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STATE SPACES OF K_0 GROUPS OF SOME RINGS

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Abstract. Let R be a ring with the Jacobson radical $J(R)$ and let $\pi\colon R \to R/J(R)$ be the canonical map. Then π induces an order preserving group homomorphism $K_0\pi\colon K_0(R) \to K_0(R/J(R))$ and an affine continuous map $S(K_0\pi)$ between the state space $St(R/J(R))$ and the state space $St(R)$. In this paper, we consider the natural affine map $S(K_0\pi)$. We give a condition under which $S(K_0\pi)$ is an affine homeomorphism. At the same time, we discuss the relationship between semilocal rings and semiperfect rings by using the affine map $S(K_0\pi)$.
Keywords: State space, K_0 group, pre-ordered group, affine map.
MSC(2010): Primary: 16E20; Secondary: 18F25, 18F30, 19A49.

1. Introduction

Throughout this paper, all rings considered are associative with identity 1 unless otherwise specified and all modules will be unitary.

A pre-ordered abelian group is a pair $(G, \leq)$ consisting of an abelian group G together with a specified translation-invariant pre-order $\leq$ on G. Denote the positive cone $\{x \in G | x \geq 0\}$ of G by G^+. An order-unit in a pre-ordered abelian group G is an element $\mu \in G^+$ such that for any $x \in G$, there is a positive integer n with $x \leq nu$. For a ring R, let $K_0(R)$ be the Grothendieck group of the category of finitely generated projective right R-modules (as defined in [11, 12, 14, 15]). Let $FP(R)$ denote the class of finitely generated projective right R-modules. Set $FP(R)/{\sim} = \{[A] | A \in FP(R)\}$, where $[A]$ denotes the equivalent class of A relative to $\sim$ (i.e., the stable isomorphism class of A). We define $K_0(R)^+ = FP(R)/{\sim}$ and use $K_0(R)^+$ to define a relation $\leq$ on $K_0(R)$, so that for $x, y \in K_0(R)$, we have $x \leq y$ if and only if $y - x \in K_0(R)^+$. Thus $\leq$ is a pre-order, called the natural pre-order on $K_0(R)$. Clearly, $[R]$ is an order-unit of $K_0(R)$ (see [8]). A state on (G, μ) is an order preserving homomorphism f from G to $\mathbb{R}$ with $f(\mu) = 1$. The state space of (G, μ) is the set $St(G, \mu)$ of all states on (G, μ). The state space $St(R)$ of a ring R is

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defined to be $St(K_0(R),[R])$. The set $St(R)$ is a compact convex set in the linear topological space of all real-valued functions on the set $K_0(R)$. Let R, H be two rings. Given any morphism $g\colon (K_0(R),[R]) \to (K_0(H),[H])$, there is a map $S(g)\colon St(K_0(H),[H]) \to St(K_0(R),[R])$ given by the rule $S(g)(f) = fg$. Clearly, $S(g)$ is affine and continuous. If $S(g)$ is a bijection, then we say that $S(g)$ is an affine homeomorphism and $St(K_0(H),[H])$ and $St(K_0(R),[R])$ are affinely homeomorphic. A complete description of $St(R)$ is given by K.R. Goodearl and R.B. Warfield in [9] for rings R in certain special classes (e.g., commutative rings, hereditary noetherian prime rings, and certain orders over Dedekind domains).

One of the most fruitful ways to study ordered groups is via the states on the group. If A is a (simple) unital C^*-algebra, the structure of the ordered group $K_0(A)$ is described by the states on $K_0(A)$ (see [4, Section 6]). In [5], B. Blackadar and M. Rϕrdam proved that every state on a subsemigroup can be extended to the whole semigroup and every state on $K_0(A)$ for a unital C^*-algebra A comes from a quasitrace. The relations between the state space of a fixed ring and the state space of a skew group ring were considered by R. Alfaro in [1]. Let R be a semilocal ring in which central idempotents can be lifted modulo the Jacobson radical $J(R)$. R. Alfaro in [2] proved that the natural map $S(K_0\pi)\colon St(R/J(R)) \to St(R)$ is an affine embedding (see [2, Theorem 4.6]). As a consequence of this theorem, Alfaro proved that if R is a semilocal noetherian ring of finite Krull dimension and the central idempotents can be lifted modulo the Jacobson radical, then $St(R)$ and $St(R/J(R))$ are affinely homeomorphic and so $St(R)$ is affinely homeomorphic to a finite dimensional simplex (see [2, Corollary 4.7]). One of our main purposes of this paper is to discuss the above problem, i.e., when the natural affine map $S(K_0\pi)$ is an affine homeomorphism from the state space $St(R/J(R))$ to the state space $St(R)$.

