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A CLASS OF TWO GENERATED AUTOMATON GROUPS ON A THREE LETTER ALPHABET

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ABSTRACT. In this paper we take first steps towards the classification of of the class of two generated automaton groups on a three letter alphabet. More precisely, we partially characterize the groups that act spherically transitively on T_3, are fractal or non-fractal, from the class of automaton groups in which each group is generated by a set of two elements each of which is composed of three automata. We determine the cardinality of each subclass as well.

1. Introduction

The notion of automaton group has received great attention of a wide range of mathematicians due to the recent works of Bartholdi-Grigorchuk [2], [3]; Grigorchuk-Zuk [12], [13], [14]; Grigorchuk [8]; Brunner-Sidki-Viera [6] and the most informative survey paper of Grigorchuk-Nekrashevych-Shushchansky [11]. So that in the international conference of group theory that was held in Gaeta-Italy in 1-6 June 2003 one of the major themes of the conference was the automaton groups.
Automaton groups are groups generated by invertible automata and act on rooted regular trees as automorphisms. In 1960's V. M.

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Glushkov suggested to apply the abstract theory of automata to the Burnside problems [1] and [16]. We got acquainted with this subject in a talk given by Grigorchuk at Sharif University of Technology in Tehran in 1994.
The first example of an automaton group belongs to the early 1970,s that was introduced to solve the Burnside problem [1], but the subject got attention of group theorists when Grigorchuk introduced in 1997 the notions of branch and fractal groups in a group theory conference in Bath(England) [10].The basic ideas for these notions were introduced by him in the beginnings of 1980 in [7] where he introduces a group that answers the Burnside problem, and has many other interesting properties, such as having sub-exponential growth, being just infinite, being just non-solvable, having infinite $L-$presentation, etc. This group which is now called the first Grigorchuk group is an automaton group generated by a 5 state automaton on the alphabet $\{0, 1\}$.
The class of automaton groups generated by the two state automata on the alphabet $\{0, 1\}$ is classified by Grigorchuk and [et.al] in [11]. Although there is no classification of other automaton groups [11], but the problem is in focus of several research groups [9].
It should be noted that the class of automaton groups under consideration may be considered as a subclass of the class of automaton groups generated by three state automata on a three letter alphabet.

2. Preliminaries

We fix the set of states $Q = \{q_1, q_2, \dots q_m\}$, the alphabet $S = \{0, 1, \dots d-1\}$, the set of all finite words S^* including the empty word $\emptyset$ on alphabet S and S_d the permutation group of S.

Definition 2.1. An automata with alphabet S and the set of states Q is a quadruple $\mathbf{A} = (\mathbf{S}, \mathbf{Q}, \varphi, \psi)$, where $\varphi : S \times Q \to Q$ and $\psi : S \times Q \to S$ are the transition and exit functions respectively. S is also called the input and output alphabet of $\mathbf{A}$. $\mathbf{A}$ is called invertible if $\psi(.,s) \in S_d$ for each $s \in Q$.

Let $\mathbf{A}$ be an invertible automaton with S and Q as above. We can use an oriented labeled graph to describe $\mathbf{A}$. To this end let Γ be a graph with m vertices labeled $q_1, q_2, \dots q_m$. Now we join the vertex r to the vertex s with an edge of label $i \in S$ if $\varphi(i,r) = s$. besides the labels mentioned above for the vertices we again label the vertex $s \in Q$ by the permutation $\psi(.,s) \in S_d$.
The task we define for the invertible automaton $\mathbf{A}$ is to read the words in S^* as inputs, letter by letter from left to right, and produce words of the same length in S^* as outputs . However since it has no initial states in order to activate it we have to initialize it by choosing any of it's states as an initial state. Doing so we get m initial automata, one corresponding to each state in Q. The initial automaton with initial state p is denoted by $A_p = (S, Q, \varphi, \psi, p)$.
Now given the word $u_1 u_2 \dots u_n$ in S^* and the state $s \in Q$ the initial automaton A_s immediately calculates $r = \varphi(u_1, s) \in Q$ and $v_1 = \psi(u_1, s) \in D$ and passes through the edge with label u_1 to the state r. Now the initial automaton A_r acts on the word $u_2 u_3 \dots u_n$ as did A_s on $u_1 u_2 \dots u_n$ and this process continues until the automaton finishes reading the word by reading its last letter u_n. The word $v_1 v_2 \dots v_n$ we obtain in this way is the output of this activity .
An initial automaton is called neutral if its action on S^* is as the identity function. By composing the initial automata $A_p = (S, Q, \varphi, \psi, p)$ and $A_q = (S, Q, \varphi, \psi, q)$, resulting from a given invertible automaton A we obtain another initial automaton $A_{(p,q)} = (S, Q \times Q, \phi, \chi, (p,q))$ on the same alphabet S, where $\phi : S \times Q \times Q \to Q \times Q$ and $\chi : S \times Q \times Q \to S$ are defined as follows . $\phi(i,(r,s)) = (\varphi(i,r), \varphi(\psi(i,r), s))$ and $\chi(i,(r,s)) = \psi((\psi(i,r), s)$. The automaton $A_{(p,q)}$ is denoted by $A_p * A_q$ and is said a composite automaton. On the set of all initial automata resulting from $\mathbf{A}$ in this way we define the equivalence of two initial automata as follows: We say that two initial automata are equivalent if they determine the same map on the set S^*. The set of equivalence classes form a group know as the **automaton group** generated by the automaton $\mathbf{A}$ and is denoted by $G(\mathbf{A})$ or $G(\mathbf{A}) =< \mathbf{A_{q_1}}, \cdots, \mathbf{A_{q_m}} >$.

