Bulletin of the Iranian Mathematical Society Vol. 30 *No. 1 (2004), pp 21-29.*

EXISTENCE OF AN ALGEBRA NORM ON CERTAIN SUBALGEBRAS OF C(X)

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ABSTRACT. Let X be a topological space and A be a subalgebra of $C(X)$. We provide conditions on A so that it admits an algebra norm (not necessarily complete). When X is hemicompact and $(A, (p_n))$ is a functionally continuous regular Fréchet function algebra on X, we show that A is a Q-algebra and does not contain any unbounded function if it admits an algebra norm. Moreover, any uniform norm on A is continuous. Finally we prove that in some particular cases, the existence of an algebra norm on A leads to the compactness of X.

1. Introduction and Preliminaries

If X is a compact Hausdorff space then $C(X)$, the algebra of all continuous complex valued functions on X, is obviously a Banach algebra with the supremum norm and any Banach algebra norm on $C(X)$ is equivalent to the supremum norm. When X is an arbitrary topological space, it has been shown in [7] and [8] that $C(X)$ admits an algebra norm (not necessarily complete) if and only if it does not contain any unbounded function (also see [5]). For a completely regular space X, this is indeed equivalent to the compactness of the

MSC(2000): Primary 46J05, 46J10; Secondary 46E25

Keywords: Algebra norms, Regular Fréchet function algebras, Topological algebras, Q-algebras

Received: 21 October 2003, Revised: 12 April 2004

real compactification of X. The proof of this result in [8] is based on the following theorem of Kaplansky.

Theorem 1.1. [6] *Let $(A, \|.\|)$ be a commutative Banach algebra which is isometrically isomorphic to $C_0(X)$, for some locally compact space X. Then any algebra norm $|.|$ on A satisfies $|f| \geq \|f\|$, for every $f \in A$.*

In this paper we consider the same problem for certain subalgebras of $C(X)$, where X is mainly a hemicompact space. In general, the existence of an algebra norm on a subalgebra A of $C(X)$ does not imply that all elements in A are bounded functions. For example, $\|f\| = \sup\{|f(z)| : |z| < 1\}$ defines an algebra norm on the algebra $\mathrm{Hol}(\mathbb{C})$ of all holomorphic functions on $\mathbb{C}$.

Before proving our results, we state some preliminaries. For more details one can refer to [1], [3] and [4].

By a *hemicompact* space we mean a Hausdorff topological space X with a sequence (K_n) of increasing compact subsets of X such that every compact subset of X is contained in some K_n. The sequence (K_n) is called an *admissible exhaustion* of X. A *k-space* is a Hausdorff space in which every subset intersecting each compact subset in a closed set is itself closed.

In this paper we assume that all algebras are unital.

By a *Fréchet algebra* we mean a locally multiplicatively convex topological algebra which is also metrizable and complete. Thus the topology of a Fréchet algebra can be defined by an increasing sequence of submultiplicative seminorms . Unlike the theory of Banach algebras, in the theory of Fréchet algebras we do not know whether each complex homomorphism on a Fréchet algebra is continuous or not (Michael's problem). A commutative Fréchet algebra $(A, (p_n))$ is called *functionally continuous* if all complex homomorphisms on A are automatically continuous. The set of all nonzero complex homomorphisms on A is denoted by S_A and the set of all nonzero continuous complex homomorphisms on A, which is called the *spectrum* of A, is denoted by M_A or $M_{(A,(p_n))}$. As usual, we always endow M_A with the Gelfand topology and for every $x \in A$, $\hat{x}$ is the Gelfand transform of x. A commutative

Fréchet algebra A is *regular* on its spectrum if for any closed subset F of M_A and $\varphi \in M_A \backslash F$ there is an element $x \in A$ with $\hat{x}(\varphi) = 1$ and $\hat{x}(F) = \{0\}$. It is well known that each commutative regular Fréchet algebra is *(Gelfand) normal*, that is, for every pair of disjoint closed subsets K and F of M_A there exists $x \in A$ such that $\hat{x}(F) = \{0\}$ while $\hat{x}(K) = \{1\}$.

A topological algebra A is called a *Q-algebra* if the set A^{-1} of all invertible elements of A is an open set. A *uniform topological (Fréchet) algebra* is a topological (Fréchet) algebra whose topology is determined by a separating family ρ of uniform seminorms, that is, $p(x^2) = p(x)^2$ for all p in ρ and $x \in A$. For example, if X is a hemicompact k-space with an admissible exhaustion (K_n), then $(C(X), (\|.\|_{K_n}))$ is a uniform Fréchet algebra , where $\|.\|_{K_n}$ is the supremum norm on K_n. Indeed, a commutative algebra A is a uniform Fréchet algebra if and only if there is a hemicompact space X such that A is topologically and algebraically isomorphic to a point separating and complete subalgebra of $C(X)$ which also contains the constants ([4, Theorem 4.1.3]).