2. **State spaces of K_0**

To prove our main results, we need the following useful definitions and preliminary results.

A ring R is said to have the *invariant basis property* or *invariant basis number* (IBN) if every free R-module has a unique rank. A ring R is said to have *unbounded generating number* (UGN) if $R^m \oplus P \cong R^n$ implies $m \leq n$, for any R-module P and positive integers m, n. Further, R is said to be *weakly finite* if for each n, $R^n \oplus P \cong R^n$ implies $P = 0$ (see [6]). Let G be an abelian group. Then $G \otimes_{\mathbb{Z}} \mathbb{Q}$ is a $\mathbb{Q}$-vector space. Define $rank(G) = dim_{\mathbb{Q}}(G \otimes_{\mathbb{Z}} \mathbb{Q})$.

We write P^n for the direct sum of n copies of a module P. A finitely generated module P over a ring R is called *stably free* if $P \oplus R^m$ is free for a positive integer m. A module P is called the power stably free if there is a positive integer r such that P^r is stably free. We call r *the stably free order* of P if r is the least positive integer such that P^r is stably free. Then we write $s.f.o(P) = r$. If $P^r \oplus R^n \cong R^{n+k}$ for some $r, n \in \mathbb{Z}^+$ and $k \in \mathbb{Z}$, then k/r is called the *stably free rank* of P. This is denoted by $s.f.rank(P) = k/r$. Evidently, for any IBN ring this definition is well-defined and if P and Q are two power stably free modules over an IBN ring R then $s.f.rank(P \oplus Q) = s.f.rank(P) + s.f.rank(Q)$ (see [21, 16]). Given a ring R, each finitely generated projective R-module is power stably free if and only if $rank(K_0(R)) = 1$ when R has IBN, or $rank(K_0(R)) = 0$ when R does not have IBN (see [17, Lemma1.1]).

Definition 2.1. [17, Definition 2.1], [18, Definition 1.1] or [20, Definition 1.1] An ideal J of a ring R is called *indecomposable* if J is not a direct sum of two non-zero ideals of R. An ideal J is called a *block ideal* of R if J is a direct summand of R and J is indecomposable. Evidently, if J is a block ideal of a ring R, then J is an indecomposable ring.

If J is a block ideal of a ring R, then there exists a centrally primitive idempotent $e \in R$ such that $J = eR$. If J_1 and J_2 are two block ideals of R, then either $J_1 = J_2$ or $J_1 \cap J_2 = \{0\}$.

Definition 2.2. Given a ring R, set

$$BI(R) = \{J | J \text{ is a block ideal of } R \text{ and has } IBN\}.$$
$$UGN(R) = \{J | J \text{ is a block ideal of } R \text{ and has } UGN\}.$$
$$BI(R) = \{J | J \text{ is a block ideal of } R \text{ and has } IBN\}.$$
$$UGN(R) = \{J | J \text{ is a block ideal of } R \text{ and has } UGN\}.$$

We say that IBN rank of R is n and denote it by $IBN\text{-}rank(R) = n$ if $R = R_0 \oplus R_1 \oplus \cdots \oplus R_n$ in which $R_i \in BI(R)$ $(1 \leq i \leq n)$ and R_0 does not have IBN. Clearly, $IBN\text{-}rank(R) = 0$ if and only if R does not have IBN. The number of block ideals J in $UGN(R)$ is called the UGN *rank* of R and is denoted by $UGN\text{-}rank(R)$.

If $IBN\text{-}rank(R) = n < \infty$, then $R = R_0 \oplus R_1 \oplus \cdots \oplus R_n$ with that $R_1, \cdots, R_n$ are indecomposable IBN rings and R_0 doesn't have IBN. If R is a left (right) noetherian (artinian) ring, then the IBN-$rank(R) < \infty$ and $R_0 = 0$. If R is a commutative ring, then $R_0 = 0$. Clearly, $IBN\text{-}rank(R) \geq UGN\text{-}rank(R)$. If R is a commutative ring, then $IBN\text{-}rank(R) = UGN\text{-}rank(R)$.