There is a close relationship between automaton groups and wreath products. To match our needs in this paper we describe this relationship in some detail. Consider the groups $G(\mathbf{A})$, S and the group $G(\mathbf{A})^{\mathbf{S}}$ consisting of all functions from S to $G(\mathbf{A})$. The function $f \in G(\mathbf{A})^{\mathbf{S}}$ is determined by its values $f_0, f_1, \dots f_{d-1}$. Therefore we can write $f = (f_0, f_1, \dots f_{d-1})$. Define the right action of S_d on S by

$$(i, \sigma) = \sigma^{-1} i, i \in S, \sigma \in S_d.$$

Therefore S_d also acts on $G(\mathbf{A})^{\mathbf{S}}$ via

$$(f, \sigma) = (f_{\sigma^{-1}(0)}, f_{\sigma^{-1}(1)}, \dots f_{\sigma^{-1}(d-1)}).$$

Using these data we can define the wreath product $G(\mathbf{A}) \wr \mathbf{S_d}$ as follows: The elements of $G(\mathbf{A}) \wr \mathbf{S_d}$ are the elements of the cartesian product $G(\mathbf{A})^{\mathbf{S}} \times \mathbf{S_d}$ and the composition of (f, σ) and (g, δ) with $f = (f_0, f_1, \dots f_{d-1})$ and $g = (g_0, g_1, \dots g_{d-1})$ is

$$(f, \sigma)(g, \delta) = (h, \sigma\delta)$$

where $h = (h_0, h_1, \dots h_{d-1})$ and $h_i = f_i g_{\sigma^{-1}(i)}$. The element (f, σ) of the wreath product will be written $f\sigma$ or $(f_0, f_1, \dots f_{d-1})\sigma$ if $f = (f_0, f_1, \dots f_{d-1})$.

We embed $G(\mathbf{A})$ in the wreath product $G(\mathbf{A}) \wr \mathbf{S_d}$ as follows: Consider the initial automaton A_q and let the label of the vertex q in Γ be $\sigma_q = \psi(., q)$. Then the initial automaton A_q will be connected to the initial automata $A_{q_0} = A_{\varphi(0,q)}, A_{q_1} = A_{\varphi(1,q)}, \dots A_{q_{d-1}} = A_{\varphi(d-1,q)}$ in Γ. Now the group $G(\mathbf{A})$ embeds in the wreath product $G(\mathbf{A}) \wr \mathbf{S_d}$ via the map

$$A_q \to (A_{q_0}, A_{q_1}, \dots A_{q_{d-1}})\sigma_q.$$

By abuse of language we write this relation as $A_q = (A_{q_0}, A_{q_1}, \dots A_{q_{d-1}})\sigma_q$.

To facilitate the study of G we now define other concepts that are necessary for this purpose. The length $|u|$ of $u \in S^*$ is the number of letters that constitute u . Now for any fixed $u \in S^*$ the elements of the subset $\{uv | v \in S^*\}$ of S^* can be arranged as the vertices of a rooted tree T_u with root u, in which there is an edge $e = (uv, uw)$ between two vertices uv and uw if and only if $w = vi$ for some $i \in S$. This tree for $u = \emptyset$ is denoted by T_d and is called a d-ary tree with root $\emptyset$. Let $n \geq 0$ be an integer, the set of all vertices of T_d with length n is denoted by L_n and is called the nth level of T_d. The stabilizer subgroups of G are the most important subgroups in our study of G. These are stabilizer of a vertex of T_d, stabilizer of a level of T_d , stabilizer of an element of the boundary of T_d (the so called parabolic subgroups of G[12]), rigid stabilizer of a vertex of T_d and the rigid stabilizer of the n-th level of T_d.

Definition 2.2. We denote the subgroup of G that stabilizes the vertex u of T_d by $St_G(u)$, i.e.

$$St_G(u) = \{g \in G | ug = u\}.$$

Also the subgroup of G that stabilizes the level L_n of T_d is denoted by $St_G(n)$. We have

$$St_G(n) = \{g \in G | ug = u, u \in L_n\}.$$

Given a vertex u of T_d a subgroup $rist_G(u)$ of G that acts trivially on the complement of the tree T_u is called the rigid stabilizer of u. The rigid stabilizer of the level L_n is a subgroup of G generated by the rigid stabilizers of the vertices of this level and is denoted by $rist_G(n)$.

The fact that the subgroups $St_G(n), n = 1, 2, \ldots$ and $rist_G(n), n = 0, 1, \ldots$ are normal is obvious. Of particular interest is the subgroup $St_G(1)$. Considering $g \in St_G(1)$ as an automaton we observe that g corresponds to a $d+1$-tuple $(g_0, g_1, \ldots g_{d-1}, i) = (g_0, g_1, \ldots g_{d-1})i = (g_0, g_1, \ldots g_{d-1})$ in the wreath product $G(A) \wr S_d$, i.e. the label of the start space of g is the identity permutation $i \in S_d$ (this is the crucial fact from which we conclude that g fixes the vertices $0, 1, \cdots d-1$), and we connect g to g_0, g_1 and g_{d-1} with edges labeled $0, 1, \cdots d-1$ respectively. Consequently there is a well defined embedding

$$\omega : St_G(1) \to G \times \cdots \times G, \omega(g) = (g_0, g_1, \ldots g_{d-1})$$

and hence well defined canonical projections $\pi_i : St_G(1) \to G, i = 0, 1 \ldots d-1; \pi_i(g) = g_i, i = 0, 1, \ldots d-1$ form $St_G(1)$ to the base group G.

Similarly one can define the projections $\pi_u : St_G(u) \to G$ for any vertex u.

Definition 2.3. A group G that acts by automorphisms on a rooted tree T is called fractal if for every vertex u, $\pi_u(St_G(u)) = G$ after the identification of the tree T_d with the subtree T_u with root at u.

3. Table of Generators

To study the two generated automaton groups G on alphabet $S = \{0, 1, 2\}$, as mentioned in the title of the paper, we use an embedded copy of G in a wreath product $G(\mathbf{A}) \wr \mathbf{S_3}$, where $\mathbf{A}$ is a three state automaton on S and S_3 is the symmetric group of S. First of all due to its repeated applications we let $S_3 = \{i, \alpha = (01), \beta = (02), \gamma = (12), \delta = (012), \lambda = \delta^2 = (021)\}$, with table 1 as its composition table.

