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ON THE EXISTENCE OF HILBERT VALUED PERIODICALLY CORRELATED AUTOREGRESSIVE PROCESSES

N. MOHAMMADI JOUZDANI, S. MAHMOODI* AND A. PARVARDEH

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Abstract. In this paper we provide sufficient condition for existence of a unique Hilbert valued ($\mathbb{H}$-valued) periodically correlated solution to the first order autoregressive model $X_n = \rho_n X_{n-1} + Z_n$, for $n \in \mathbb{Z}$, and formulate the existing solution and its autocovariance operator. Also we specially investigate equivalent condition for the coordinate process $\langle X_n, v\rangle$, for arbitrary element v in $\mathbb{H}$, to satisfy in some autoregressive model. Finally, we extend our result to the autoregressive process with finite order.
Keywords: Second order process, autoregressive process, periodically correlated process, Hilbert valued process, linear operator.
MSC(2010): Primary: 60G12; Secondary: 62M10, 60B11.

1. Introduction

Periodically correlated (PC) processes have been introduced first by Gladyshev in [2]. Later, different authors, [4, 5, 7, 8] , studied univariate, multivariate and Hilbert valued PC processes. In this paper we mainly consider Hilbert valued ($\mathbb{H}$-valued) periodically correlated (PC) processes, and provide sufficient condition for the existence of a Hilbertian autoregressive process of finite order. The causal representation as well as the uniqueness of the solution is considered. Special case, when all operators belong to the space of compact operators with common set of eigenvalues, say $\{e_j\}$ is investigated. Moreover, we study the coordinate process $\{\langle X_n, v\rangle , n \in \mathbb{Z}\}$, for arbitrary element v in $\mathbb{H}$, and present equivalent condition for this coordinate process to satisfy in some univariate autoregressive model.

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This paper generalizes the result of Chapters 3 and 5 in [1] which defines and establishes some properties of stationary Hilbert valued autoregressive processes of finite order. Parvardeh et.al in [6] study Banach valued strictly periodically correlated processes satisfying in some AR(1) equation with i.i.d whit noise without assuming moment conditions. Here we focus on Hilbert valued processes potentially with paradigm $L^2([0,1])$ which is more practical. Our study is concerned with the strongly second order processes which admit covariance operators. In comparison to [7], our method is direct, without associating the higher dimension stationary counterpart. More important, our existing condition is weaker than what is stated in [7].

All mentioned results are provided in Sections 3 and 4 of this paper.

2. **Hilbert-valued PCAR(1) sequences**

Let $\mathbb{H}$ be a separable real Hilbert space, with the inner product and the corresponding norm $< \cdot, \cdot >$ and $\| \cdot \|$, respectively. Notation $L(\mathbb{H})$ is used to show the Banach space of bounded linear operators on $\mathbb{H}$. The space of Hilbert-Schmidt operators and the nuclear operators on $\mathbb{H}$ are denoted by $\mathcal{S}(\mathbb{H})$ and $\mathcal{N}(\mathbb{H})$, respectively. We use subscripts L, $\mathcal{S}$ and $\mathcal{N}$ to distinguish the corresponding norms and inner products (if it is the case). The subscript L is suppressed when there is no ambiguity. Let $L^2_{\mathbb{H}}(\Omega, \mathcal{F}, P)$ denote the Hilbert space of all zero mean, square integrable $\mathbb{H}$-valued random elements defined on some probability space $(\Omega, \mathcal{F}, P)$. In particular, $L^2_{\mathbb{R}}(\Omega, \mathcal{F}, P)$ or simply $L^2(\Omega, \mathcal{F}, P)$ indicates the Hilbert space of all zero mean, square integrable real valued random variables on $(\Omega, \mathcal{F}, P)$. We use notation π_1 to indicate projection on the first component of $\mathbb{H}^p$ elements, the p-fold Cartesian product of $\mathbb{H}$, i.e.,

$$\begin{aligned} \pi_1 &: \mathbb{H}^p \longrightarrow \mathbb{H} \\ \pi_1 &: (h_1, h_2, \ldots, h_p) \longrightarrow h_1. \end{aligned}$$

An $\mathbb{H}$-valued discrete time stochastic process $X = \{X_n : n \in \mathbb{Z}\}$ is called weakly second order if $\langle X_n, x\rangle \in L^2(\Omega, \mathcal{F}, P)$, for every $x \in \mathbb{H}$ and $n \in \mathbb{Z}$. It is called strongly second order if $\|X_n\| \in L^2(\Omega, \mathcal{F}, P)$, for all $n \in \mathbb{Z}$. Henceforth, we assume the processes are strongly second order.

For strongly second order $\mathbb{H}$-valued random elements X and Y, the variance operator and the covariance operator are respectively defined by

$$\begin{aligned} C_X(x) &= E\left(\langle x, X\rangle X\right), \qquad x \in \mathbb{H} \\ C_{X,Y}(x) &= E\left(\langle x, Y\rangle X\right), \qquad x \in \mathbb{H}. \end{aligned}$$

We note that C_X belongs to $\mathcal{N}(\mathbb{H})$.

Definition 2.1. Let $X = \{X_n\}_{n\in\mathbb{Z}}$ be a strongly second order $\mathbb{H}$-valued stochastic process defined on some probability space. We say X is a weakly

periodically correlated process if there is a positive integer T for which we have

$$\begin{aligned} E(X_n) &= E(X_{n+T}), \quad \forall n \in \mathbb{Z} \\ C_{X_n,X_m} &= C_{X_{n+T},X_{m+T}}, \quad \forall n,m \in \mathbb{Z}, \end{aligned}$$

where $C_{.,.}$ stands for the covariance operator of the process. The smallest such T is called the period of the process and the process is called periodically correlated with period T or T-PC for short. If $T = 1$, then the process is called stationary.

