T_\phi).$$

With above notation, the following lemma holds.

Lemma 6.2.
$$\bar{C}_\varphi^\circ = C_\phi(\widehat{G})^\circ$$

Proof. We first consider the following commutative diagram

It follows from the right vertical exact sequence above that

$$\bar{C}_\varphi^\circ = C_\varphi(\widehat{\mathcal{G}})^\circ/Z(\widehat{\mathcal{G}})^\Gamma \subset (C_\varphi(\widehat{\mathcal{G}})/Z(\widehat{\mathcal{G}})^\Gamma)^\circ \subset C_\varphi(\widehat{\mathcal{G}})/Z(\widehat{\mathcal{G}})^\Gamma.$$

Note that the index $[C_\varphi(\widehat{\mathcal{G}})/Z(\widehat{\mathcal{G}})^\Gamma : \bar{C}_\varphi^\circ]$ is finite and so is $[(C_\varphi(\widehat{\mathcal{G}})/Z(\widehat{\mathcal{G}})^\Gamma)^\circ : \bar{C}_\varphi^\circ]$. Since $\bar{C}_\varphi^\circ$ is connected due to the isomorphism $C_\varphi(\widehat{\mathcal{G}})^\circ/Z(\widehat{\mathcal{G}})^\Gamma \simeq (C_\varphi(\widehat{\mathcal{G}})^\circ \cdot Z(\widehat{\mathcal{G}}))/Z(\widehat{\mathcal{G}})$, we have

(6.5)
$$\bar{C}_\varphi^\circ = (C_\varphi(\widehat{\mathcal{G}})/Z(\widehat{\mathcal{G}})^\Gamma)^\circ.$$

From the proof of [8, Lemma 5.3.4] and the fact that $C_\varphi(\widehat{\mathcal{G}})/Z(\widehat{\mathcal{G}})^\Gamma \subset C_\phi(\widehat{G})$, we have the exact sequence

$$1 \longrightarrow C_\varphi(\widehat{\mathcal{G}})/Z(\widehat{\mathcal{G}})^\Gamma \longrightarrow C_\phi(\widehat{G}) \longrightarrow X^{\mathcal{G}}(\varphi) \longrightarrow 1,$$

where $X^{\mathcal{G}}(\varphi) := \{\mathbf{a} \in H^1(W_F, \widehat{U_1}) : \mathbf{a}\varphi \simeq \varphi\}$. Since $X^{\mathcal{G}}(\varphi)$ is finite, we have

$$(C_\varphi(\widehat{\mathcal{G}})/Z(\widehat{\mathcal{G}})^\Gamma)^\circ = C_\phi(\widehat{G})^\circ.$$

Therefore, from (6.5), Lemma 6.2 is proved. $\square$

Combining Lemmas 3.3 and 6.2, we have proved (6.4). This completes the proof of Theorem 6.1.

6.3. **Invariance of R-groups within L-packets and between inner forms.** Let $\phi \in \Phi_{\mathrm{disc}}(M)$ be given. In this section, we will discuss the behavior of Knapp-Stein R-groups within L-packets and between inner forms. We first provide the following theorem that is crucial to the invariance (Theorems 6.6 and 6.8).

Theorem 6.3. *Fix a lift $\varphi \in \Phi_{\mathrm{disc}}(\mathcal{M})$ and $\Sigma \in \Pi_\varphi(\mathcal{M})$. Then, we have*

$$W(\sigma) \simeq \{w \in W_M : {}^w\Sigma \simeq \Sigma\lambda \text{ for some } \lambda \in (\mathcal{M}/M)^D\}.$$

Proof. It suffices to consider the case of $m \geq 2$, since the equality is already true for the case $m = 1$ due to [15, Lemma 2.5]. Let $m \geq 2$. We then recall from (3.5) that

$$M \simeq (\mathrm{GL}_{n_1}(E) \times \cdots \times \mathrm{GL}_{n_k}(E)) \rtimes \mathrm{SU}_m(F).$$

Since Σ is of the form

$$\Sigma_1 \otimes \Sigma_2 \otimes \cdots \otimes \Sigma_k \otimes \Upsilon,$$

we set $\Sigma = \Sigma_0 \otimes \Upsilon$. Denote by τ is an irreducible constituent in $\mathrm{Res}^{\mathrm{U}_m}_{\mathrm{SU}_m}(\Upsilon)$. Write $V_\Sigma, V_{\Sigma_0} = \otimes^k_{i=1}V_{\Sigma_i}, V_\Upsilon, V_\tau$ for the corresponding representation $\mathbb{C}$-vector spaces.