$\circ$	1	α	β	γ	δ	λ
1	1	α	β	γ	δ	λ
α	α	1	λ	δ	γ	β
β	β	δ	1	λ	α	γ
γ	γ	λ	δ	1	β	α
δ	δ	β	γ	α	λ	1
λ	λ	γ	α	β	1	δ

Table 1

Secondly we arrange all possible three state initial automata $\{a = (x, y, z)\sigma, b = (u, v, w)\nu\}$ with $x, y, z, u, v, w \in \{1, a, b\}$ and $\sigma, \nu \in S_3$ in table 2. Now let for the fixed element $1 = (1, 1, 1)i$ and any two elements a and b from this table $\mathbf{A(a, b, 1)}$ be an automaton with three initial automata two of which are a and b, and the third one is the neutral initial automaton. The group generated by $\mathbf{A(a, b, 1)}$ is denoted by $G(a, b, 1)$. As 1 is the neutral automaton this is actually a two generated group. We observe that according to table 2 we have 162 choices for each a and b. Since the automata $\mathbf{A(a, b, 1)}$ and $\mathbf{A(b, a, 1)}$ are identical there are $\frac{162\times162}{2} = 13122$ automata that generate the groups we interested in.

1	$(1,1,1)$	43	$(1,a,a)$	85	(b,b,b)	127	(b,a,a)
2	$(1,1,1)\alpha$	44	$(1,a,a)\alpha$	86	$(b,b,b)\alpha$	128	$(b,a,a)\alpha$
3	$(1,1,1)\beta$	45	$(1,a,a)\beta$	87	$(b,b,b)\beta$	129	$(b,a,a)\beta$
4	$(1,1,1)\gamma$	46	$(1,a,a)\gamma$	88	$(b,b,b)\gamma$	130	$(b,a,a)\gamma$
5	$(1,1,1)\delta$	47	$(1,a,a)\delta$	89	$(b,b,b)\delta$	131	$(b,a,a)\delta$
6	$(1,1,1)\lambda$	48	$(1,a,a)\lambda$	90	$(b,b,b)\lambda$	132	$(b,a,a)\lambda$
7	$(1,a,1)$	49	$(1,1,b)$	91	$(1,b,a)$	133	(b,a,b)
8	$(1,a,1)\alpha$	50	$(1,1,b)\alpha$	92	$(1,b,a)\alpha$	134	$(b,a,b)\alpha$
9	$(1,a,1)\beta$	51	$(1,1,b)\beta$	93	$(1,b,a)\beta$	135	$(b,a,b)\beta$
10	$(1,a,1)\gamma$	52	$(1,1,b)\gamma$	94	$(1,b,a)\gamma$	136	$(b,a,b)\gamma$
11	$(1,a,1)\delta$	53	$(1,1,b)\delta$	95	$(1,b,a)\delta$	137	$(b,a,b)\delta$
12	$(1,a,1)\lambda$	54	$(1,1,b)\lambda$	96	$(1,b,a)\lambda$	138	$(b,a,b)\lambda$
13	$(a,1,1)$	55	$(b,1,1)$	97	$(1,a,b)$	139	(a,a,b)
14	$(a,1,1)\alpha$	56	$(b,1,1)\alpha$	98	$(1,a,b)\alpha$	140	$(a,a,b)\alpha$
15	$(a,1,1)\beta$	57	$(b,1,1)\beta$	99	$(1,a,b)\beta$	141	$(a,a,b)\beta$
16	$(a,1,1)\gamma$	58	$(b,1,1)\gamma$	100	$(1,a,b)\gamma$	142	$(a,a,b)\gamma$
17	$(a,1,1)\delta$	59	$(b,1,1)\delta$	101	$(1,a,b)\delta$	143	$(a,a,b)\delta$
18	$(a,1,1)\lambda$	60	$(b,1,1)\lambda$	102	$(1,a,b)\lambda$	144	$(a,a,b)\lambda$
19	$(a,a,1)$	61	$(1,b,1)$	103	$(a,b,1)$	145	(a,b,a)
20	$(a,a,1)\alpha$	62	$(1,b,1)\alpha$	104	$(a,b,1)\alpha$	146	$(a,b,a)\alpha$
21	$(a,a,1)\beta$	63	$(1,b,1)\beta$	105	$(a,b,1)\beta$	147	$(a,b,a)\beta$
22	$(a,a,1)\gamma$	64	$(1,b,1)\gamma$	106	$(a,b,1)\gamma$	148	$(a,b,a)\gamma$
23	$(a,a,1)\delta$	65	$(1,b,1)\delta$	107	$(a,b,1)\delta$	149	$(a,b,a)\delta$
24	$(a,a,1)\lambda$	66	$(1,b,1)\lambda$	108	$(a,b,1)\lambda$	150	$(a,b,a)\lambda$
25	$(1,1,a)$	67	$(b,1,b)$	109	$(b,1,a)$	151	(b,b,a)
26	$(1,1,a)\alpha$	68	$(b,1,b)\alpha$	110	$(b,1,a)\alpha$	152	$(b,b,a)\alpha$
27	$(1,1,a)\beta$	69	$(b,1,b)\beta$	111	$(b,1,a)\beta$	153	$(b,b,a)\beta$
28	$(1,1,a)\gamma$	70	$(b,1,b)\gamma$	112	$(b,1,a)\gamma$	154	$(b,b,a)\gamma$
29	$(1,1,a)\delta$	71	$(b,1,b)\delta$	113	$(b,1,a)\delta$	155	$(b,b,a)\delta$
30	$(1,1,a)\lambda$	72	$(b,1,b)\lambda$	114	$(b,1,a)\lambda$	156	$(b,b,a)\lambda$
31	$(a,1,a)$	73	$(b,b,1)$	115	$(a,1,b)$	157	(a,b,b)
32	$(a,1,a)\alpha$	74	$(b,b,1)\alpha$	116	$(a,1,b)\alpha$	158	$(a,b,b)\alpha$
33	$(a,1,a)\beta$	75	$(b,b,1)\beta$	117	$(a,1,b)\beta$	159	$(a,b,b)\beta$
34	$(a,1,a)\gamma$	76	$(b,b,1)\gamma$	118	$(a,1,b)\gamma$	160	$(a,b,b)\gamma$
35	$(a,1,a)\delta$	77	$(b,b,1)\delta$	119	$(a,1,b)\delta$	161	$(a,b,b)\delta$
36	$(a,1,a)\lambda$	78	$(b,b,1)\lambda$	120	$(a,1,b)\lambda$	162	$(a,b,b)\lambda$
37	(a,a,a)	79	$(1,b,b)$	121	$(b,a,1)$		
38	$(a,a,a)\alpha$	80	$(1,b,b)\alpha$	122	$(b,a,1)\alpha$		
39	$(a,a,a)\beta$	81	$(1,b,b)\beta$	123	$(b,a,1)\beta$		
40	$(a,a,a)\gamma$	82	$(1,b,b)\gamma$	124	$(b,a,1)\gamma$		
41	$(a,a,a)\delta$	83	$(1,b,b)\delta$	125	$(b,a,1)\delta$		
42	$(a,a,a)\lambda$	84	$(1,b,b)\lambda$	126	$(b,a,1)\lambda$		