Similarly by a *Fréchet function algebra* on a hemicompact space X we mean a point separating subalgebra A of $C(X)$ which contains the constants and is a Fréchet algebra under some topology such that all evaluation homomorphisms φ_x, for $x \in A$, are continuous. That is, $\varphi_x \in M_A$ for all $x \in A$. In fact, the class of all commutative semisimple unital Fréchet algebras coincides with the class of Fréchet function algebras, since the spectrum of each commutative Fréchet algebra is a hemicompact space.

If $(A, (p_n))$ is a Fréchet function algebra on a hemicompact space X and (K_n) is an admissible exhaustion of X then it is easy to see that for each n there is an integer $m \geq n$ such that

$$\|f\|_{K_n} \leq p_m(f)$$

for all $f \in A$ [9]. This shows that the identity map from $(A, (p_n))$ into $(C(X), \|.\|_{K_n})$ is continuous. For each n, let $i(n) \geq n$ be the smallest integer for which the above inequality holds. Since (p_n) is an increasing sequence and $(p_{i(n)})$ generates the same topology, we may assume that $i(n) = n$.

2. The Results

When X is a hemicompact k-space with an admissible exhaustion (K_n), the existence of an algebra norm on the uniform Fréchet algebra $(C(X), (\|.\|_{K_n}))$ implies the compactness of X. Because, as it was mentioned before, in this case all functions in $C(X)$ are bounded by [7] or [8]. That is, $C_b(X) = C(X)$. Since $(C_b(X), \|.\|_X)$ is a semisimple Banach algebra, by Carpenter's Theorem, it has a unique topology as a Fréchet algebra. Hence the topologies induced by the sequence $(\|.\|_{K_n})$ of seminorms and the supremum norm $\|.\|_X$ are equivalent on $C(X)$. That is, $(C(X), (\|.\|_{K_n}))$ is a Banach algebra and thus X is compact (see [4]).

Now let A be a subalgebra of $C(X)$ for some topological space X and set $A_b = A \cap C_b(X)$. As the following proposition shows, if there exists an algebra norm $\|.\|$ on A such that either all evaluation homomorphisms are $\|.\|$ continuous or the set A^{-1} is open with respect to this norm, then $A_b = A$.

Proposition 2.1. *Let S be a non-empty set and let A be an algebra of functions on S. If there exists an algebra norm on A such that each evaluation functional is continuous then all functions in A are bounded. The same conclusion holds if there is a topology on A making it a Q-algebra.*

Proof. Let $\|.\|$ be an algebra norm on A such that φ_x is continuous on $(A, \|.\|)$ for all $x \in X$. The completion $\bar{A}$ of $(A, \|.\|)$ is a Banach algebra and since every φ_x is $\|.\|$ continuous on A it can be extended to a $\|.\|$ continuous homomorphism on $\bar{A}$. Hence for $f \in A$, $f(X)$ is contained in $\sigma_{\bar{A}}(f)$, the spectrum of f in $\bar{A}$. Since $\sigma_{\bar{A}}(f)$ is compact, f is a bounded function.

In a Q-algebra each element has a bounded spectrum, so that the result follows from the inclusion $f(X) \subseteq \sigma_A(f)$. □

An immediate consequence of the above proposition is that when $(A, (\|.\|_{K_n}))$ is a uniform Fréchet algebra on a hemicompact space X (with (K_n) as an admissible exhaustion) such that (A, τ) is a Q-algebra, for some topology τ, then $(A, (\|.\|_{K_n}))$ is a Banach algebra.

Because $A_b = A$ by Proposition 2.1, and since $(A, (\|.\|_{K_n}))$ is a uniform Fréchet algebra, it is easy to verify that $(A_b, \|.\|_X)$ is a Banach algebra. Hence by the Carpenter's theorem the topologies induced by $\|.\|_X$ and $(\|.\|_{K_n})$ are equivalent on A. That is, A is a Banach algebra.

It is well known that for any algebra norm $|.|$ on a regular uniform Banach algebra $(A, \|.\|)$ we have $\|.\| \leq |.|$ (see [3, Corollary 4.1.28] or [2]). In the sequel we confine ourselves to hemicompact spaces. In the first part of the following theorem we use a similar idea to [2] to get a result concerning the existence of an algebra norm on a functionally continuous regular Fréchet function algebra.

Theorem 2.2. *Let X be a topological space and let $(A, (p_n))$ be a commutative semisimple unital Fréchet algebra whose spectrum is equal to X. Suppose that A is regular and functionally continuous.*

(i) If A admits an algebra norm then $(A, (p_n))$ is a Q-algebra.

(ii) Any uniform norm on A is continuous.