Definition 2.3 ([18, Definition 1.3])**.** Let R be a ring and let P be a finitely generated projective R-module. We say that P is a *relative power stably free* R-module if for each $R_i \in BI(R)$, $P \otimes_R R_i$ is power stably free as an R_i-module and $P \otimes_R R_0$ is also power stably free as an R_0-module.

Clearly, each power stably free R-module is relative power stably free. If P and Q are relative power stably free, then $P \oplus Q$ is also relative power stably free. Let R be a ring with $IBN\text{-}rank(R) = n < \infty$ and let P be a relative power stable free R-module. Then $P \otimes_R R_i$ is power stably free as an R_i-module. So $(P \otimes_R R_i)^{r_i} \oplus {R_i}^{m_i} \cong {R_i}^{n_i}$ for some $r_i,\ m_i,\ n_i \in \mathbb{Z}$. In this case, we write $s.f.rank_{R_i}(P \otimes_R R_i) = \frac{n_i - m_i}{r_i}$ for $R_i \in BI(R)$. For a relative power stably free R-module P, $s.f.rank_{R_i}(P \otimes_R R_i)$ is called the *relative power stably free rank* of P *with respect to* $R_i \in BI(R)$. Clearly,

$$r_i[P \otimes_R R_i] = (n_i - m_i)[R_i]$$

in $K_0(R)$. Further, we have

$$(P \otimes_R R_i \otimes_R R_i/J(R_i))^{r_i} \oplus R_i/J(R_i)^{m_i} \cong R_i/J(R_i)^{n_i}.$$

Then

$$\begin{aligned} s.f.rank_{R_i/J(R_i)}(P \otimes_R R_i \otimes_R R_i/J(R_i)) &= \frac{n_i - m_i}{r_i} \\ &= s.f.rank_{R_i}(P \otimes_R R_i). \end{aligned}$$

Moreover, if $R_i \in UGN(R)$, then $s.f.rank_{R_i}(P \otimes_R R_i) = \frac{n_i - m_i}{r_i} \geq 0$. Let M be a finitely generated module over a semilocal ring R. Using the relative power stably free rank of M, Zhu [20] obtained some conditions under which M is a free R-module.

Lemma 2.4. [18, Lemma 1.2] *Let R be a ring. If any two of the following three conditions are satisfied, then the third one holds*:

(a) *$rank(K_0(R)) = m$,*
(b) *IBN-$rank(R) = m$,*
(c) *Each finitely generated projective R-module is relative power stably free.*

Lemma 2.5. [18, Theorem 1.1] *Given a ring R, the following statements are equivalent:*

(1) *IBN-$rank(R) = IBN$-$rank(R/J(R))$ and each finitely generated projective $R/J(R)$-module is relative power stably free.*
(2) *$rank(K_0(R)) = rank(K_0(R/J(R)))$ and each finitely generated projective R-module is relative power stably free.*

Recall that a ring R is said to be semilocal if $R/J(R)$ is a left artinian ring, or, equivalently, if $R/J(R)$ is semisimple. A ring R is said to be *semiperfect* if R is semilocal, and idempotents of $R/J(R)$ can be lifted to R. Let R be a ring and I an ideal of R. The canonical map $\pi\colon R \to R/I$ induces a natural map $K_0\pi\colon K_0(R) \to K_0(R/I)$. If $I \subseteq J(R)$, then $K_0\pi$ is an injection (see [11, 14, 15]). For a semilocal ring R, Zhu [19] gave some equivalent conditions under which $S(K_0\pi)\colon St(R/J(R)) \to St(R)$ is an affine homeomorphism.

Next, for a ring R, we consider the problem when the natural affine map $S(K_0\pi)$ is an affine homeomorphism from the state space $St(R/J(R))$ to the state space $St(R)$. We give a suitable condition under which $S(K_0\pi)$ is an affine homeomorphism. Further, for a ring R in which central idempotents can be lifted modulo the Jacobson radical $J(R)$, we show that R is semiperfect if and only if R is semilocal and for each $f \in St(R/J(R))$, $Im(f) = Im(S(K_0\pi)(f))$.