Table2

To get nearer our aim we partition the entries of table 2 in five sets I, A, B, C and D, that contain the entries $1-6$, $7-48$, $49-90$, $91-126$ and $127-162$ respectively. Observe that each of these sets has its own characteristics. For example all of the coordinates of elements of I are 1, and each element of A has coordinates 1's and a's but not all 1's, and so on. For later use we also note that if $|X|$ is the cardinality of X then $|I|=6$, $|A|=|B|=42$ and $|C|=|D|=36$.
Let $A' \subset A(B' \subset B)$ be those elements in $A(B)$ that have no coordinate other than $a(b)$. Then for any $a, b \in A' \cup B' \cup D$ the group generated by the automaton $\mathbf{A(a,b,1)}$ is in fact a two generated automaton group on a two letter alphabet. Since $|A'|=|B'|=6$ and $|D|=36$ the total number of these automata is 2404.

4. Main Result

Theorem 4.1. *There are more than* 5000 *three state automata* $\mathbf{A(a,b,1)}$ *on* $S=\{0,1,2\}$ *that generate fractal groups acting spherically transitively on* T_3*.*
This result says nothing about isomorphic groups in our list of fractal groups. To obtain non-isomorphic fractal groups one can invoke the following general lemma and its corollaries.

Definition 4.2. Two automata $\mathbf{A}=(\mathbf{S},\mathbf{Q},\varphi,\psi)$ and $\mathbf{B}=(\mathbf{S},\mathbf{Q}', \varphi',\psi')$ on the same alphabet S are said to be isomorphic if there is a bijection $\Pi : Q \to Q'$ such that
$\Pi\varphi(i,q)=\varphi'(i,\Pi(q))$ and $\psi(i,q)=\psi'(i,\Pi(q))$ for all $i \in S$ and $q \in Q$.

Lemma 4.3. *Isomorphic automata on the same alphabet generate the same group of tree automorphisms.*

Proof. Let $\mathbf{A}=(\mathbf{S},\mathbf{Q},\tau,\mu)$ and $\mathbf{B}=(\mathbf{S},\mathbf{Q}',\tau',\mu')$ be isomorphic automata with isomorphism $' : Q \to Q'$, where $Q=\{q_1,\dots q_n\}$,$Q'=\{q'_1,\dots q'_n\}$ and $S=\{0,1,\dots,d-1\}$. Then to the initial automaton

$A_q = (A_{\tau(0,q)}, A_{\tau(1,q)}, \ldots, A_{\tau(d-1,q)})\mu(,q)$ in $\mathbf{A}$ corresponds the initial automaton $B_{q'} = (A_{\tau'(0,q')}, A_{\tau'(1,q')}, \ldots, A_{\tau'(d-1,q')})\mu'(,q')$, and this correspondence is a bijection. In fact we have a renaming in automata $\mathbf{A}$ to obtain automata $\mathbf{B}$. And therefore every relation in $G(\mathbf{A})$ is renamed to the same relation in $G(\mathbf{B})$. Now define $F : G(\mathbf{A}) :\to \mathbf{G}(\mathbf{B})$ by defining $F(A_q) = B_{q'}$ and extend it to a homomorphism. It is clear that F is an isomorphism of groups. □

Corollary 4.4. *If one of the two isomorphic automaton groups G and H on the same alphabet S and the same state set Q is fractal the other is also fractal.*

Corollary 4.5. *If one of the two isomorphic automaton groups G and H on the same alphabet S and the same state set Q acts transitively spherically on S^* then so does the other.*

Proof. of theorem 4.1 The following sections are devoted to the proof of this theorem. □

5. Non-Fractal Automaton Groups

According to the definition of a fractal group each automaton group is either fractal or non-fractal. In this section we formulate some statements that characterize some non-fractal three generated automaton groups on S.

Theorem 5.1. *Let $G(a, b, 1)$ be the group generated by the three state automaton $\mathbf{A}(\mathbf{a}, \mathbf{b}, \mathbf{1})$ with $a, b \in I \cup A$ or $a, b \in I \cup B$, then G is not fractal, except for the following cases*

(1) *Trivial case $a = (1, 1, 1)$ and $b = (1, 1, 1)$,*
(2) *The cyclic groups generated by $\mathbf{A}(\mathbf{x}, \mathbf{1}, \mathbf{1})$ with $x = (x, 1, 1)\sigma$, $x = (1, x, 1)\sigma, x = (1, 1, x)\sigma$, where $\sigma \in \{\delta, \lambda\}$ and $x \in \{a, b\}$,*
(3) *the group generated by automaton $\mathbf{A}(\mathbf{x}, \mathbf{x^2}, \mathbf{1})$ with x as in item 2.*

Proof. For $a, b \in I$ the group G is isomorphic to a subgroup of S_3 and so is finite. For various values of a and b we have brought these subgroups in table 3.

$*$	a_1	a_2	a_3	a_4	a_5	a_6
a_1	$\{1\}$					
a_2	$\mathbb{Z}_{\not\models}$	$\mathbb{Z}_{\not\models}$				
a_3	$\mathbb{Z}_{\not\models}$	S_3	$\mathbb{Z}_{\not\models}$			
a_4	$\mathbb{Z}_{\not\models}$	S_3	S_3	$\mathbb{Z}_{\not\models}$		
a_5	$\mathbb{Z}_{\nVdash}$	S_3	S_3	S_3	$\mathbb{Z}_{\nVdash}$	
a_6	$\mathbb{Z}_{\nVdash}$	S_3	S_3	S_3	$\mathbb{Z}_{\nVdash}$	$\mathbb{Z}_{\nVdash}$,

Table 3

where $a_1 \cdots a_6$ are elements of I. Obviously none of these is a fractal group.