Proof. (i) Let $|.|$ be an algebra norm on A and let $\bar{A}$ be the completion of $(A, |.|)$. Take Y as the set of all points in X such that φ_x is $|.|$ continuous on A. Note that since A is functionally continuous, Y can be identified with the spectrum of $\bar{A}$. We first show that Y is dense in X. Assume that $\bar{Y} \neq X$ and choose, by the regularity of A, an open subset G of X with $\bar{G} \subseteq X \backslash \bar{Y}$. Since A is regular and hence normal, there exists an element $f \in A$ with $\hat{f}|_{\bar{Y}} = 1$ and $\hat{f}|_{\bar{G}} = 0$. We claim that f is invertible in $\bar{A}$. If f is not so, there exists an element y in the spectrum of the Banach algebra $\bar{A}$ with $y(f) = 0$. That is, $y \in Y$ and $\hat{f}(y) = 0$ which is impossible. Now let $z \in G$ and choose an element $g \in A$ with $\hat{g}|_{X \backslash G} = 0$ and $\hat{g}(z) = 1$ so that supp $\hat{g} \subseteq \bar{G}$ and $\hat{g} \neq 0$. Then $\hat{f}\hat{g} = 0$ on X and hence $fg = 0$ in A and consequently in $\bar{A}$. But this is impossible because f is invertible in $\bar{A}$ and $g \neq 0$. Thus $\bar{Y} = X$.

As we noted before Y can be identified with the spectrum of $\bar{A}$ and so Y is a compact space with respect to the Gelfand topology from $\bar{A}$ which is the same as the relative topology from X. Hence Y is a compact subset of X, so that $Y = X$. It now follows that A

is a Q-algebra (see [10] or [1, 4.12-3]) . In particular, each element of A has a finite spectral radius and $\|\hat{f}\|_X = r_A(f) = r_{\bar{A}}(f) \leq |f|$, for all $f \in A$.

(ii) Let $|.|$ be a uniform norm on A. By Theorem 1 in [2], $|.|$ is an algebra norm and so by (i), $(A,(p_n))$ is a Q-algebra and $\|\hat{f}\|_X \leq |f|$, for all $f \in A$. Let $N \in \mathbb{N}$ be such that $V = \{f \in A : p_N(1-f) < 1/N\} \subseteq A^{-1}$. If $f \in A$ and $p_N(f) = 0$ then for each non zero scalar λ, $1 - \lambda^{-1}f \in V$ and so $\lambda \notin \sigma_A(f)$. Therefore, $r_A(f) = r_{\bar{A}}(f) = |f| = 0$, where $\bar{A}$ is the completion of $(A,|.|)$. Hence $f = 0$. This shows that p_N is indeed a norm.

Now let f be a non zero element of A. Then for each scalar λ with $|\lambda| > 1$ we have $p_N(\frac{f}{N\lambda p_N(f)}) < 1/N$ and so $\lambda.1 - \frac{f}{Np_N(f)} \in A^{-1}$. This shows that $r_A(\frac{f}{Np_N(f)}) = r_{\bar{A}}(\frac{f}{Np_N(f)}) = |\frac{f}{Np_N(f)}| \leq 1$. Note that since $|.|$ is a uniform norm it follows easily that $|f| \leq p_N(f)$, for all $f \in A$. □

In the following theorem we obtain a similar result for regular Fréchet function algebras with locally compact spectrum without the functional continuity assumption.

Theorem 2.3. *Let X be a hemicompact space and let $(A,(p_n))$ be a regular Fréchet function algebra whose spectrum is locally compact. If there is a continuous algebra norm on A then all functions in A are bounded.*

Proof. Let $\|.\|$ be a continuous algebra norm on A and let $(\bar{A},\|.\|)$ be the completion of $(A,\|.\|)$. Then the inclusion map $id : (A,(p_n)) \longrightarrow (\bar{A},\|.\|)$ is continuous with a dense range so that its transpose map $id^* : M_{(\bar{A},\|.\|)} \longrightarrow M_{(A,(p_n))}$, defined by $\varphi \longmapsto \varphi|_A$, is a continuous injective map. Since $M_{(A,(p_n))}$ is locally compact, it follows from Theorem 2.6 in [9] that $id^*(M_{(\bar{A},\|.\|)})$ is dense in $M_{(A,(p_n))}$ and hence $M_{(A,(p_n))}$ is compact, since $M_{(\bar{A},\|.\|)}$ is compact. Therefore, $(A,(p_n))$ is a Q-algebra by [10] and all elements of A are bounded by Proposition 2.1. Note that in this case, $(A_b,\|.\|_X)$ may not be a Banach algebra. □