Definition 2.2. Let $X = \{X_n\}_{n\in\mathbb{Z}}$ be an $\mathbb{H}$-valued periodically correlated process with period T. Then X is said to be autoregressive of order one (T-PCAR(1)) if it satisfies

$$X_n = \rho_n X_{n-1} + Z_n, \quad n \in \mathbb{Z}, \tag{2.1}$$

where $Z = \{Z_n\}_{n\in\mathbb{Z}}$ is a white noise process with $\sigma^2 = E\,\|Z_n\|^2_{\mathbb{H}}$, for each n and $\{\rho_n\}_{n\in\mathbb{Z}}$ is a nonzero T-periodic sequence of bounded operators on $\mathbb{H}$, i.e., for all $n \in \mathbb{Z}$, $\rho_n = \rho_{n+T}$.

Let us introduce the following notations.

$$A^s_r := \rho_s \cdots \rho_{r+1}\rho_r, \quad r \le s,$$

with the convention that $A^s_r = I$, if $r > s$. Moreover, let

$$B = A^{T-1}_0 = \rho_{T-1} \cdots \rho_0.$$

It is plain to arrive at the following so called autoregressive moving average, ARMA, modelling for X using the AR(1) equation. We heavily use it in this article.

When $X = \{X_n\}_{n\in\mathbb{Z}}$ is a T-PCAR(1) satisfying recursive formula (2.1), then for all $n \in \mathbb{Z}$, $m \ge 1$, and $q = 0, \ldots, T-1$, the process X satisfies

$$X_n - \sum_{k=0}^{mT+q-1} A^n_{n+1-k} Z_{n-k} = A^n_{n-mT-q+1} X_{n-mT-q}. \tag{2.2}$$

Moreover, for $n = lT + k$, $k = 0, \ldots, T-1$ we have

$$\begin{aligned} A^n_{n-mT-q+1} &= A^{lT+k}_{lT+k-mT-q+1} \\ &= \rho_k \ldots \rho_1\rho_0 B^{m'} \rho_{T-1} \cdots \rho_{T-(q-k)_T}, \end{aligned} \tag{2.3}$$

where

$$\begin{aligned} m' &= \left[\frac{mT+q-k-1}{T}\right] = \begin{cases} m & q > k \\ m-1 & q \le k \end{cases} \\ (q-k)_T &= \begin{cases} q-k-1 & q > k \\ T+q-k-1 & q \le k \end{cases}. \end{aligned}$$

Therefore,

(2.4)

$$A^n_{n-mT-q+1} = \rho_{n-\left[\frac{n}{T}\right]T} \cdots \rho_0 B^{\left[\frac{mT+q-n+\left[\frac{n}{T}\right]T-1}{T}\right]} \rho_{T-1} \cdots \rho_{T-(q-n+\left[\frac{n}{T}\right]T)_T}$$

Now let us introduce the following conditions.

Condition (I) There exist constants $a \in (0,\infty)$ and $b \in (0,1)$ such that bounded linear operator B satisfies,

$$\|B^m\| \leq ab^m \qquad m \geq 1.$$

This condition is corresponding to the condition (c_1) (or (c_0)) in [1], page 74, and then it is equivalent to the following condition.

Condition (II) There is a positive integer k such that $\|B^k\| < 1$.

3. Main result

The following theorem is derived by [7]. Their proof very much relies on associating the $T - PCAR(1)$ in $\mathbb{H}$ to a class of $AR(1)$ processes in $\mathbb{H}^T$. The proof that we present is direct and rather transparent.

Theorem 3.1. *Under Condition (I), AR(1) equation (2.1) has a unique periodically correlated solution $X = \{X_n\}_{n\in\mathbb{Z}}$ in the form*

$$X_n = \sum_{k=0}^{\infty} A^n_{n+1-k} Z_{n-k}, \qquad n \in \mathbb{Z}, \tag{3.1}$$

where the series converges in $L^2_{\mathbb{H}}(\Omega,\mathcal{F},P)$ and almost surely. Moreover, Z is the innovation process of $X = \{X_n\}_{n\in\mathbb{Z}}$. (cf. [1], page 73)

Remark 3.2. Note that the existing condition in [7] that is Assumption A_1, $\sum_{i=0}^{T-1} \|\rho_i\|^{k_i} < 1$, for some positive integers k_i, equivalent to $\|\rho_i\| < 1$, for each $i = 0,\ldots,T-1$, is stronger than the existing Condition (I) or equivalently Condition (II) assumed in Theorem 3.1.