Theorem 2.6. *Given a ring R satisfying the following conditions:*

$$\begin{aligned} IBN\text{-}rank(R) &= IBN\text{-}rank(R/J(R)) = UGN\text{-}rank(R/J(R)) \\ &= rank(K_0(R/J(R))) = n < \infty, \end{aligned}$$

then $St(R/J(R))$ is affinely homeomorphic to $St(R)$.

Proof. The canonical map $\pi\colon R \to R/J(R)$ induces an order preserving group homomorphism $K_0\pi\colon K_0(R) \to K_0(R/J(R))$ and an affine continuous map $S(K_0\pi)\colon St(R/J(R)) \to St(R)$. Since

$$\begin{aligned} \text{IBN-}rank(R) &= \text{IBN-}rank(R/J(R)) \\ &= rank(K_0(R/J(R))) = n < \infty, \end{aligned} \tag{2.1}$$

we have a ring decomposition of R, $R = R_0 \oplus R_1 \oplus \cdots \oplus R_n$ in which $R_i \in BI(R)$ $(1 \le i \le n)$ and R_0 does not have IBN. Then

$$R/J(R) = R_0/J(R_0) \oplus R_1/J(R_1) \oplus \cdots R_n/J(R_n) \tag{2.2}$$

is a ring decomposition of $R/J(R)$ in which each $R_i/J(R_i)$ has IBN $(1 \le i \le n)$ and $R_0/J(R_0)$ does not have IBN. By Lemmas 2.4 and 2.5 each finitely generated projective R-module and each finitely generated projective $R/J(R)$-module are relative power stably free. Moreover, we have that IBN-$rank(R_i/J(R_i)) = rank(K_0(R_i/J(R_i))) = 1$ by (2.1) and (2.2). So each finitely generated projective $R_i/J(R_i)$-module is power stably free. Clearly, for each finitely generated projective R_i-module P_i $(1 \le i \le n)$,

$$s.f.rank_{R_i}(P_i) = s.f.rank_{R_i/J(R_i)}(P_i \otimes_{R_i} R_i/J(R_i)). \tag{2.3}$$

First, we prove that $S(K_0\pi)$ is surjective. For any $s \in St(R)$, define a map

$$\overline{s}\colon K_0(R/J(R)) \to \mathbb{R}$$

by

$$\overline{s}([\overline{P}]) = \sum_{i=1}^{n} s.f.rank_{R_i/J(R_i)}(\overline{P} \otimes_{R/J(R)} R_i/J(R_i))s[R_i]$$

and

$$\overline{s}([\overline{P}] - [\overline{Q}]) = \overline{s}([\overline{P}]) - \overline{s}([\overline{Q}]),$$

where $\overline{P}$ and $\overline{Q}$ are finitely generated projective $R/J(R)$-modules. Clearly, $\overline{s}$ is a group homomorphism. Moreover, $\overline{s}([R/J(R)] = \sum_{i=1}^{n} s([R_i]) = s([R]) = 1$

and for any $[\overline{P}] \in K_0(R/J(R))^+$, $\overline{s}[\overline{P}] \geq 0$ (note that IBN-$rank(R/J(R)) =$ UGN-$rank(R/J(R))$). Then $\overline{s} \in St(R/J(R))$. For any $[P] \in K_0(R)^+$,

$$\begin{aligned} S(K_0\pi)(\overline{s})([P]) &= \overline{s}K_0\pi([P]) = \overline{s}([P \otimes_R R/J(R)]) \\ &= \sum_{i=1}^{n} s.f.rank_{R_i/J(R_i)}([P \otimes_R R/J(R) \otimes_{R/J(R)} R_i/J(R_i)])s([R_i]) \\ &= \sum_{i=1}^{n} s.f.rank_{R_i/J(R_i)}([P \otimes_R R_i \otimes_{R_i} R_i/J(R_i)])s([R_i]) \\ &\overset{(2.3)}{=} \sum_{i=1}^{n} s.f.rank_{R_i}([P \otimes_R R_i])s([R_i]) = \sum_{i=1}^{n} s([P \otimes_R R_i]) = s([P]). \end{aligned}$$

Thus $S(K_0\pi)(\overline{s}) = s$ and so $S(K_0\pi)$ is surjective.