Let $a, b \in I \cup A$ and $a \neq b$ and b^{-1}. Then the projection of any element of St_G in any of it's coordinates gives either 1 or some power of a. Therefore St_G is projected on 1, $\mathbb{Z}_{\gtrdot}$ for some integer m or $\mathbb{Z}$, which are not equal to G because G is not cyclic.

In case when $a = b$ or $a = b^{-1}$, G is cyclic. Let $a = (x, y, z)\sigma$ with $x, y, z \in \{1, a\}$ and without loss of generality let $\sigma = 1, \alpha$ or δ. Therefore for any choice of (x, y, z) there are three cases for a.

For $x = y = z = 1$, G is trivial, of order 2 or of order 3 according to the choices $\sigma = 1, \alpha$ or δ respectively. In these cases $St_G(1)$ is trivial. For $x = y = z = a$, G is trivial or $St_G(1)$ is trivial.

For $x = y = a$ and $z = 1$, G is trivial or of order two when $\sigma = 1$ or $\sigma = \alpha$, and when $\sigma = \delta$ we have $St_G =< a^3 >$ and so $\varphi_i(St_G(1)) =< a^2 >$ which is not equal to G.

For $x = a$ and $y = z = 1$, G is trivial for $\sigma = 1$, $\varphi_2(St_G(1)) = \{1\}$ for $\sigma = \alpha$, and finally for $\sigma = \delta$ we have $Stab_G(1) =< (a, a, a) >$ and therefore $\varphi_i(St_G(1)) = G, i = 0, 1, 2$. $\square$

Corollary 5.2. *The groups G generated by* $\mathbf{A}(\mathbf{x}, \mathbf{1}, \mathbf{1})$ *and* $\mathbf{A}(\mathbf{x}, \mathbf{x^2}, \mathbf{1})$ *with x as in item 2 of theorem 5.1 act spherically transitively on T_3.*

Corollary 5.3. *The number of three state automata* $\mathbf{A}(\mathbf{a}, \mathbf{b}, \mathbf{1})$ *with* $a, b \in I \cup A$ *or* $a, b \in I \cup B$ *that generate non-fractal groups is* 2333.

Proof. Obviously the trivial group generated by $a = (1, 1, 1)$ and $b = (1, 1, 1)$ is fractal. therefore theorem 5.1 shows that there are 7 fractal groups among the groups under consideration. Now the assertion follows since we have totally $48 \times 48 + 48 + 6 \times 6/2 + 3$ groups. □

Definition 5.4. We say that the pair $(a, b) \in B \times A$ is exceptional of first kind if

(1) $a = (1, 1, b)\alpha, b = (1, a, 1)\beta$
(2) $a = (1, 1, b)\alpha, b = (a, 1, 1)\gamma$
(3) $a = (1, b, 1)\beta, b = (1, 1, a)\alpha$
(4) $a = (1, b, 1)\beta, b = (a, 1, 1)\gamma$
(5) $a = (b, 1, 1)\gamma, b = (1, 1, a)\alpha$
(6) $a = (b, 1, 1)\gamma, b = (1, a, 1)\beta$
(7) $a = (b, b, b)\sigma, b \in A; \sigma \in S_3 \setminus \{i\}$.
(8) $b = (a, a, a)\sigma, a \in B; \sigma \in S_3 \setminus \{i\}$.

The group $G(a, b, 1)$ generated by the three state automaton $\mathbf{A}(\mathbf{a}, \mathbf{b}, \mathbf{1})$, with (a, b) an exceptional pair of first kind is called an exceptional group of first kind.

Proposition 5.5. *The first kind exceptional groups are not fractal.*

Proof. We prove the theorem in two cases. Other cases treated in the same way.
case 1. Let $G =< a, b >$, where $a = (1, 1, b)\alpha, b = (1, a, 1)\beta$. We have $a^2 = (1, 1, b^2)$ and $b^2 = (1, a^2, 1)$, which imply $a^2 = b^2 = 1$. Therefore any other relation in G will be of the form $(ab)^n = 1$, for some n a multiple of 3. Let $n = 3k$ be the smallest positive value of n satisfying this condition. Then we have $1 = (1, 1, 1) = (ab)^{3k} = ((ab)^k, (ba)^k, (ba)^k)$, which for $0 < k < n$ imply $(ab)^k = 1$. Therefore $G = \mathbb{Z}_{\not\models} * \mathbb{Z}_{\not\models}$.
Now according to $(ab)^3(ba)^3 = 1$ we have $St_G(1) =< (ab)^3 >$. From this we conclude that G is not fractal.

case 2. Assume $G = < a, b >$, where $a = (b, b, b)\alpha, b = (a, a, a)\gamma$. We have $a^2 = (b^2, b^2, b^2)$ and $b^2 = (a^2, a^2, a^2)$, which imply $a^2 = b^2$. We have

$$St_G(1) = < a^2, (ab)^3, (ba)^3 > .$$

This implies G is not fractal. □

Corollary 5.6. *There are* 416 *three state automata* $\mathbf{A}(\mathbf{a}, \mathbf{b}, \mathbf{1})$ *that generate non-fractal groups of first kind.*

We now introduce our second set of exceptional pairs.

Definition 5.7. We say that the pair $(a, b) \in I \times D \cup C \cup B \cup A \cup I$ is exceptional of second kind if

(1) $a = (1, 1, 1)\alpha, b = (x, y, z)\alpha, x, y, z \in \{1, a, b\}$
(2) $a = (1, 1, 1)\beta, b = (x, y, z)\beta, x, y, z \in \{1, a, b\}$
(3) $a = (1, 1, 1)\gamma, b = (x, y, z)\gamma, x, y, z \in \{1, a, b\}$.

The group $G(a, b, 1)$ generated by the three state automaton $\mathbf{A}(\mathbf{a}, \mathbf{b}, \mathbf{1})$, with (a, b) an exceptional pair of second kind is called an exceptional group of second kind.