Proof. To see the convergence of the series appearing in (3.1) it is enough to show that the sequence $\left\{\sum_{k=0}^{j} A^n_{n+1-k} Z_{n-k}, \quad j \in \mathbb{Z}\right\}$, for each n, forms a Cauchy sequence in the Hilbert space $L^2_{\mathbb{H}}(\Omega,\mathcal{F},P)$. Observe that by the orthogonality of Z_n's we have

$$\begin{aligned} \left\| \sum_{k=m}^{j} A^n_{n+1-k} Z_{n-k} \right\|^2_{L^2_{\mathbb{H}}(\Omega,\mathcal{F},P)} &= \sum_{k=m}^{j} \left\| A^n_{n+1-k} Z_{n-k} \right\|^2_{L^2_{\mathbb{H}}(\Omega,\mathcal{F},P)} \\ &\leq \sigma^2 \sum_{k=m}^{j} \left\| A^n_{n+1-k} \right\|^2. \end{aligned} \tag{3.2}$$

For big enough m and j, (3.2) equals

$$
\begin{aligned}
& \sigma^2 \sum_{l=1}^{\infty} \sum_{k=lT}^{lT+T-1} \left\| A_{n+1-k}^{n} \right\|^2 I(m \leq k \leq j) \\
= \quad & \sigma^2 \sum_{l=1}^{\infty} \sum_{q=0}^{T-1} \left\| A_{n+1-lT-q}^{n} \right\|^2 I(m \leq q+lT \leq j) \\
= \quad & \sigma^2 \sum_{l=1}^{\infty} \sum_{q=0}^{T-1} \left\| A_0^{n-\left[\frac{n}{T}\right]T} B^{\left[\frac{lT+q-(n-\left[\frac{n}{T}\right]T)-1}{T}\right]} A_{T-(q-(n-\left[\frac{n}{T}\right]T))_T}^{T-1} \right\|^2 \\
\times \quad & I(m \leq q+lT \leq j).
\end{aligned}
$$

Since $(L(\mathbb{H}), \|.\|_L)$ forms a Banach Algebra, we get the upper bound

$$
\begin{aligned}
& \sigma^2 \sum_{l=1}^{\infty} \sum_{q=0}^{T-1} \left\| A_0^{n-\left[\frac{n}{T}\right]T} B^{\left[\frac{lT+q-(n-\left[\frac{n}{T}\right]T)-1}{T}\right]} A_{T-(q-(n-\left[\frac{n}{T}\right]T))_T}^{T-1} \right\|^2 \\
\times \quad & I(m \leq q+lT \leq j) \\
\leq \quad & \sigma^2 \sum_{l=1}^{\infty} \sum_{q=0}^{T-1} \left\| A_0^{n-\left[\frac{n}{T}\right]T} \right\|^2 \left\| B^{\left[\frac{lT+q-(n-\left[\frac{n}{T}\right]T)-1}{T}\right]} \right\|^2 \left\| A_{T-(q-(n-\left[\frac{n}{T}\right]T))_T}^{T-1} \right\|^2 \\
\times \quad & I(m \leq q+lT \leq j) \\
\leq \quad & \sigma^2 \sum_{l=1}^{\infty} \sum_{q=0}^{T-1} M^2 \left\| B^{\left[\frac{lT+q-(n-\left[\frac{n}{T}\right]T)-1}{T}\right]} \right\|^2 M^2 I(m \leq q+lT \leq j) \\
= \quad & M^4 \sigma^2 \sum_{l=1}^{\infty} \sum_{q=0}^{T-1} \left\| B^{\left[\frac{lT+q-(n-\left[\frac{n}{T}\right]T)-1}{T}\right]} \right\|^2 I(m \leq q+lT \leq j) \\
\leq \quad & M^4 \sigma^2 \sum_{l=1}^{\infty} \sum_{q=0}^{T-1} \left(\left\| B^{l-1} \right\|^2 + \left\| B^{l} \right\|^2 \right) I(m \leq q+lT \leq j) \\
= \quad & M^4 \sigma^2 \sum_{l=1}^{\infty} \sum_{k=lT}^{lT+T-1} \left(\left\| B^{l-1} \right\|^2 + \left\| B^{l} \right\|^2 \right) I(m \leq k \leq j) \\
\leq \quad & M^4 T \sigma^2 \sum_{k=\left[\frac{m}{T}\right]}^{\left[\frac{j}{T}\right]} \left(\left\| B^{k-1} \right\|^2 + \left\| B^{k} \right\|^2 \right),
\end{aligned}
$$

where

$$M := \max\left\{\|A_r^s\|\,; r = 0,\dots,T-1, s = 0,\dots,T-1, 0 \le s-r \le T-1\right\}.$$

Using (2.3) leads to the upper bound. And then we have

$$\le \quad c_0 a^2 M^4 T\sigma^2 \sum_{k=m}^{j}\left(b^{\frac{2(k-1)}{T}} + b^{\frac{2k}{T}}\right) \to 0, \quad \text{as } m, j \to \infty,$$

for some $c_0 > 0$. Thus it follows from Cauchy's criterion that the series $\sum_{k=0}^{\infty} A_{n+1-k}^n Z_{n-k}$ converges in $L^2_{\mathbb{H}}(\Omega, \mathcal{F}, P)$. To prove the almost sure convergence of the series (3.1) it is enough to show its mean square is finite. In more detail,

$$\begin{aligned} & E\left(\left\|\sum_{k=T}^{\infty} A_{n+1-k}^n Z_{n-k}\right\|\right)^2 \\ \le\ & E\left(\sum_{k=T}^{\infty}\|A_{n+1-k}^n\|\,\|Z_{n-k}\|\right)^2 \\ =\ & \sum_{k=T}^{\infty}\sum_{l=T}^{\infty} E\,\|Z_{n-k}\|\,\|Z_{n-l}\|\,\|A_{n+1-k}^n\|\,\|A_{n+1-l}^n\|. \end{aligned}$$

