ISSN: 1017-060X (Print)

ISSN: 1735-8515 (Online)

Bulletin of the

Iranian Mathematical Society

Vol. 43 (2017), No. 7, pp. 2547–2565

Title:

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Published by the Iranian Mathematical Society
http://bims.ims.ir

Bull. Iranian Math. Soc.
Vol. 43 (2017), No. 7, pp. 2547–2565
Online ISSN: 1735-8515

EXISTENCE AND CONVERGENCE RESULTS FOR MONOTONE NONEXPANSIVE TYPE MAPPINGS IN PARTIALLY ORDERED HYPERBOLIC METRIC SPACES

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(Communicated by Ali Abkar)

Abstract. We present some existence and convergence results for a general class of nonexpansive mappings in partially ordered hyperbolic metric spaces. We also give some examples to show the generality of the mappings considered herein.

Keywords: Hyperbolic metric space, nonexpansive mapping, condition (C).
MSC(2010): Primary: 47H10; Secondary: 54H25, 47H09.

1. **Introduction**

Nonexpansive mappings are those which have Lipschitz constant equal to one. A nonexpansive mapping need not have a fixed point in a complete space. However, if the space is endowed with rich geometric properties then the existence of fixed point can be ensured. In 1965, three mathematicians, Browder [3, 4], Göhde [11] and Kirk [17] obtained the first existence result for nonexpansive mappings, independently (see also [8]).

The study of fixed point theory for nonexpansive mappings in a hyperbolic metric space setting was initiated by Takahashi [32]. He used the term convex metric space to describe members of this class (see also [22]). Goebel and Kirk [7] used hyperbolic type spaces, which contain spaces with hyperbolic metric (see also [12]). Reich and Shafrir [28] introduced hyperbolic metric spaces on general infinite dimensional manifolds and studied iteration processes for nonexpansive mappings in these spaces using an additional condition on the hyperbolic metric. Accommodating previous definitions of hyperbolic metric spaces, Kohlenbach [19] introduced a more general definition. Busemann spaces [5] are well-known examples of hyperbolic metric spaces. Leuştean [20] showed

Article electronically published on December 30, 2017.
Received: 10 December 2016, Accepted: 23 December 2017.
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that CAT(0) spaces are uniformly convex hyperbolic metric spaces. Recently, Bin Dehaish and Khamsi [2] obtained a fixed point theorem for monotone nonexpansive mappings in the setting of partially ordered hyperbolic metric spaces.

On the other hand, generalizing nonexpansive mappings, Suzuki [31] introduced the following new class of mappings and obtained some existence and convergence results:

Definition 1.1 ([31])**.** Let E be a Banach space and K a nonempty subset of E. A mapping $T : K \to K$ is said to satisfy condition (C) if for all $u, v \in K$

$$\frac{1}{2}\|u - T(u)\| \leq \|u - v\| \text{ implies } \|T(u) - T(v)\| \leq \|u - v\|.$$

Theorem 1.2 ([31])**.** *Let K be a nonempty convex subset of a Banach space E and $T : K \to K$ a mapping satisfying the condition (C). Assume also that one of the following holds:*

- *K is compact;*
- *K is weakly compact and E has the Opial property.*

Then T has a fixed point.

In this paper, we present a more general version of Bin Dehaish and Khamsi's theorem in a partially ordered hyperbolic metric space. We also obtain some Δ-convergence (see Definitions 2.6, 2.8 below) and strong convergence theorems for a monotone mapping satisfying condition (C) in partially ordered hyperbolic metric spaces. In this way certain results from [2, 20, 23, 25, 27, 30, 31] are extended and generalized.

2. Preliminaries

The following definition is due to Kohlenbach [19].

Definition 2.1. A triplet $(\mathcal{M}, d, W)$ is said to be a hyperbolic metric space if $(\mathcal{M}, d)$ is a metric space and $W : \mathcal{M} \times \mathcal{M} \times [0, 1] \to \mathcal{M}$ is a function satisfying

(H1) $d(z, W(u, v, \beta)) \leq (1 - \beta)d(z, u) + \beta d(z, v)$;
(H2) $d(W(u, v, \beta), W(u, v, \gamma)) = |\beta - \gamma|d(u, v)$;
(H3) $W(u, v, \beta) = W(v, u, 1 - \beta)$;
(H4) $d(W(u, z, \beta), W(v, w, \beta)) \leq (1 - \beta)d(u, v) + \beta d(z, w)$,

for all $u, v, z, w \in \mathcal{M}$ and $\beta, \gamma \in [0, 1]$. The set

$$\text{seg}[u, v] := \{W(u, v, \beta) : \beta \in [0, 1]\}$$

is called the metric segment with endpoints u and v.