Next we prove that $S(K_0\pi)$ is injective. In fact, if $\overline{s}$, $\overline{t} \in St(R/J(R))$ with $S(K_0\pi)(\overline{s}) = S(K_0\pi)(\overline{t})$, then

$$\begin{aligned} \overline{s}([R_i/J(R_i)]) &= \overline{s}K_0\pi([R_i]) = S(K_0\pi)(\overline{s})([R_i]) \\ &= S(K_0\pi)(\overline{t})([R_i]) = \overline{t}K_0\pi([R_i]) = \overline{t}([R_i/J(R_i)]). \end{aligned}$$

For any $[M] \in K_0(R/J(R))^+$,

$$\begin{aligned} \overline{s}([M]) &= \sum_{i=1}^{n} s.f.rank_{R_i/J(R_i)}(M \otimes_{R/J(R)} R_i/J(R_i))\overline{s}[R_i/J(R_i)] \\ &= \sum_{i=1}^{n} s.f.rank_{R_i/J(R_i)}(M \otimes_{R/J(R)} R_i/J(R_i))\overline{t}[R_i/J(R_i)] = \overline{t}[M]. \end{aligned}$$

Thus $\overline{s} = \overline{t}$ and so $S(K_0\pi)$ is injective.

Therefore, $St(R/J(R))$ is affinely homeomorphic to $St(R)$. □

Let R be a ring. An element $e \in R$ is called *central* if and only if $eRf = fRe = 0$, where $f = 1 - e$ is the complementary idempotent of e. Further, e is called *centrally primitive* in R if $e \neq 0$ and e cannot be written as a sum of two nonzero orthogonal central idempotents in R. By [10, Section 22], a ring R can be expressed as a finite direct product of indecomposable rings if $R = e_1R \oplus \cdots \oplus e_rR$ where each e_i $(1 \leq i \leq r)$ is a centrally primitive idempotent of R. We call this finite direct product a finite block decomposition of R, and each e_iR a *block* of R. If I is an ideal of R, we say that a central idempotent $\overline{g} \in R/I$ *can be lifted* to R if there exists a central idempotent $e \in R$ whose image under the natural map $R \to R/I$ is $\overline{g}$. We say that the central idempotents *can be lifted modulo* I, if every central idempotent in R/I can be lifted to a central idempotent in R (see [10]).

Lemma 2.7. *Let R be a ring with IBN-rank$(R/J(R)) < \infty$. If the central idempotents can be lifted modulo the Jacobson radical $J(R)$, then IBN-rank$(R) =$ IBN-rank$(R/J(R))$.*

Proof. Assume that IBN-$rank(R) = n < \infty$ and $R = R_0 \oplus R_1 \oplus \cdots \oplus R_n$ in which $R_i \in BI(R)$ $(1 \le i \le n)$ and R_0 does not have IBN. Then $R/J(R) = R_0/J(R_0) \oplus R_1/J(R_1) \oplus \cdots R_n/J(R_n)$. Clearly, each $R_i/J(R_i)$ has IBN $(1 \le i \le n)$ and $R_0/J(R_0)$ does not have IBN. Since central idempotents can be lifted modulo the Jacobson radical $J(R)$, each $R_i/J(R_i)$ $(1 \le i \le n)$ is an indecomposable ideal of $R/J(R)$. In fact, if there is some $R_i/J(R_i)$ $(i \ne 0)$ which is not indecomposable, then there is a non-trivial central idempotent $u_i \in R_i/J(R_i)$. By the assumption, there exists a non-trivial central idempotent $e_i \in R_i$ such that $\overline{e_i} = u_i$, i.e., R_i can be decomposed a direct sum of two non-zero ideals of R. This is a contradiction with $R_i \in BI(R)$ $(i \ne 0)$. So $R_i/J(R_i) \in BI(R/J(R))$ $(1 \le i \le n)$. Thus IBN-$rank(R/J(R)) = n$. □

Proposition 2.8. *Let R be a semilocal ring. Assume that the central idempotents can be lifted modulo the Jacobson radical $J(R)$. Then $St(R/J(R))$ is affinely homeomorphic to $St(R)$. Hence $St(R)$ is affinely homeomorphic to a finite dimensional simplex.*