Using Cauchy Schwart's inequality leads to the upper bound

$$\begin{aligned} & \sigma^2 \sum_{k=T}^{\infty}\sum_{l=T}^{\infty}\|A_{n+1-k}^n\|\,\|A_{n+1-l}^n\| \\ =\ & \sigma^2\left(\sum_{k=T}^{\infty}\|A_{n+1-k}^n\|\right)^2 \\ =\ & \sigma^2\left(\sum_{j=1}^{\infty}\sum_{q=0}^{T-1}\left\|A_{n+1-jT-q}^n\right\|\right)^2 \\ =\ & \sigma^2\left(\sum_{j=1}^{\infty}\sum_{q=0}^{T-1}\left\|A_0^{n-\left[\frac{n}{T}\right]T} B^{\left[\frac{jT+q-(n-\left[\frac{n}{T}\right]T)-1}{T}\right]} A_{T-(q-(n-\left[\frac{n}{T}\right]T))_T}^{T-1}\right\|\right)^2 \\ \le\ & \sigma^2\left(\sum_{j=1}^{\infty}\sum_{q=0}^{T-1}\left(\left\|B^j\right\| + \left\|B^{j-1}\right\|\right)\left\|A_0^{n-\left[\frac{n}{T}\right]T}\right\|\left\|A_{T-(q-(n-\left[\frac{n}{T}\right]T))_T}^{T-1}\right\|\right)^2 \\ =\ & \sigma^2\left(\sum_{q=0}^{T-1}\left\|A_0^{n-\left[\frac{n}{T}\right]T}\right\|\left\|A_{T-(q-(n-\left[\frac{n}{T}\right]T))_T}^{T-1}\right\|\right)^2\left(\sum_{j=1}^{\infty}\left(\left\|B^j\right\| + \left\|B^{j-1}\right\|\right)\right)^2 \end{aligned}$$

which is bounded by finite series

$$\sigma^2 a^2 M^4 T^2 \sum_{j=1}^{\infty} \left(b^j + b^{j-1}\right).$$

Therefore, $\sum_{k=0}^{\infty} A^n_{n+1-k} Z_{n-k}$ converges almost surely and the periodically correlated process

$$Y_n := \sum_{k=0}^{\infty} A^n_{n+1-k} Z_{n-k}$$

satisfies the AR(1) equation (2.1).

It remains to show the uniqueness of the solution. Indeed, if $X = \{X_n\}_{n\in\mathbb{Z}}$ is any arbitrary solution to the AR(1) equation, then by the equality

$$X_n - \sum_{k=0}^{mT+q-1} A^n_{n+1-k} Z_{n-k} = A^n_{n-mT-q+1} X_{n-mT-q}, \qquad m > 0,$$

we obtain

$$\begin{aligned}
& E\left[\left\| X_n - \sum_{k=0}^{mT+q-1} A^n_{n+1-k} Z_{n-k} \right\|^2\right] \\
= \;& E\left[\left\| A^n_{n-mT-q+1} X_{n-mT-q} \right\|^2\right] \\
\leq \;& \left\| A^n_{n-mT-q+1} \right\|_L^2 E\left[\left\| X_{n-mT-q} \right\|^2\right] \\
= \;& \left\| A_0^{n-\left[\frac{n}{T}\right]T} B^{\left[\frac{mT+q-(n-\left[\frac{n}{T}\right]T)-1}{T}\right]} A^{T-1}_{T-(q-(n-\left[\frac{n}{T}\right]T))_T} \right\|^2 E\left[\left\| X_{n-q} \right\|^2\right] \\
\leq \;& \left(\left\| A_0^{n-\left[\frac{n}{T}\right]T} \right\|^2 \left\| A^{T-1}_{T-(q-(n-\left[\frac{n}{T}\right]T))_T} \right\|^2 E\left[\left\| X_{n-q} \right\|^2\right] \right) \\
\times \;& \left(\|B^m\|^2 + \left\|B^{m-1}\right\|^2 \right) \\
\leq \;& \left(M^4 E\left[\left\| X_{n-q} \right\|^2\right] \right) \left(\|B^m\|^2 + \left\|B^{m-1}\right\|^2 \right)
\end{aligned}$$

which tends to zero by Condition (I) as m tends to infinity. That completes the proof. □

Remark 3.3. We can replace the assumption of being second order by a weaker assumption, and draw a similar conclusion. Indeed, if random elements belong to L^p-space, for $1 \leq p < 2$, modifying the proof leads to the convergence of the series in L^p. Obviously the limiting random elements satisfy the AR(1) equation, however in this case there is no certainty of existing the covariance operators.

Since definition of the PC processes is based on autocorrelation functions, we review the covariance operator of a $T-PCAR(1)$ process in the following theorem and show it satisfies in a specific recursive functional equation.

Theorem 3.4. *Assume that Condition* (I) *holds and* $X=\{X_n; n\in\mathbb{Z}\}$ *is the unique solution to the* $AR(1)$ *equation, and* $C_n(\cdots)$ *and* $C_Z(\cdots)$ *are the variance operator of* X_n *and* Z_0 *respectively. Then*

$$C_n=\rho_n C_{n-1}\rho_n^*+C_Z \tag{3.3}$$

$$C_n=\sum_{k=0}^{\infty}A_{n-k+1}^n C_Z\left(A_{n-k+1}^n\right)^*, \tag{3.4}$$

in which the convergence is with respect to the nuclear norm $\|.\|_{\mathcal{N}}$. *Furthermore,*