Remark 2.2. If only condition (H1) is satisfied, then $(\mathcal{M}, d, W)$ is a convex metric space in the sense of Takahashi [32]. Conditions (H1)-(H3) are equivalent to $(\mathcal{M}, d, W)$ being a space of hyperbolic type in the sense of Goebel and

Kirk [7]. Condition (H4) was considered by Itoh [13] as condition III and later used in [28] (with restriction on β, $\beta = 1/2$) to define the class of hyperbolic metric spaces. Condition (H3) ensures that $\text{seg}[u, v]$ is an isometric image of the real line segment $[0, d(u, v)]$.

We shall adopt the customary notations and write $W(u, v, \beta) = (1-\beta)u \oplus \beta v$. We shall say that a subset $\mathcal{K}$ of $\mathcal{M}$ is convex if $u, v \in \mathcal{K}$ implies that $(1-\beta)u \oplus \beta v \in \mathcal{K}$ for all $\beta \in [0, 1]$. We shall use $(\mathcal{M}, d)$ for $(\mathcal{M}, d, W)$ when there is no ambiguity. All normed linear spaces and Hilbert ball equipped with the hyperbolic metric are some examples of hyperbolic metric spaces [9].

Throughout, we shall assume that order intervals are closed convex subsets of a hyperbolic metric space $\mathcal{M}$. We denote these as follows:

$$[a, \to) := \{u \in \mathcal{M} : a \preceq u\} \text{ and } (\leftarrow, b] := \{u \in \mathcal{M} : u \preceq b\},$$

for any $a, b \in \mathcal{M}$ (cf. [2]).

Definition 2.3 ([10, 15])**.** Let $(\mathcal{M}, d)$ be a hyperbolic metric space. For any $a \in \mathcal{M}$, $r > 0$ and $\varepsilon > 0$, set

$$\delta(r, \varepsilon) = \inf \left\{ 1 - \frac{1}{r} d\left(\frac{1}{2}u \oplus \frac{1}{2}v, a\right) : d(u, a) \le r,\ d(v, a) \le r,\ d(u, v) \ge r\varepsilon \right\}.$$

We say that $\mathcal{M}$ is uniformly convex if $\delta(r, \varepsilon) > 0$, for any $r > 0$ and $\varepsilon > 0$.

Definition 2.4 ([14])**.** A hyperbolic metric space $(\mathcal{M}, d)$ is said to satisfy property (R) if for each decreasing sequence $\{C_n\}$ of nonempty bounded closed convex subsets of $\mathcal{M}$, $\bigcap_{n=1}^{\infty} C_n \neq \emptyset$.

Uniformly convex hyperbolic spaces enjoy the property (R) [2].

Definition 2.5 ([29])**.** Let $\mathcal{K}$ be a subset of a metric space $(\mathcal{M}, d)$. A mapping $T : \mathcal{K} \to \mathcal{K}$ is said to satisfy Condition (I) if there exists a nondecreasing function $f : [0, \infty) \to [0, \infty)$ satisfying $f(0) = 0$ and $f(r) > 0$ for all $r \in (0, \infty)$ such that $d(u, T(u)) \ge f(\text{dist}(u, F(T)))$ for all $u \in \mathcal{K}$, where $\text{dist}(u, F(T))$ denotes the distance from u to $F(T)$.

Let $\mathcal{K}$ be a nonempty subset of a hyperbolic metric space $(\mathcal{M}, d)$ and $\{u_n\}$ a bounded sequence in $\mathcal{M}$. For each $u \in \mathcal{M}$, define:

(i) asymptotic radius of $\{u_n\}$ at u as $r(\{u_n\}, u) := \limsup_{n\to\infty} d(u_n, u)$;

(ii) asymptotic radius of $\{u_n\}$ relative to $\mathcal{K}$ as

$$r(\{u_n\}, \mathcal{K}) := \inf\{r(\{u_n\}, u) : u \in \mathcal{K}\};$$

(iii) asymptotic centre of $\{u_n\}$ relative to $\mathcal{K}$ by

$$A(\{u_n\}, \mathcal{K}) := \{u \in \mathcal{K} : r(\{u_n\}, u) = r(\{u_n\}, \mathcal{K})\}.$$

Lim [21] introduced the concept of Δ-convergence in metric spaces. Kirk and Panyanak [18] used Lim's concept in CAT(0) spaces and showed that many Banach spaces results involving weak convergence have precise analogs in this setting.