Proof. By the Wedderburn-Artin theorem,

$$R/J(R) \cong \mathbb{M}_{t_1}(D_1) \times \cdots \times \mathbb{M}_{t_n}(D_n)$$

for suitable division rings $D_1, \cdots, D_n$ and positive integers $t_1, \cdots, t_n$. Then

$$\text{IBN-}rank(R/J(R)) = \text{UGN-}rank(R/J(R)) = rank(K_0(R/J(R))) = n.$$

By Lemma 2.7, IBN-$rank(R) =$ IBN-$rank(R/J(R))$. Using Theorem 2.6, we have that $St(R/J(R))$ is affinely homeomorphic to $St(R)$. Furthermore, since $R/J(R)$ is semisimple artinian, $St(R/J(R))$ is affinely homeomorphic to a finite dimensional simplex (see [9]). □

Theorem 2.9. *Let R be a ring in which the central idempotents can be lifted modulo the Jacobson radical $J(R)$. Then R is semiperfect if and only if R is semilocal and for each $f \in St(R/J(R))$, $Im(f) = Im(S(K_0\pi)(f))$.*

Proof. "$\Rightarrow$" By [3, Corollary 3.7] and the definition of the semiperfect ring.

"$\Leftarrow$" Since R is a semilocal ring, $K_0\pi\colon K_0(R) \to K_0(R/J(R))$ is injective (see [11, 14, 15]). So it is sufficient to prove that $K_0\pi$ is surjective by [3, Corollary 3.7]. Lemma 2.7 gives that IBN-$rank(R) =$ IBN-$rank(R/J(R))$. Assume that IBN-$rank(R) = n < \infty$ and $R = R_1 \oplus \cdots \oplus R_n$ in which $R_i \in BI(R)$ $(1 \le i \le n)$. Then $R/J(R) = R_1/J(R_1) \oplus \cdots R_n/J(R_n)$. Clearly, $R_i/J(R_i) \in BI(R/J(R))$ $(1 \le i \le n)$. In fact, $R_i/J(R_i) \in UGN(R/J(R))$ $(1 \le i \le n)$. Define

$$f_i\colon K_0(R/J(R)) \to \mathbb{R} \text{ by } f_i([R_j/J(R_j)]) = \begin{cases} 1, & i = j \\ 0, & i \ne j \end{cases}$$

and

$$\begin{aligned} f_i([\overline{P}]-[\overline{Q}]) &= \sum_{j=1}^{n} s.f.rank_{R_j/J(R_j)}(\overline{P}\otimes_{R/J(R)} R_j/J(R_j))f_i([R_j/J(R_j)]) \\ &- \sum_{j=1}^{n} s.f.rank_{R_j/J(R_j)}(\overline{Q}\otimes_{R/J(R)} R_j/J(R_j))f_i([R_j/J(R_j)]) \\ &= s.f.rank_{R_i/J(R_i)}(\overline{P}\otimes_{R/J(R)} R_i/J(R_i)) \\ &- s.f.rank_{R_i/J(R_i)}(\overline{Q}\otimes_{R/J(R)} R_i/J(R_i)) \end{aligned}$$

for all $[\overline{P}],\ [\overline{Q}] \in K_0(R/J(R))^+$. It is not difficult to see that $f_i \in St(R/J(R))$ for all $1 \le i \le n$. Since $Im(f_i) = Im(S(K_0\pi)(f_i))$, for each $[\overline{P}] \in K_0(R/J(R))^+$ there is $x_i = [P_i] - t_i[R] \in K_0(R)$ such that

$$\begin{aligned} f_i([\overline{P}]) &= S(K_0\pi)(f_i)(x_i) \\ &= f_iK_0\pi(x_i) = f_i([P_i\otimes_R R/J(R)]) - t_i[R/J(R)]) \\ &= f_i(\sum_{j=1}^{n}[P_i\otimes_R R_j/J(R_j)] - t_i\sum_{j=1}^{n}[R_j/J(R_j)]) \\ &= f_i(\sum_{j=1}^{n}[P_i\otimes_R R_j\otimes_{R_j} R_j/J(R_j)] - t_i\sum_{j=1}^{n}[R_j/J(R_j)]) \\ &= s.f.rank_{R_i/J(R_i)}(P_i\otimes_R R_i\otimes_{R_i} R_i/J(R_i)) - t_i \\ &= f_iK_0\pi([P_i\otimes_R R_i] - t_i[R_i]). \end{aligned}$$