$$C_{X_n,X_m}=A_{m+1}^n C_{X_m,X_m}, \qquad n>m \tag{3.5}$$

$$C_{X_m,X_n}=C_{X_m,X_m}\left(A_{m+1}^n\right)^*, \qquad n>m. \tag{3.6}$$

Proof. Note that for each h in $\mathbb{H}$, we have

$$\begin{aligned}
\langle C_n(h),h\rangle &= E\left[\langle h,X_n\rangle\langle X_n,h\rangle\right]\\
&= E\left[\langle \rho_n X_{n-1}+Z_n,h\rangle^2\right]\\
&= E\left[\langle \rho_n X_{n-1},h\rangle^2\right]+E\left[\langle Z_n,h\rangle^2\right]\\
&+ 2\left\langle C_{\rho_n X_{n-1},Z_n}(h),h\right\rangle\\
&= E\left[\langle X_{n-1},\rho_n^*h\rangle^2\right]+\langle C_Z(h),h\rangle\\
&= \langle C_{n-1}(\rho_n^*h),\rho_n^*h\rangle+\langle C_Z(h),h\rangle\\
&= \langle \rho_n C_{n-1}\rho_n^*(h),h\rangle+\langle C_Z(h),h\rangle.
\end{aligned}$$

This leads to the functional equation (3.3), since C_n and $\rho_n C_{n-1}\rho_{n-1}^*+C_Z$ are symmetric operators. To prove the infinite series representation (3.4) we recall the representation

$$X_n=A_{n-mT-q+1}^n X_{n-mT-q}+\sum_{k=0}^{mT+q-1}A_{n+1-k}^n Z_{n-k},$$

which entails

$$C_n=A_{n-mT-q+1}^n C_{n-mT-q}\left(A_{n-mT-q+1}^n\right)^*+\sum_{k=0}^{mT+q-1}A_{n+1-k}^n C_Z\left(A_{n+1-k}^n\right)^*.$$

Since $A^n_{n-mT-q+1}C_{n-mT-q}\left(A^n_{n-mT-q+1}\right)^*$, as a covariance operator, is nuclear and considering $l_0 = \sup_{0\le l\le T-1}\left\|C_l^{\frac{1}{2}}\right\|_{\mathcal{S}}$, we conclude that

$$\begin{aligned}
&\left\|C_n - \sum_{k=0}^{mT+q-1} A^n_{n+1-k}C_Z\left(A^n_{n+1-k}\right)^*\right\|_{\mathcal{N}}\\
=\ &\left\|A^n_{n-mT-q+1}C_{n-mT-q}\left(A^n_{n-mT-q+1}\right)^*\right\|_{\mathcal{N}}\\
=\ &\left\|A^n_{n-mT-q+1}C^{\frac{1}{2}}_{n-mT-q}\right\|_{\mathcal{S}}\\
\le\ &\left\|A^n_{n-mT-q+1}\right\|_L\left\|C^{\frac{1}{2}}_{n-mT-q}\right\|_{\mathcal{S}}\\
\le\ &M^2\left\|B^{m'}\right\|_L\left\|C^{\frac{1}{2}}_{n-mT-q}\right\|_{\mathcal{S}}\\
\le\ &aM^2b^{m-1}\left\|C^{\frac{1}{2}}_{n-mT-q}\right\|_{\mathcal{S}}\\
\le\ &l_0aM^2b^{m-1}\longrightarrow 0,\quad \text{as}\ \ m\longrightarrow\infty,
\end{aligned}$$

giving the desired result. Finally, we consider the representation

$$X_n = A^n_{m+1}X_m + \sum_{k=0}^{n-m-1} A^n_{n-k+1}Z_{n-k},\qquad m<n$$

Thus

$$\begin{aligned}
C_{X_n,X_m} &= A^n_{m+1}C_{X_m,X_m} + \sum_{k=0}^{n-m-1} A^n_{n-k+1}C_{Z_{n-k},X_m}\\
&= A^n_{m+1}C_{X_m,X_m}.
\end{aligned}$$

By using the relation $C_{X_n,X_m} = C^*_{X_m,X_n,}$ we deduce the last equality (3.6). □

Following theorem presents necessary and sufficient condition for the coordinate process $\{\langle X_n, v\rangle\}_{n\in\mathbb{Z}}$, where $v\in\mathbb{H}$, to satisfy in some AR(1) model.

Theorem 3.5. *Let $X=\{X_n\}_{n\in\mathbb{Z}}$ be the T-PC solution to the AR(1) equation and v be an element of $\mathbb{H}$ for which $\langle C_Z(v),v\rangle>0$. Then $\{\langle X_n,v\rangle\}_{n\in\mathbb{Z}}$ is a T-PCAR(1) real valued process with noise $\{\langle Z_n,v\rangle\}_{n\in\mathbb{Z}}$ if and only if there exists a real T-periodic sequence $\{\alpha_j\}_{j\in\mathbb{Z}}$ such that*

$$\langle X_{n-1},\rho_n^*v\rangle = \langle X_{n-1},\alpha_n v\rangle \quad a.s.\ n\in\mathbb{Z}. \tag{3.7}$$

If the condition holds, then $|\alpha_0\cdots\alpha_{T-1}|<1$ and $\{\langle X_n,v\rangle\}_{n\in\mathbb{Z}}$ has the autoregressive representation

$$\langle X_n,v\rangle = \alpha_n\langle X_{n-1},v\rangle + \langle Z_n,v\rangle.$$