Definition 2.6 ([18])**.** A bounded sequence $\{u_n\}$ in $\mathcal{M}$ is said to Δ-converge to a point $u \in \mathcal{M}$ if u is the unique asymptotic centre of every subsequence $\{u_{n_k}\}$ of $\{u_n\}$.

Definition 2.7 ([2])**.** Let $\mathcal{K}$ be a nonempty subset of a hyperbolic metric space $(\mathcal{M}, d)$. A function $\tau : \mathcal{K} \to [0, \infty)$ is said to be a type function if there exists a bounded sequence $\{u_n\}$ in $\mathcal{M}$ such that

$$\tau(u) = \limsup_{n\to\infty} d(u_n, u)$$

for any $u \in \mathcal{K}$.

We note that every bounded sequence generates a unique type function.

Now we rephrase the concept of Δ-convergence in hyperbolic metric spaces.

Definition 2.8. A bounded sequence $\{u_n\}$ in $\mathcal{M}$ is said to Δ-converge to a point $z \in \mathcal{M}$ if z is unique and the type function generated by every subsequence $\{u_{n_k}\}$ of $\{u_n\}$ attains its infimum at z.

Agarwal *et al.* [1] introduced an iteration process known as S-iteration process, which can be defined in the framework of hyperbolic metric spaces as follows:

$$\begin{cases} u_1 \in \mathcal{K} \\ v_n = \gamma_n T(u_n) \oplus (1-\gamma_n)u_n \\ u_{n+1} = \beta_n T(v_n) \oplus (1-\beta_n)T(u_n), \end{cases} \tag{2.1}$$

where $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $(0, 1)$.

3. Existence results

First, we recall the following definitions and preliminary results:

Definition 3.1 ([2])**.** Let $(\mathcal{M}, d, \preceq)$ be a partially ordered metric space and $T : \mathcal{M} \to \mathcal{M}$ a mapping. The mapping T is said to be monotone if

$$u \preceq v \text{ implies } T(u) \preceq T(v).$$

Definition 3.2 ([2])**.** Let $(\mathcal{M}, d, \preceq)$ be a partially ordered metric space and $T : \mathcal{M} \to \mathcal{M}$ a mapping. The mapping T is said to be monotone nonexpansive if T is monotone and

$$d(T(u), T(v)) \leq d(u, v), \tag{3.1}$$

for all $u, v \in \mathcal{M}$ such that u and v are comparable.

We extend Definition 1.1 from Banach spaces to partially ordered hyperbolic metric spaces as follows:

Definition 3.3. Let $(\mathcal{M}, d, \preceq)$ be a partially ordered metric space and $T : \mathcal{M} \to \mathcal{M}$ a monotone mapping. The mapping T is said to satisfy condition (C) if

$$\frac{1}{2}d(u, T(u)) \leq d(u, v) \text{ implies } d(T(u), T(v)) \leq d(u, v), \tag{3.2}$$

for all $u, v \in \mathcal{M}$ such that u and v are comparable.

Remark 3.4. Every nonexpansive mapping satisfies the condition (C) (c.f. [31]).

The following lemma will be useful in our results.

Lemma 3.5 ([2]). *Let $(\mathcal{M}, d)$ be a uniformly convex hyperbolic metric space and $\mathcal{K}$ a nonempty closed convex subset of $\mathcal{M}$. Let $\tau : \mathcal{K} \to [0, \infty)$ be a type function. Then τ is continuous. Moreover, there exists a unique minimum point $z \in \mathcal{K}$ such that $\tau(z) = \inf\{\tau(u) : u \in \mathcal{K}\}$.*

We need the following two propositions to accomplish our main results.

Proposition 3.6 ([7]). *Let $(\mathcal{M}, d)$ be a hyperbolic type metric space and $\{\beta_n\} \subset [0, 1)$. Suppose $\{u_n\}$ and $\{v_n\}$ are sequences in $\mathcal{M}$ which satisfy for all $n \in \mathbb{N}$,*

(i) $u_{n+1} \in \text{seg}[u_n, v_n]$ *with* $d(u_n, u_{n+1}) = \beta_n d(u_n, v_n)$, *and*
(ii) $d(v_{n+1}, v_n) \leq d(u_{n+1}, u_n)$.