Proposition 5.8. *The second kind exceptional groups are not fractal.*

Proof. We prove the theorem in two cases. Other cases treated in the same way.
case 1. Let $G(a, b, 1)$, where $a = (1, 1, 1)\alpha, b = (b, a, b)\alpha$. We have $St_G(1) = < ab, ba, b^2 >$. We observe that the projection of this stabilizer on the third coordinate is the infinite cyclic group, and so G is not fractal.
case 2. Assume $G(a, b, 1)$, where $a = (1, 1, 1)\alpha, b = (1, b, a)\alpha$. We observe that the projection of this stabilizer on the third coordinate is the cyclic group of order 2, and so G is not fractal. □

Corollary 5.9. *There are* 60 *three state automata* $\mathbf{A}(\mathbf{a}, \mathbf{b}, \mathbf{1})$ *that generate exceptional groups of second kind.*

Definition 5.10. We say that the pair $(a, b) \in A \times C$ is exceptional of third kind if

(1) $a = (1, 1, a)\alpha, b = (1, a, b)\beta$, or $b = (b, a, 1)\beta$,
(2) $a = (1, 1, a)\alpha, b = (a, 1, b)\gamma$, or $b = (a, b, 1)\gamma$,
(3) $a = (1, a, 1)\beta, b = (1, b, a)\alpha$, or $b = (b, 1, a)\alpha$,
(4) $a = (1, a, 1)\beta, b = (a, 1, b)\gamma$, or $b = (a, b, 1)\gamma$,
(5) $a = (a, 1, 1)\gamma, b = (1, b, a)\alpha$, or $b = (b, 1, a)\alpha$,
(6) $a = (a, 1, 1)\gamma, b = (1, a, b)\beta$, or $b = (b, a, 1)\beta$,
(7) $a = (a, a, a)\sigma, \sigma \in S_3 \setminus \{\delta, \lambda\}$, $b = (x, y, z)\tau \in C$ with $\tau \neq i$.

In the same way the third kind exceptional elements in $B \times C$ are defined. The group $G(a, b, 1)$ generated by the three state automaton $\mathbf{A}(\mathbf{a}, \mathbf{b}, \mathbf{1})$, with (a, b) an exceptional pair of third kind is called an exceptional group of third kind.

Proposition 5.11. *The exceptional groups of third kind are not fractal.*

Proof. We prove this proposition in cases when $a = (1, 1, a)\alpha, b = (1, a, b)\beta$ and $a = (a, a, a)\alpha, b = (a, b, 1)\beta$. in the first case we have $a^2 = 1$ and

$$St_G(1) =< b^2, ab^2a, (ab)^3, (ba)^3 > .$$

From which we observe that $\varphi_1 St_G(1) =< aba, b > \neq G$. Hence G is not fractal.
In the second case we observe that $St_G(1) =< b^2, ab^2a >$. This implies that $\varphi_2 St_G(1) =< a >$ which is isomorphic to $\mathbb{Z}_{\not{2}}$.

Corollary 5.12. *There are* 312 *three state automata* $\mathbf{A}(\mathbf{a}, \mathbf{b}, \mathbf{1})$ *that generate exceptional groups of third kind.*

□

6. Fractal Automata Groups

In this section we determine fractal groups $G(a, b, 1)$ generated by a three state automaton $\mathbf{A}(\mathbf{a}, \mathbf{b}, \mathbf{1})$, with $(a, b) \in X \times Y$, where $X, Y \in \{I, A, B, C, D\}$.

Proposition 6.1. *Let $a = (x, y, z)\sigma \in A$ and $b = (u, v, w)\tau \in B$ be such that $\sigma, \tau \in \{\delta, \lambda\}$; not all of x, y, z are a and 1; and not all of u, v, w are b and 1. Then $G(a, b, 1)$ is fractal.*

Proof. We prove the proposition in three cases, the remaining cases are quite similar.
case 1. $a = (1, 1, a)\delta$ and $b = (1, 1, b)\lambda$. We have $St_G =< ab, ba, a^3, b^3 >$ and so G is fractal.
case 2. $a = (1, 1, a)\delta$ and $b = (1, b, b)\lambda$. In this case $St_G(1) =< a^3, ab, ba, b^3 >$, which implies G is fractal.
case 3. $a = (1, a, a)\delta$ and $b = (1, b, b)\lambda$. Here we have

$$St_G(1) =< a^3, b^3, ab, ba, [a^{-1}, b], [b^{-1}, a] > .$$

From this we observe that G is fractal.□
We observe that there are 8 elements in A and 8 elements in B that satisfy the conditions of the above proposition. Therefore we have the following corollary:

Corollary 6.2. *There are 64 three state automata $\mathbf{A(a, b, 1)}$ that generate the groups in proposition above. These groups all act spherically transitively on T_3.*

Proposition 6.3. *If $(a, b) \in B \times A$ is not exceptional(of first kind) then $G(a, b)$ is fractal.*

Proof. We prove this proposition through the following two lemmas

Lemma 6.4. *Let $a = (1, 1, b)\alpha$ and $b = (1, a, 1)\gamma$. Then $G(a, b)$ is fractal.*

Proof. From the relations
$a^2 = (1, 1, b^2), b^2 = (1, a, a), b^{-2}(ab)^3 = (ab, b, a^{-1}ba), (ab)^3b^{-2} = (ab, aba^{-1}, b)$

$$a^{-1}b^2a = (a, 1, b^{-1}ab),$$

$$a^{-1}b^{-2}a(ab)^3b^{-2} = (b, aba^{-1}, b^{-1}a^{-1}b^2) \in St_G(1),$$

we deduce that G is fractal. □

Lemma 6.5. *Let* $a = (1, b, b)\alpha$ *and* $b = (a, 1, 1)\gamma$*. Then* $G(a, b, 1)$ *is fractal.*

Proof. From the relations

$$a^2 = (b, b, b^2), ababab = (bab, bab, b^2a), a^{-1}babab = (ab, ab, a),$$
$$a^{-1}babab = (b, ab, a), a^{-1}bababa^{-2} = (1, a, ab^{-2}),$$
$$b^{-1}a^2b = (a^{-1}ba, b^2, b), bababa^{-3} = (a, 1, bab^{-3}) \in St_G(1),$$

we deduce that G is fractal.

□

The above proposition provides us with a great number of three state automata that generate fractal groups acting spherically transitively on T_3. In fact by the corollary 5.7 we have $48 \times 48 - 416$ three state automata $\mathbf{A(a, b, 1)}$ generating fractal groups provided by the above proposition. We have proved:

Corollary 6.6. *There are* 1788 *three state automata* $\mathbf{A(a, b, 1)}$ *that generate fractal groups resulting from the proposition* 6.3 *all of which act spherically transitively on* T_3.