Proof. Sufficiency: We note that

$$\begin{aligned}\langle X_n, v\rangle &= \langle \rho_n X_{n-1}, v\rangle + \langle Z_n, v\rangle \\ &= \langle X_{n-1}, \rho_n^* v\rangle + \langle Z_n, v\rangle \\ &= \langle X_{n-1}, \alpha_n v\rangle + \langle Z_n, v\rangle \\ &= \alpha_n \langle X_{n-1}, v\rangle + \langle Z_n, v\rangle ,\end{aligned}$$

the desired equation. Moreover,

$$\begin{aligned}E\langle X_n, v\rangle^2 &= \alpha_n^2 E\langle X_{n-1}, v\rangle^2 + E\langle Z_n, v\rangle^2 \\ &= \alpha_n^2 \cdots \alpha_{n-T+1}^2 E\langle X_{n-T}, v\rangle^2 + \sum_{k=1}^{T-1} \alpha_n^2 \cdots \alpha_{n-k+1}^2 E\langle Z_{n-k}, v\rangle^2 \\ &+ E\langle Z_n, v\rangle^2 .\end{aligned}$$

Since $E\langle X_n, v\rangle^2$ equals $E\langle X_{n-T}, v\rangle^2$ and the last two terms are positive we conclude

$$1 - |\alpha_0 \cdots \alpha_{T-1}|^2 > 0.$$

Necessity: Assume $\{\langle X_n, v\rangle\}_{n\in\mathbb{Z}}$ is a T-PCAR(1) real process with noise $\{\langle Z_n, v\rangle\}_{n\in\mathbb{Z}}$, then there exists a real T-periodic sequence $\{\alpha_j\}_{j\in\mathbb{Z}}$ with $|\alpha_0 \cdots \alpha_{T-1}| < 1$ such that

$$\langle X_n, v\rangle = \alpha_n \langle X_{n-1}, v\rangle + \langle Z_n, v\rangle . \tag{3.8}$$

Therefore, we obtain the desired result by drawing a comparison between equation $\langle X_n, v\rangle = \langle X_{n-1}, \rho_n^* v\rangle + \langle Z_n, v\rangle$ and the AR(1) equation (3.8). □

Remark 3.6. Note that to hold condition (3.7), it is enough for element v to be a common eigenfunction of the T operators $\rho_0^*, \cdots, \rho_{T-1}^*$ satisfying $\langle C_Z(v), v\rangle > 0$. In the next example we will choose operators $\rho_0, \cdots, \rho_{T-1}$ as coefficients of some constant operator, ρ say, and hence it is easy to find common eigenfunctions.

Example 3.7. Assume ρ is a uniformly exponentially stable operator on $\mathbb{H}$ and $\rho_j = \alpha_j \rho$ for $j = 0, \ldots, T-1$, where each α_j is a real number and $|\alpha_0 \cdots \alpha_{T-1}| < 1$. Then Condition (II) satisfies and the AR(1) equation has a unique T-PC solution in the form

$$X_n = \sum_{l=0}^{\infty} (\alpha_0 \cdots \alpha_{T-1})^l \rho^{lT} \sum_{q=0}^{T-1} \left(\prod_{j=0}^{q-1} \alpha_{n-lT-j} \right) \rho^q Z_{n-lT-q}.$$

Let us put light on Hilbert-valued T-PCAR(1) processes with symmetric compact autocorrelation operators. Indeed, we focus on the special case when operators ρ_j, and consequently the operator B, are symmetric and compact on

$\mathbb{H}$ and assume the spectral representation based on a common orthonormal basis, say $\{e_j\}_{j\in\mathbb{N}}$. In more detail, we have the following eigenvalue decomposition for each ρ_j.

$$\rho_j = \sum_{k=1}^{\infty} \alpha_{j,k} e_k \otimes e_k, \qquad j = 0, \ldots, T-1.$$

Consequently B admits the decomposition

$$B = \sum_{k=1}^{\infty} \beta_k e_k \otimes e_k,$$

where $\beta_k = \alpha_{0,k} \cdots \alpha_{T-1,k}$. By rearranging $\{e_j\}_{j\in\mathbb{N}}$, if it is necessary, we can assume $\{|\beta_j|\}_{j\in\mathbb{N}}$ is a decreasing sequence converging to zero. Then we define a symmetric compact operator

$$B_Z = \sum_{k=k_0}^{\infty} \beta_k e_k \otimes e_k,$$

where k_0 is the smallest positive integer k for which at least one of the terms

$$\{E \langle \rho_t \cdots \rho_s Z_0, e_k \rangle^2 : -1 \leq t - s \leq T-2\}$$

is positive. If such a k_0 does not exist, we set B_Z to be the zero operator.

In this special case, the following theorem specifies condition to the existence of an AR(1) solution and formulate the existing series. This informative theorem states that, similar to univariate case, $\|B_Z\| < 1$ is necessary and sufficient for the existence of a T-PCAR(1) solution and presents its causal form.

Theorem 3.8. *Let ρ_j for $j = 0, \ldots, T-1$, B, and B_Z be as above. Assume $Z = \{Z_n\}_{n\in\mathbb{Z}}$ is an $\mathbb{H}$-valued white noise. Then the equation*

$$X_n = \rho_n X_{n-1} + Z_n, \qquad n \in \mathbb{Z}$$

has a T-PC solution if and only if $\|B_Z\| < 1$.