Then, for all $i, n \in \mathbb{N}$,

$$\begin{aligned}\left(1 + \sum_{s=i}^{i+n-1} \beta_s\right) d(v_i, u_i) \quad &\leq \quad d(v_{i+n}, u_i) \\ &+ \quad \prod_{s=i}^{i+n-1} (1 - \beta_s)^{-1} [d(v_i, u_i) - d(v_{i+n}, u_{i+n})].\end{aligned}$$

Proposition 3.7. *Let $(\mathcal{M}, d, \preceq)$ be a partially ordered hyperbolic metric space and $\mathcal{K}$ a bounded convex subset of $\mathcal{M}$ not reduced to one point. Let $T : \mathcal{K} \to \mathcal{K}$ be a monotone mapping satisfying the condition (C). Let $u_1 \in \mathcal{K}$ be such that u_1 and $T(u_1)$ are comparable and $\beta_n \in [1/2, 1)$. Define a sequence $\{u_n\}$ in $\mathcal{K}$ by*

$$u_{n+1} = \beta_n T(u_n) \oplus (1 - \beta_n) u_n, \quad n \in \mathbb{N}. \tag{3.3}$$

Then $\lim_{n\to\infty} d(u_n, T(u_n)) = 0.$

Proof. Without loss of generality, we may assume that $u_1 \preceq T(u_1)$. By the convexity of order interval, we have $u_1 \preceq u_2 \preceq T(u_1)$. Since T is monotone, we get $T(u_1) \preceq T(u_2)$. This implies $u_1 \preceq u_2 \preceq T(u_1) \preceq T(u_2)$. Continuing in this way, we get

$$u_n \preceq u_{n+1} \preceq T(u_n) \preceq T(u_{n+1}).$$

This implies that $u_{n+1} \in \text{seg}[u_n, T(u_n)]$. From (3.3) u_{n+1} is a unique point of $\text{seg}[u_n, T(u_n)]$ such that

$$\begin{aligned} d(u_{n+1}, T(u_n)) &= (1-\beta_n) d(u_n, T(u_n)) \text{ and} \\ d(u_{n+1}, u_n) &= \beta_n d(u_n, T(u_n)). \end{aligned} \tag{3.4}$$

Since $\beta_n \in \left[\frac{1}{2}, 1\right)$, by (3.4),

$$\frac{1}{2} d(u_n, T(u_n)) \leq \beta_n d(u_n, T(u_n)) = d(u_{n+1}, u_n).$$

Now (3.2) implies that

$$d(T(u_{n+1}), T(u_n)) \leq d(u_{n+1}, u_n). \tag{3.5}$$

By Proposition 3.6, it follows that

$$\begin{aligned} \left(1 + \sum_{s=i}^{i+n-1} \beta_s\right) d(T(u_i), u_i) \leq d(T(u_{i+n}), u_i) + \prod_{s=i}^{i+n-1} (1-\beta_s)^{-1} \\ [d(T(u_i), u_i) - d(T(u_{i+n}), u_{i+n})]. \end{aligned}$$

This implies that

$$\begin{aligned} \left(1 + \frac{1}{2}(n-1)\right) d(T(u_i), u_i) \leq d(T(u_{i+n}), u_i) + \prod_{s=i}^{i+n-1} (1-\beta_s)^{-1} \\ [d(T(u_i), u_i) - d(T(u_{i+n}), u_{i+n})]. \end{aligned} \tag{3.6}$$

We will show that $\{d(u_{n+1}, u_n)\}$ is a decreasing sequence. Since X is a hyperbolic metric space, by (3.5), we have

$$\begin{aligned} d(u_{n+2}, u_{n+1}) &= d(\beta_n T(u_{n+1}) \oplus (1-\beta_n) u_{n+1}, \beta_n T(u_n) \oplus (1-\beta_n) u_n) \\ &\leq \beta_n d(T(u_{n+1}), T(u_n)) + (1-\beta_n) d(u_{n+1}, u_n) \\ &\leq \beta_n d(u_{n+1}, u_n) + (1-\beta_n) d(u_{n+1}, u_n) = d(u_{n+1}, u_n), \end{aligned}$$

for all $n \in \mathbb{N}$. Thus $\{d(u_n, T(u_n))\}$ is a decreasing sequence. Let $\theta = \lim_{n\to\infty} d(u_n, T(u_n))$. Letting $i \to \infty$ in (3.6), we get

$$\left(1 + \frac{1}{2}(n-1)\right) \theta \leq \delta(\mathcal{K})$$

for all $n \in \mathbb{N}$, where $\delta(\mathcal{K}) = \sup\{d(u, v); u, v \in \mathcal{K}\} < \infty$. This implies that $\theta = 0$. □

Now, we present some existence results on a partially ordered hyperbolic metric space. For more details on ordered metric spaces and applications one may refer to [24, 26].