Proposition 6.7. *Let* $a = (1, 1, 1)\sigma$ *and* $b \in C$*, where* $\sigma \in \{\delta, \lambda\}$*. Then* $G(a, b, 1)$ *is fractal.*

Proof. We prove the proposition in four cases. Other cases are proved similarly. We use $a^3 = 1$ in the proof.
case 1. $a = (1, 1, 1)\delta$ and $b = (a, 1, b)$. We observe that $b, a^{-1}ba = (1, b, a), aba^{-1} = (b, 1, a) \in St_G(1)$. Therefore G is fractal.
case 2. For $a = (1, 1, 1)\delta$ and $b = (a, 1, b)\alpha$ we calculate

$$a^{-1}ba = (1, b, a)\beta, a^{-1}b^2a = (a, b2, a), ab^2a^{-1} = (b^2, a, a),$$

and

$$(ab)^2 = (b, a^2, b), (ba)^2 = (a^2, b, b)$$

and conclude that G is fractal.
case 3. $a = (1, 1, 1)\delta$ and $b = (a, 1, b)\delta$ is easily handled because of $aba = (b, a, 1)$, $a^{-1}b = (1, b, a)$ and $ba^{-1} = (a, 1, b)$.

case 4. $a = (1,1,1)\delta$ and $b = (1,a,b)\alpha$. Using the elements $b^2 = (a,a,b^2), a^{-1}b^2a = (a,b^2,a), ab^2a^{-1} = (b^2,a,a)$, $abab = (ba,1,ab)$, $baba = (1,ab,ba)$ and $a^2baba^{-1} = (ab,ba,1)$ we observe that G is fractal. □

Corollary 6.8. *The group* $G(a,b,1)$ *with* $a = (1,1,1)\delta$ *or* $a = (1,1,1)\lambda$ *and* $b \in C$ *acts spherically transitively on* T_3*.*

Proof. Fractal plus spherical action on the first level of T_3 imply spherical transitivity on T_3 [5]. □

Corollary 6.9. *The group* $G(a,b,1)$ *with* $a = (x,y,z) \in C$ *and* $b = (x,y,z)\sigma \in C$ *with* $\sigma = \delta$ *or* $\sigma = \lambda$ *is fractal and acts spherically transitively on* T_3*.*

Corollary 6.10. *The proposition* 6.7 *yields* $2 \times 36 + 12 = 84$ *three state automata* $\mathbf{A(a,b,1)}$ *generating fractal groups that act spherically transitively on* T_3*.*
Considering the generators from $B \times C (A \times C)$ we have the following two propositions

Proposition 6.11. *The group* $G(a,b,1)$ *is fractal when* $a = (t,t,t)\sigma$ *,* $b = (x,y,z)$*, where* $t \in \{a,b\}$*,* $\sigma \in \{\delta,\lambda\}$ *and one of* x,y,z *is* a *the other is* b *the third is 1.*

Proof. Without loss of generality assume $a = (a,a,a)\delta$ and $b = (a,b,1)$, we then have $a^3 = 1$ and

$$St_G(1) =< b = (a,b,1), aba^{-1} = (1,a,aba^{-1}), a^{-1}ba = (a^{-1}ba,1,a) > .$$

Which implies G is fractal. □

Corollary 6.12. *There are* 24 *three state automata* $\mathbf{A(a,b,1)}$ *resulting from the proposition* 6.11 *that generate fractal groups acting spherically transitively on* T_3.

Proposition 6.13. *Let* $G(a,b,1)$ *be a two generated group generated by* $a \in A$ *and* $b \in C$*, if the pair* (a,b) *is not exceptional(of third kind) then* G *is fractal.*

Remark. Note that $a = (x, y, z) \in A$ with $x, y, z \in \{1, a\}$ all represent the identity $i = (1, 1, 1)$ of group and therefore in this case G will not be considered as a two generated group .

Proof. Without loss of generality we will prove the proposition in 6 cases. In any case we have to show how $\varphi_i St_G(1) = G$ for $i = 0, 1, 2$.
case 1. $a = (1, 1, a)\delta, b = (1, a, b)$. We have $b, a^3 = (a, a, a)$, $a^2ba^{-2} = (1, b, 1)$ and $aba^{-1} = (b, 1, a) \in St_G(1)$. From which we conclude that G is fractal.
case 2. $a = (1, 1, a)\alpha, b = (1, a, b)\gamma$. We list some elements of $Stab_G(1)$. $a^2 = 1$, $b^2 = (1, ab, ba)$, $(ab^{-1})^3 = (b^{-1}, b^{-1}, b^{-1})$, $ab^2a^{-1} = (ab, 1, ab)$ From these we conclude that G is fractal.
case 3. $a = (1, a, 1)\alpha, b = (1, a, b)\beta$. We observe that $a^2 = (a, a, 1)$, $b^2 = (b, a^2, b)$, $(a^{-1}b)^3 = (b, b, b)$ and $ba^2b^{-1} = (1, a, bab^{-1})$ all belong to $St_G(1)$. Therefore G is fractal.
case 4. $a = (1, a, a)\alpha, b = (1, a, b)\beta$. Here are some elements of $St_G(1)$ from which we conclude that G is fractal. $a^2 = (a, a, a^2), b^2 = (b, a^2, b), (a^{-1}b)^3 = (a^{-1}b, a^{-1}b, a^{-1}b)$.
case 5. $a = (1, a, a)\alpha, b = (a, 1, b)\beta$. We have $a^2 = (a, a, a^2)$, $(a^{-1}b)^3 = (a^{-1}b, ba^{-1}, a^{-1}b)$ and $b^{-1}a^2b = (b^{-1}a^2b, a, a)$. These imply that G is fractal.
case 6. $a = (1, a, a)\alpha, b = (a, 1, b)\gamma$. The relations $a^2 = (a, a, a^2)$, $b^2 = (a^2, b, b), (a^{-1}b)^3 = (a^{-1}ba, b, a^{-1}ba)$ and $b^{-1}a^2b = (a, b^{-1}a^2b, a)$ show that G is fractal. □

Corollary 6.14. *There are* 3024 *three state automata* $\mathbf{A(a, b, 1)}$ *resulting from proposition* 6.13 *that generate fractal groups acting spherically transitively on* T_3
Now we consider the groups $G(a, b, 1)$ where $a \in I \cup A \cup B \cup C \cup D$ and $b \in D$