Proof. First, note that Z is a solution of the AR(1) equation if and only if $\|B_Z\| = 0$. Assume $0 \neq \|B_Z\| < 1$. We will generally show that $\sum_{k=0}^{\infty} A^n_{n+1-k} Z_{n-k}$ is the solution of the equation, where the series converges in $L^2_{\mathbb{H}}(\Omega, \mathcal{F}, P)$ norm. By orthogonality of Z_n s, we have

$$\begin{aligned} \left\| \sum_{k=m}^{j} A^n_{n+1-k} Z_{n-k} \right\|^2_{L^2_{\mathbb{H}}(\Omega,\mathcal{F},P)} &= \sum_{k=m}^{j} \left\| A^n_{n+1-k} Z_{n-k} \right\|^2_{L^2_{\mathbb{H}}(\Omega,\mathcal{F},P)} \\ &= \sum_{k=m}^{j} \left\| A^n_{n+1-k} Z_0 \right\|^2_{L^2_{\mathbb{H}}(\Omega,\mathcal{F},P)}, \end{aligned}$$

where for big enough m and j and using the definition of B_Z, is equal to

$$
\begin{aligned}
& \sum_{l=1}^{\infty} \sum_{k=lT}^{lT+T-1} \left\| A_{n+1-k}^{n} Z_0 \right\|_{L^2_{\mathbb{H}}(\Omega,\mathcal{F},P)}^2 I(m \le k \le j) \\
= \; & \sum_{l=1}^{\infty} \sum_{q=0}^{T-1} \left\| A_0^{n-\left[\frac{n}{T}\right]T} B^{\left[\frac{lT+q-(n-\left[\frac{n}{T}\right]T)-1}{T}\right]} A_{T-(q-(n-\left[\frac{n}{T}\right]T))_T}^{T-1} Z_0 \right\|^2 \\
\times \; & I(m \le q+lT \le j) \\
& \sum_{l=1}^{\infty} \sum_{q=0}^{T-1} \left\| A_0^{n-\left[\frac{n}{T}\right]T} B_Z^{\left[\frac{lT+q-(n-\left[\frac{n}{T}\right]T)-1}{T}\right]} A_{T-(q-(n-\left[\frac{n}{T}\right]T))_T}^{T-1} Z_0 \right\|^2 \\
\times \; & I(m \le q+lT \le j) \\
\le \; & \sigma^2 \sum_{l=1}^{\infty} \sum_{q=0}^{T-1} \left\| A_0^{n-\left[\frac{n}{T}\right]T} \right\|^2 \left\| B_Z^{\left[\frac{lT+q-(n-\left[\frac{n}{T}\right]T)-1}{T}\right]} \right\|^2 \left\| A_{T-(q-(n-\left[\frac{n}{T}\right]T))_T}^{T-1} \right\|^2 \\
\times \; & I(m \le q+lT \le j) \\
\le \; & M^4\sigma^2 \sum_{l=1}^{\infty} \sum_{q=0}^{T-1} \left\| B_Z^{\left[\frac{lT+q-(n-\left[\frac{n}{T}\right]T)-1}{T}\right]} \right\|^2 I(m \le q+lT \le j) \\
\le \; & M^4\sigma^2 \sum_{l=1}^{\infty} \sum_{q=0}^{T-1} \left(\left\| B_Z^{l-1} \right\|^2 + \left\| B_Z^{l} \right\|^2 \right) I(m \le q+lT \le j) \\
& M^4\sigma^2 \sum_{l=1}^{\infty} \sum_{k=lT}^{lT+T-1} \left(\|B_Z\|^{2(l-1)} + \|B_Z\|^{2l} \right) I(m \le k \le j) \\
\le \; & M^4 T\sigma^2 \sum_{k=m}^{j} \left(\|B_Z\|^{2(k-1)} + \|B_Z\|^{2k} \right) \to 0, \qquad \text{as } m, j \to \infty.
\end{aligned}
$$

Cauchy's criterion entails the desired result.

Conversely, assume X is a T-PC solution to the AR(1) equation. Then,

$$
\begin{aligned}
E\left\langle X_n, e_{k_0} \right\rangle^2 &= \alpha_{n,k_0}^2 E\left\langle X_{n-1}, e_{k_0} \right\rangle^2 + E\left\langle Z_n, e_{k_0} \right\rangle^2 \\
&= \alpha_{n,k_0}^2 \cdots \alpha_{n-T+1,k_0}^2 E\left\langle X_{n-T}, e_{k_0} \right\rangle^2 + \sum_{k=1}^{T-1} \alpha_{n,k_0}^2 \cdots \alpha_{n-k+1,k_0}^2 \\
&\quad E\left\langle Z_{n-k}, e_{k_0} \right\rangle^2 + E\left\langle Z_n, e_{k_0} \right\rangle^2,
\end{aligned}
$$

hence (by considering the definition of k_0),

$$1 - |\alpha_{n,k_0} \cdots \alpha_{n-T+1,k_0}|^2 > 0$$

or equivalently

$$\|B_Z\| = |\beta_{k_0}| = |\alpha_{n,k_0} \cdots \alpha_{n-T+1,k_0}| < 1,$$

which completes the proof. □

4. Hilbert-valued PCAR(p) sequences

The present section devotes to the more practical $\mathbb{H}$-valued T-PC autoregressive models with finite order which are defined similarly. We establish in this section the $\mathbb{H}$-valued PCAR(p) processes can be treated as a PCAR(1) model in the space $\mathbb{H}^p$, cartesian product of p copies of the space $\mathbb{H}$. The space $\mathbb{H}^p$ equipped with the inner product

$$\langle h, g\rangle_{\mathbb{H}^p} = \sum_{j=0}^{p-1} \langle h_j, g_j\rangle_{\mathbb{H}}, \qquad h = (h_0, \ldots, h_{p-1})^T, g = (g_0, \ldots, g_{p-1})^T \in \mathbb{H}^p,$$

forms a real separable Hilbert space. In the following, we rigorously define the $\mathbb{H}$-valued PCAR(p) models and its relation to $\mathbb{H}^p$-valued PCAR(1) models in general.