Theorem 2.4. *Let X be a hemicompact space, (K_n) an admissible exhaustion of X and $(A,(p_n))$ a functionally continuous regular Fréchet function algebra on X. If the sequence (p_n) is such that for $f \in A$ and $n \in \mathbb{N}$, $f|_{K_n} = 0$ implies $p_n(f) = 0$, then A admits an algebra norm if and only if X is compact.*

Proof. Assume that there is an algebra norm on A. Then by Theorem 2.2, $(A,(p_n))$ is a Q-algebra and hence the set of all invertible elements of A is open. So that there is an integer N such that $V = \{f \in A : p_N(1-f) < 1/N\} \subseteq A^{-1}$. Now if X is not compact, we can choose $x \in X \backslash K_N$. The regularity of A shows that there is an element $f \in A$ with $\hat{f}(\varphi_y) = 1$, $y \in K_N$ and $\hat{f}(\varphi_x) = 0$. Hence $(1-f)|_{K_N} = 0$ and so by the hypothesis $p_N(1-f) = 0$, which shows that f is invertible in A. But this is impossible because $f(x) = 0$. □

Let X be a hemicompact k-space, (K_n) an admissible exhaustion of X and (A_n) a sequence of Banach function algebras such that for each n, A_n is a Banach function algebra on K_n with respect to $\|.\|_n$, $A_{n+1}|_{K_n} \subseteq A_n$ and $\|f|_{K_n}\|_n \leq \|f\|_{n+1}$, for all $f \in A_{n+1}$. Then the subalgebra

$$A = \{f \in C(X) : f|_{K_n} \in A_n,\ n \in \mathbb{N}\}$$

of C(X) is a Fréchet algebra with respect to the topology defined by the sequence (p_n) of seminorms, where $p_n(f) = \|f|_{K_n}\|_n$, $f \in A$. Moreover, if A separates the points of X then it is a Fréchet function algebra on X (see [9]). Clearly, in this case for each n and $f \in A$, $f|_{K_n} = 0$ implies $p_n(f) = 0$. Thus if A is regular and functionally continuous then the existence of an algebra norm on A is equivalent to the compactness of X by Theorem 2.4. An example of this situation is described as follows:

Example 2.5. Let (X,d) be a hemicompact metric space with the admissible exhaustion (K_n) and let $0 < \alpha \leq 1$. For each n, let $A_n = \mathrm{Lip}(\mathrm{K_n}, \alpha)$ be the Banach algebra of all Lipschitz functions of order α on K_n with pointwise addition and multiplication, which is

endowed with the following norm:

$$\|f\|_n = \|f\|_{K_n} + \sup_{\substack{x,y\in K_n \\ x\neq y}} \frac{|f(x)-f(y)|}{d^\alpha(x,y)}.$$

By the above argument the subalgebra $A = \mathrm{FLip(X,\alpha)} = \{\mathrm{f} \in \mathrm{C(X)} : \mathrm{f}|_{\mathrm{K_n}} \in \mathrm{A_n}, \mathrm{n} \in \mathbb{N}\}$ of $C(X)$ is a Fréchet function algebra on X. Since A is a symmetric *-algebra with $f \mapsto \bar{f}$ as an involution, it is functionally continuous. Moreover, M_A is homeomorphic to X [9] and it is easy to see that A is regular on its spectrum. Since $M_A \cong X$, the algebra A does not admit a Banach algebra norm if X is not compact. Now Theorem 2.4 shows that in this case A does not admit an algebra norm (not necessarily complete).

Acknowledgment

The author would like to thank the referee for his helpful comments.

References

[1] E. Beckenstein, L. Narici and C. Suffel, *Topological Algebras*, Notas Mat. 60, North-Holland, Amsterdam, 1977.

[2] S. J. Bhatt and D. J. Karia, Uniqueness of the uniform norm with an application to topological algebras, *Proc. Amer. Math. Soc.* 116 (2) (1992), 499-503.

[3] H. G. Dales, *Banach Algebras and Automatic Continuity*, Oxford University Press, Oxford, 2000.

[4] H. Goldmann, *Uniform Fréchet Algebras*, North-Holland, Amsterdam, 1990.

[5] K. Jarosz, Norms on $C(X)$, preprint.

[6] L. Kaplansky, Normed algebras, *Duke Math. J.*, 16 (1949), 399-418.

[7] M. J. Meyer, Some algebras without submultiplicative norms or positive functionals, *Studia Math.* 116 (3) (1995), 299-302.

[8] A. R. Pruss, A remark on non-existence of an algebra norm for the algebra of continuous functions on a topological space admitting an unbounded continuous function, *Studia Math.* 116 (3) (1995), 295-297.

[9] F. Sady, Projective limit of a sequence of Banach function algebras as a Fréchet function algebra, *Bull. Korean Math. Soc.* 39 (2) (2002), 259-267.

[10] W. Zelazko, On maximal ideals in commutative m-convex algebras, *Studia Math.* 58 (1976), 291-298.

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