Proposition 6.15. *Let* $a = (1, 1, 1)\delta$ *or* $a = (1, 1, 1)\lambda$ *and* $b \in D$, *Then* $G(a, b, 1)$ *is fractal.*

Proof. We prove the proposition in four cases. The proof of remaining cases are similar.

case 1. $b = (a, b, a)$. We have $a^{-1}ba = (b, a, a)$ and $aba^{-1} = (a, a, b)$. So that $b, aba^{-1}, a^{-1}ba \in St_G(1)$. Therefore G is fractal.
case 2. $b = (a, b, a)\alpha$. In this case we have $b^2 = (ab, ba, a^2)$ and $a^3 = 1$.Therefore $b^{-2} = (b^{-1}a^{-1}, a^{-1}b^{-1}, a) \in St_G(1)$. Conjugating shows that $a^{-1}b^{-2}a = (a^{-1}b^{-1}, a, b^{-1}a^{-1})$ and , $ab^{-2}a^{-1} = (a, b^{-1}a^{-1}, a^{-1}b^{-1}) \in St_G(1)$. On the other hand we have

$$ab^2a^{-1} = (a^2, ab, ba), ab^2a^{-1}b^2 = (b, ab^2a, b),$$
$$b^2a^{-1}b^2a = (ab^2a, b, b) \in St_G(1).$$

Therefore G is fractal.
case 3. $b = (a, b, a)\delta$. We have $ba^{-1} = (a, b, a)$, $a^{-1}b = (b, a, a)$ and $aba = (a, a, b)$ and so G is fractal.
cas4. $b = (b, b, a)\alpha$. Using $a^3 = 1$,$b^2 = (b^2, b^2, a^2)$, $a^{-1}b^{-2}a$, $ab^{-2}a^{-1}$, $b^{-1}aba$ and $b^{-1}a^{-1}ba^{-1}$, we observe that G is fractal. □

Corollary 6.16. *Proposition* 6.15 *yields* 84 *three state automata* $\mathbf{A(a, b, 1)}$ *that generate fractal groups acting spherically transitively on* T_3

Proposition 6.17. *Let* $a = (x, y, z)\sigma \in A$ *where not all* x, y, z *are* 1 *and not all are a; and let* $b = (u, v, w)\tau \in D$ *with* $\sigma \in \{\delta, \lambda\}$ *and* $\tau \in \{i, \delta, \lambda$. *Then* $G(a, b, 1)$ *is fractal.*

Proof. We prove the proposition in for cases.
case 1. $a = (1, 1, a)\delta, b = (b, b, a)\lambda$. We have

$$a^3 = (a, a, a), ab = (a, b, ab), ba = (b, ba, a).$$

Therefore G is fractal.
case 2. $a = (1, 1, a)\delta, b = (a, b, a)\lambda$. We have

$$St_G(1) =< b, a^3, a^{-1}ba, aba^{-1} > .$$

This imply that G is fractal.
case 3. $a = (1, 1, a)\delta, b = (b, a, b)\lambda$. A little calculation shows that $St_G(1) =< a^3, ab, ba, b^3 >$ and so G is fractal.
case 4. $a = (1, a, a)\delta, b = (a, b, a)$. In this case we have $a^3 = (a^2, a^2, a^2), ab = (b, a^2, ab), ba = (a^2, ba, a)$. From these we deduce that G is fractal. □

In the same way one can use B instead of A in the above proposition, so that we obtain the following corollary

Corollary 6.18. *The proposition* 6.17 *yields* 432 *three state automata* $\mathbf{A(a,b,1)}$ *that generate fractal groups acting spherically transitively on* T_3

Proposition 6.19. *For* $a = (x,y,z)\sigma, b = (x,y,z)\tau \in D$*, where* $\sigma \in S_3$ *,* $\tau = \delta$ *or* $\tau = \lambda$ *and* $\tau \neq \sigma$ *the group* $G(a,b,1)$ *generated by the automaton* $\mathbf{A(a,b,1)}$ *is fractal.*

Proof. We prove the proposition in three cases. The proof of remaining cases are similar.
case 1. $a = (b,b,a)$ and $b = (b,b,a)\delta$. Let $c = a^{-1}b = (1,1,1)\delta$. We have $cac^{-1} = (a,b,b)$ and $c^{-1}ac = (b,a,b)$. Since a is also in $St_G(1)$, the proof of case 1 is complete.
case 2. $a = (b,a,a)\alpha$ and $b = (b,a,a)\delta$. Let $c = a^{-2}b^3 = (a,a,b)$ and $u = a^{-2}b^2 = (1,1,1)\lambda$, We have:

$$ucu^{-1} = (a,b,a), u^{-1}cu = (b,a,a).$$

Therefore $G(a,b,1)$ is fractal.
case 3. $a = (b,a,a)\lambda$ and $b = (b,a,a)\delta$. Let $c = b^{-1}a = (1,1,1)\delta$. We have $ca = (a,b,a)$, $c^2b = (a,a,b)$ and $ac = (b,a,a)$. Therefore G is fractal. □

Corollary 6.20. *There are more than* 55 *three state automata* $\mathbf{A(a,b,1)}$ *with* $a,b \in D$ *that generate fractal groups acting spherically transitively on* T_3

Proof. One can see easily that the automaton $\mathbf{A(a,b,1)}$ with $a = (a,a,b)$ and $b = (a,b,b)\delta$ also generate a group with above properties. □

Calculating the number of automata $\mathbf{A(a,b,1)}$ we have obtained in the course of the proof of propositions above we observe that there are more than 5000 automata that generate fractal groups acting spherically transitively on T_3. This completes the proof of the theorem 3.1.

Remark. According to propositions 5.5 and 6.19 and some calculations, except the trivial group the groups generated by automata $\mathbf{A}(\mathbf{a}, \mathbf{b}, \mathbf{1})$ with $(a, b) \in X \times Y$, where $X, Y \in \{A', B', D\}$, are not fractal if $X \neq D$ or $Y \neq D$. Therefore the only possibility that the automata $\mathbf{A}(\mathbf{a}, \mathbf{b}, \mathbf{1})$ may generate fractal groups is $(a, b) \in D \times D$. Using corollary 6.20 and it's proof one can count at least 100 of these groups.

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