Theorem 3.8. *Let $(\mathcal{M}, d, \preceq)$ be a uniformly convex partially ordered hyperbolic metric space and $\mathcal{K}$ a nonempty closed convex bounded subset of $\mathcal{M}$ not reduced to one point. Let $T : \mathcal{K} \to \mathcal{K}$ be a monotone mapping satisfying the condition (C). Assume that there exists $u_1 \in \mathcal{K}$ such that u_1 and $T(u_1)$ are comparable. Then T has a fixed point.*

Proof. Without loss of generality, we may assume that $u_1 \preceq T(u_1)$. We can consider the sequence $\{u_n\}$ defined by (3.3) with initial point $u_1 \in \mathcal{K}$ and $\beta_n \in [1/2, 1)$. Since $\mathcal{M}$ is uniformly convex, it satisfies the property (R) and by the construction of $\{u_n\}$, we have

$$\mathcal{K}_\infty = \bigcap_{n=1}^{\infty} [u_n, \to) \cap \mathcal{K} = \bigcap_{n=1}^{\infty} \{u \in \mathcal{K}; u_n \preceq u\} \neq \emptyset.$$

Let $u \in \mathcal{K}_\infty$. Then $u_n \preceq u$. Since T is monotone, we have $u_n \preceq T(u_n) \preceq T(u)$, for all $n \in \mathbb{N}$. This implies that $T(\mathcal{K}_\infty) \subset \mathcal{K}_\infty$. Let $\tau : \mathcal{K}_\infty \to [0, \infty)$ be a type function generated by $\{u_n\}$, that is,

$$\tau(u) = \limsup_{n\to\infty} d(u_n, u).$$

From Lemma 3.5, there exists a unique element $z \in \mathcal{K}_\infty$ such that

$$\tau(z) = \inf\{\tau(u); u \in \mathcal{K}_\infty\}.$$

Since $z \in \mathcal{K}_\infty$, $u_n \preceq z$ for all $n \in \mathbb{N}$. If $u_n = u_{n+1}$, then $d(u_n, u_{n+1}) \le d(u_n, z)$ for all $n \in \mathbb{N}$. Again if $u_n \prec u_{n+1}$, then $u_n \prec u_{n+1} \preceq z$. Thus in the both cases we have $d(u_n, u_{n+1}) \le d(u_n, z)$ for all $n \in \mathbb{N}$. By (3.4) we have $\frac{1}{2} d(u_n, T(u_n)) \le d(u_n, z)$. Since T satisfies the condition (C), from (3.2)

$$d(T(u_n), T(z)) \le d(u_n, z). \tag{3.7}$$

By the triangle inequality, (3.7) and Proposition 3.7, we have

$$\begin{aligned}\tau(T(z)) &= \limsup_{n\to\infty} d(u_n, T(z)) \\ &\le \limsup_{n\to\infty} d(u_n, T(u_n)) + \limsup_{n\to\infty} d(T(u_n), T(z)) \\ &\le \limsup_{n\to\infty} d(u_n, z).\end{aligned}$$

Since $\tau(z) = \inf\{\tau(u); u \in \mathcal{K}_\infty\}$, it follows that $T(z) = z$, and z is a fixed point of T. ☐

Corollary 3.9 ([2, Theorem 3.1])**.** *Let $(\mathcal{M}, d, \preceq)$ be a uniformly convex partially ordered hyperbolic metric space and $\mathcal{K}$ a nonempty closed convex bounded subset of $\mathcal{M}$ not reduced to one point. Let $T : \mathcal{K} \to \mathcal{K}$ be a monotone nonexpansive mapping. Assume that there exists $u_1 \in \mathcal{K}$ such that u_1 and $T(u_1)$ are comparable. Then T has a fixed point.*

Lemma 3.10. *Let $(\mathcal{M}, d, \preceq)$ be a partially ordered hyperbolic metric space and $\mathcal{K}$ a nonempty closed convex subset of $\mathcal{M}$. Let $T : \mathcal{K} \to \mathcal{K}$ be a monotone mapping. Let $u_1 \in \mathcal{K}$ be such that $u_1 \preceq T(u_1)$ (or $T(u_1) \preceq u_1$). Then for the sequence $\{u_n\}$ defined by (2.1), we have*

(a) $u_n \preceq T(u_n) \preceq u_{n+1}$ *(or $u_{n+1} \preceq T(u_n) \preceq u_n$);*

(b) $u_n \preceq p$ *(or $p \preceq u_n$), provided $\{u_n\}$ Δ-converges to a point $p \in \mathcal{K}$,*

for all $n \in \mathbb{N}$.