Definition 4.1. Let $X = \{X_n\}_{n\in\mathbb{Z}}$ be an $\mathbb{H}$-valued periodically correlated process with period T. Then X is said to be autoregressive of order p (T-PCAR(p)) if it satisfies

$$X_n = \phi_{1,n}X_{n-1} + \phi_{2,n}X_{n-2} + \cdots + \phi_{p,n}X_{n-p} + \varepsilon_n, \quad n \in \mathbb{Z}, \tag{4.1}$$

where $\varepsilon = \{\varepsilon_n\}_{n\in\mathbb{Z}}$ is a white noise process with $\sigma^2 = E\|\varepsilon_n\|^2_{\mathbb{H}}$, for each n. For each $j = 1, \ldots, p$ the sequence $\{\phi_{j,n}\}_{n\in\mathbb{Z}}$ is a T-periodic sequence of bounded linear operators on $\mathbb{H}$. Moreover, (for identifiability of p) we assume the sequence $\{\phi_{p,n}\}_{n\in\mathbb{Z}}$ to be a nonzero sequence of operators.

Lemma 4.2. *Let $X = \{X_n\}_{n\in\mathbb{Z}}$ be an $\mathbb{H}$-valued PCAR(p) random sequence satisfying (4.1). Then the sequence $Y = \{Y_n = (X_n, \ldots, X_{n-p+1})^T\}_{n\in\mathbb{Z}}$ admits a PCAR(1) representation in the space $\mathbb{H}^p$.*

Proof. Define the T-periodic sequence of $p \times p$ operator matrix $\{\rho_n\}_{n\in\mathbb{Z}}$ on $\mathbb{H}^p$, as bellow

$$\rho_n = \begin{pmatrix} \phi_{1,n} & \phi_{2,n} & \ldots & \phi_{p,n} \\ I_{\mathbb{H}} & 0 & 0 & 0 \\ 0 & I_{\mathbb{H}} & 0 & 0 \\ 0 & 0 & I_{\mathbb{H}} & 0 \end{pmatrix}, \quad n \in \mathbb{Z}, \tag{4.2}$$

where $I_{\mathbb{H}}$ denotes the identity operator on $\mathbb{H}$. We also define the $\mathbb{H}^p$-valued white noise $Z = \{Z_n\}_{n\in\mathbb{Z}}$ in the form

$$Z_n = (\varepsilon_n, 0, \ldots, 0)^T, \quad n \in \mathbb{Z}.$$

Now, a similar argument to the proof of [1, Lemma 5.1] entails the sequence $Y = \{Y_n\}_{n\in\mathbb{Z}}$ is a T-PC sequence satisfying the PCAR(1) equation (2.1) with innovation $Z = \{Z_n\}_{n\in\mathbb{Z}}$. In other words, $Y = \{Y_n\}_{n\in\mathbb{Z}}$ is a T-PC sequence with the following representation

$$Y_n = \rho_n Y_{n-1} + Z_n, \quad n \in \mathbb{Z}. \tag{4.3}$$

□

Theorem 4.3. *Assume Condition* (I) *or equivalently Condition* (II) *holds for the T-periodic sequence of operators $\{\rho_n\}_{n\in\mathbb{Z}}$ defined in* (4.2). *We then conclude the equation* (4.1) *admits a unique T-PC solution with the following representation*

$$X_n = \sum_{k=0}^{\infty} \left(\pi_1 A^n_{n+1-k}\right)(Z_{n-k}), \quad n \in \mathbb{Z},$$

where the convergence holds in mean square sense and with probability one.

Proof. According to Theorem 3.1 we can easily conclude the (4.3) has the unique T-PC solution

$$Y_n = \sum_{k=0}^{\infty} A^n_{n+1-k} Z_{n-k}, \quad n \in \mathbb{Z}, \tag{4.4}$$

where the series converges in mean square sense and with probability one. Hence, applying the projection operator π_1 on both sides of (4.4) and using the continuity of this operator entails the solution

$$\begin{aligned} X_n &= \pi_1(Y_n) = \pi_1\left(\sum_{k=0}^{\infty} A^n_{n+1-k} Z_{n-k}\right) \\ &= \sum_{k=0}^{\infty} \left(\pi_1 A^n_{n+1-k}\right)(Z_{n-k}), \quad n \in \mathbb{Z}, \end{aligned}$$

to the AR(p) equation (4.1). To see the uniqueness, assume $\breve{X}_n$ to be any solution to the equation (4.1). Thus, according to Lemma 4.2 the stochastic process $\breve{Y}_n = \left(\breve{X}_n, \breve{X}_{n-1}, \ldots, \breve{X}_{n-p+1}\right)^T$ as a T-PC AR(1) solution to the (4.3) equals (4.4) with probability one. Which verifies the uniqueness of the existing solution. □

4.1. **Discussion.** Theorem 3.1 answers the main question posed in this paper for Hilbert valued PCAR processes of order one and the last section extends our result to each PCAR process of finite order. Theorem 3.5 deals with the coordinate process and discusses conditions that this process can be real valued PCAR process. Future work can be pursued in the direction of parameter estimation and prediction. The extension of the research to ARMA processes and investigating the Yule-Walker equation sounds interesting. Also Recently [3] have investigated Hilbertian spatial periodical correlated Autoregressive processes (HPCAR, in abbreviation). A future work would be extending our result to spatial HPCAR models.

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