Let $y = \sum\limits_{j=1}^{n}([P_j\otimes_R R_j] - t_j[R_j])$. Then

$$\begin{aligned} f_iK_0\pi(y) &= f_i(\sum_{j=1}^{n}([P_j\otimes_R R_j\otimes_R R/J(R)] - t_j[R_j\otimes_R R/J(R)])) \\ &= f_i(\sum_{j=1}^{n}([P_j\otimes_R R_j\otimes_{R_j} R_j/J(R_j)] - t_j[R_j\otimes_{R_j} R_j/J(R_j)]) \\ &= s.f.rank_{R_i/J(R_i)}(P_i\otimes_R R_i\otimes_{R_i} R_i/J(R_i)) - t_i = f_i([\overline{P}]). \end{aligned}$$

So $w = [\overline{P}] - K_0\pi(y) \in \cap_{i=1}^{n} ker f_i$. On the other hand, there are $q \in \mathbb{N}$ and $q_1\ \ \cdots,\ q_n \in \mathbb{Z}$ such that $qw = q_1[R_1/J(R_1)] + \cdots + q_n[R_n/J(R_n)]$. Hence $0 = f_i(w) = \frac{q_i}{q}$. Then $q_i = 0$ for $1 \le i \le n$, i.e., $qw = 0$. So $w = 0$ since $Tor(K_0(R/J(R))) = 0$ (note that $K_0(R/J(R)) \cong \mathbb{Z}^n$). Thus $[\overline{P}] = K_0\pi(y)$ and so $K_0\pi$ is surjective. Therefore, $K_0\pi$ is an isomorphism. Using [3, Corollary 3.7] again, we have that R is semiperfect. □

Corollary 2.10. *Let R be a semilocal ring and $Im(f) = Im(S(K_0\pi)(f))$ for each $f \in St(R/J(R))$. If the central idempotents can be lifted modulo Jacobson radical $J(R)$, then the idempotents can be lifted modulo Jacobson radical $J(R)$.*

Proof. This follows by Theorem 2.9 and the definition of the semeperfect ring. □

An element a in a ring R is called a *left morphic* if $R/Ra \cong l(a)$ where $l(a)$ denotes the left annihilator of a in R. The ring itself is called a left morphic ring if every element is left morphic. Nicholson and Sánchez characterized the left morphic rings in [13]. Zhu proved that if R is a semiperfect and left morphic ring, then the central idempotents in $R/J(R)$ can be lifted to the central idempotents in R (see [18, Corollary 2.2]). Using this result, Zhu gave a simple proof of [13, Theorem 29]. Next we give a generalization of [18, Corollary 2.2].

Corollary 2.11. *Let R be a left morphic ring. Then R is a semiperfect ring if and only if R is a semilocal ring in which the central idempotents can be lifted modulo Jacobson radical $J(R)$ and $Im(f) = Im(S(K_0\pi)(f))$ for each $f \in St(R/J(R))$.*

Proof. This follows by Theorem 2.9 and [18, Lemma 2.1]. □

Remark 2.12. (1) The converse of Corollary 2.10, in general, is false. For instance, in the ring R of $n \times n$ upper triangular matrices over a field k, the quotient

$$R/J(R) \cong k \times \cdots \times k \text{ (n copies)}$$

has 2^n central idempotents. However, there are no nontrivial central idempotents in R. Thus the nontrivial central idempotents in $R/J(R)$ cannot be lifted to central idempotents in R. But idempotents in $R/J(R)$ can be lifted to R since $J(R)$ is nilpotent.

(2) The condition "$Im(f) = Im(S(K_0\pi)(f))$ for each $f \in St(R/J(R))$" in Theorem 2.9 is not redundant. For example, consider the ordered group with order-unit $(\mathbb{Z}, m)$ where m is a positive integer greater than one. For the subgroup $m\mathbb{Z}$ of $\mathbb{Z}$, we have an embedding of ordered groups with order-unit $(m\mathbb{Z}, m) \subset (\mathbb{Z}, m)$. By [7, Theorem 6.3], there is a semilocal ring R such that the embedding $K_0\pi\colon K_0(R) \hookrightarrow K_0(R/J(R))$ coincides with the above embedding. Since $K_0\pi$ is not an isomorphism, the ring R is not semiperfect. On the other hand, $St(R/J(R))$ is affinely homeomorphic to $St(R)$ and the central idempotents in $R/J(R)$ can be lifted to central idempotents in R. But there is $f \in St(R/J(R))$ such that $Im(f) \neq Im(S(K_0\pi)(f))$.

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