Given $(g, h) \in M$ with $g = (g_1, \cdots, g_k) \in \mathrm{GL}_{n_1}(E) \times \cdots \times \mathrm{GL}_{n_k}(E)$ and $h \in \mathrm{SU}_m(F)$, $v'_0 = v_1 \otimes \cdots \otimes v_k \in V_{\Sigma_0}$, and $v_0 \in V_\tau$, we define a representation σ^* of M

$$(6.6) \qquad \sigma^*((g, h))(v'_0 \otimes v_0) = \Sigma_0(g)v'_0 \otimes \Upsilon(\alpha_m(g)^{-1})\tau(h)v_0.$$

It turns out from [15, p. 353] that σ^* is an irreducible constituent in $\mathrm{Res}^{\mathrm{U}_n}_{\mathrm{SU}_n}(\Sigma)$.

In what follows, using the idea in the proof of [15, Proposition 2.9], we will show

$$(6.7) \qquad W(\sigma^*) = \{w \in W_M : {}^w\Sigma \simeq \Sigma\lambda \text{ for some } \lambda \in (\mathcal{M}/M)^D\}.$$

The inclusion $\subset$ is obvious. Suppose that ${}^w\Sigma \simeq \Sigma\lambda$ for some $\lambda \in (\mathcal{M}/M)^D$. From (4.2), we set $w = sc$ with $s \in S_k$ and $c \in \mathbb{Z}_2^k$. From Sections 4.1 and 4.2, we have

$$^w\Sigma \simeq \bigotimes_{i=1}^k \varepsilon_i(\Sigma_{s(i)}) \otimes \Upsilon,$$

where ε_i is either ε or trivial. Since ${}^w\Sigma \simeq \Sigma\lambda$, we have $\varepsilon_i(\Sigma_{s(i)}) \simeq \Sigma_i\lambda$ for each i, and $\Upsilon \simeq \Upsilon\lambda$. We fix intertwining maps $T_i : V_{\Sigma_{s(i)}} \to V_{\Sigma_i}$ with $\Sigma_i\lambda T_i = T_i\varepsilon_i(\Sigma_{s(i)})$, and $T_\lambda : V_\Upsilon \to V_\Upsilon$ with $\Upsilon\lambda T_\lambda = T_\lambda\Upsilon$. Write

$$T = \bigotimes_{i=1}^k T_i \otimes T_\lambda.$$

Note that $\Sigma T = T^w \Sigma$. Then, from the definition (6.6) of σ^*, we have

$$T^w \sigma^*(g,h)(v_0' \otimes v_0)$$

$$= \bigotimes_{i=1}^{k} T_i \varepsilon_i(\Sigma_{s(i)})(g_i)v_i \otimes T_\lambda \Upsilon(\alpha_m(^w g)^{-1})\tau(\alpha_m(g)^{-1}\alpha_m(^w g)h)v_0$$

$$= \bigotimes_{i=1}^{k} T_i \varepsilon_i(\Sigma_{s(i)})(g_i)v_i \otimes T_\lambda \Upsilon(\alpha_m(g)^{-1})\tau(h)v_0$$

$$= \bigotimes_{i=1}^{k} \Sigma_i(g_i)\lambda(g_i)T_i(v_i) \otimes \Upsilon(\alpha_m(g)^{-1})\lambda(\alpha_m(g)^{-1})\tau(h)T_\lambda(v_0)$$

$$= \sigma^*\lambda(g,h)T(v_0' \otimes v_0)$$

$$= \sigma^* T(v_0' \otimes v_0),$$

since

$$\lambda(g,h) \overset{(3.6)}{=} \lambda\left(\begin{pmatrix} g & & \\ & \alpha_m(g)^{-1}h & \\ & & \varepsilon(g) \end{pmatrix}\right) = 1.$$

Thus, we have $w \in W(\sigma^*)$, which implies (6.7). From the proof of [10, Theorem 4.19], we note

$$W(\sigma) \simeq W(\sigma^*),$$

since both are in the same restriction $\text{Res}^{U_n}_{SU_n}(\Sigma)$. Therefore, we proved the theorem. $\square$

Remark 6.4. The difference between $W(\sigma)$ and $W(\Sigma)$ turns out to be whether the twisting by a character in $(\mathcal{M}/M)^D$ is allowed or not. We also refer to [15, Theorem 3.7] for another description of this difference.

Remark 6.5. We note that Theorem 6.3 holds for F-inner forms $\mathbf{M}'$ as well. To be precise, given $\sigma' \in \Pi_\phi(M')$, we fix $\Sigma' \in \Pi_\varphi(\mathcal{M}')$. Then, we have

$$W(\sigma') \simeq \{w \in W_{M'} : {}^w\Sigma' \simeq \Sigma'\lambda \text{ for some } \lambda \in (\mathcal{M}'/M')^D\}.$$

Theorem 6.6. *Given* σ_1, $\sigma_2 \in \Pi_\phi(M)$, *we have*

$$R_{\sigma_1} \simeq R_{\sigma_2}.$$

Furthermore, given σ_1', $\sigma_2' \in \Pi_\phi(M')$, *we have*

$$R_{\sigma_1'} \simeq R_{\sigma_2'}.$$

Proof. Since $W^\circ(\sigma_1) = W^\circ(\sigma_2)$, it is enough to show that

$$W(\sigma_1) \simeq W(\sigma_2).$$

Fix Σ^1 and Σ^2 in $\Pi_\varphi(\mathcal{M})$ such that $\sigma_1 \hookrightarrow \text{Res}^{U_n}_{SU_n}(\Sigma^1)$ and $\sigma_2 \hookrightarrow \text{Res}^{U_n}_{SU_n}(\Sigma^2)$. Suppose that ${}^w\Sigma^1 \simeq \Sigma^1\lambda$ for some $\lambda \in (\mathcal{M}/M)^D$. From (4.2), we write $w = sc$

with $s \in S_k$ and $c \in \mathbb{Z}_2^k$. Set $\Sigma^1 \simeq \Sigma_1^1 \otimes \Sigma_2^1 \otimes \cdots \otimes \Sigma_k^1 \otimes \Upsilon_1$. Then $\Sigma_i^1 \simeq \varepsilon_i(\Sigma_{s(i)}^1)\lambda$ for each i, and $\Upsilon_1 \simeq \Upsilon_1\lambda$. Set

$$\lambda' := \begin{cases} \lambda, & \text{on } \mathrm{GL}_{n_1}(E) \times \cdots \times \mathrm{GL}_{n_k}(E), \\ \mathbb{1}, & \text{on } \mathrm{U}_m(F), \end{cases}$$

which is a character on $\mathcal{M}/M$. Moreover, λ' satisfies

$$^w\Sigma^2 \simeq \Sigma^2\lambda',$$

which implies, from Theorem 6.3, that $W(\sigma_1) \hookrightarrow W(\sigma_2)$. In the same manner, one can verify $W(\sigma_2) \hookrightarrow W(\sigma_1)$. Thus, we have $R_{\sigma_1} \simeq R_{\sigma_2}$. Since the method is the same, we omit the proof for $R_{\sigma_1'} \simeq R_{\sigma_2'}$. □

Remark 6.7. The trivial character $\mathbb{1}$ in the definition of λ' can be replaced by a character λ_0 on $\mathrm{U}_m(F)$ such that $\Upsilon_2 \simeq \Upsilon_2\lambda_0$, where Υ_2 is the representation of $\mathrm{U}_m(F)$ in the decomposition $\Sigma^2 \simeq \Sigma_1^2 \otimes \Sigma_2^2 \otimes \cdots \otimes \Sigma_k^2 \otimes \Upsilon_2$.

Theorem 6.8. *Given $\sigma \in \Pi_\phi(M)$ and $\sigma' \in \Pi_\phi(M')$, we have*

$$R_\sigma \simeq R_{\sigma'}.$$

Proof. The proof is similar to Theorem 6.6. □

Remark 6.9. Due to [15, Theorem 3.7], Theorem 6.8 shows that the Knapp-Stein R-group $R_{\sigma'}$ is of the form

$$\Gamma_{\sigma'} \ltimes \mathbb{Z}_2^d,$$

for some subgroup $\Gamma_{\sigma'}$ in $R_{\sigma'}$ defined in *loc. cit.* and some integer d in Corollary 5.4. This implies that $R_{\sigma'}$ is in general non-abelian from the argument in [14, Remark p. 359].

Acknowledgements

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