Proof. We shall use induction to prove (a). By assumption, we have $u_1 \preceq T(u_1)$. By the convexity of order interval $[u_1, T(u_1)]$ and (2.1), we have

$$u_1 \preceq v_1 \preceq T(u_1). \tag{3.8}$$

Since T is monotone, we have $T(u_1) \preceq T(v_1)$. Again by the convexity of order interval $[T(u_1), T(v_1)]$ and (2.1), we have

$$T(u_1) \preceq u_2 \preceq T(v_1). \tag{3.9}$$

Combining (3.8) and (3.9), we get

$$u_1 \preceq v_1 \preceq T(u_1) \preceq u_2.$$

Thus (a) is true for $n = 1$. Now suppose it is true for n, that is, $u_n \preceq T(u_n) \preceq u_{n+1}$. By the convexity of order interval $[u_n, T(u_n)]$ and (2.1), we have

$$u_n \preceq v_n \preceq T(u_n). \tag{3.10}$$

Since T is monotone, we have $T(u_n) \preceq T(v_n)$. By the convexity of order interval $[T(u_n), T(v_n)]$ and (2.1), we have

$$T(u_n) \preceq u_{n+1} \preceq T(v_n). \tag{3.11}$$

Combining (3.10) and (3.11), we get

$$u_n \preceq v_n \preceq T(u_n) \preceq u_{n+1} \preceq T(v_n). \tag{3.12}$$

Since $v_n \preceq u_{n+1}$ and T is monotone, $T(v_n) \preceq T(u_{n+1})$. By (3.12), we have $u_{n+1} \preceq T(u_{n+1})$. By the convexity of order interval $[u_{n+1}, T(u_{n+1})]$, we have

$$u_{n+1} \preceq v_{n+1} \preceq T(u_{n+1}).$$

By the monotonicity of T, we have $T(u_{n+1}) \preceq T(v_{n+1})$. Convexity of order interval implies $T(u_{n+1}) \preceq u_{n+2} \preceq T(v_{n+1})$. Therefore

$$u_{n+1} \preceq v_{n+1} \preceq T(u_{n+1}) \preceq u_{n+2}. \tag{3.13}$$

Suppose p is a Δ-limit of $\{u_n\}$. Here the sequence $\{u_n\}$ is monotone increasing and the order interval $[u_m, \rightarrow)$ is closed and convex. We claim that $p \in [u_m, \rightarrow)$ for a fixed $m \in \mathbb{N}$. If $p \notin [u_m, \rightarrow)$, then the type function generated by subsequence $\{u_r\}$ of $\{u_n\}$ defined by leaving first $m-1$ terms of sequence $\{u_n\}$ will not attain an infimum at p, which is a contradiction to the assumption that p is a Δ-limit of the sequence $\{u_n\}$. This completes the proof. □

The following lemma is analogous to [31, Lemma 7].

Lemma 3.11. *Let $(\mathcal{M}, d, \preceq)$ be a partially ordered hyperbolic metric space and $\mathcal{K}$ a nonempty subset of $\mathcal{M}$. Let $T : \mathcal{K} \to \mathcal{K}$ be a monotone mapping satisfying the condition (C). Then for all $u, v \in \mathcal{K}$ such that u and v are comparable, we have*

$$d(u, T(v)) \leq 3d(u, T(u)) + d(u, v).$$

Theorem 3.12. *Let $(\mathcal{M}, d, \preceq)$ be a uniformly convex partially ordered hyperbolic metric space and $\mathcal{K}$ a nonempty closed convex subset of $\mathcal{M}$. Let $T : \mathcal{K} \to \mathcal{K}$ be a monotone mapping satisfying the condition (C). Assume that there exists $u_1 \in \mathcal{K}$ such that u_1 and $T(u_1)$ are comparable. Let the sequence $\{u_n\}$ defined by (2.1) be bounded, there exists a point $v \in \mathcal{K}$ such that every point of the sequence $\{u_n\}$ are comparable with v and $\lim_{n\to\infty} \inf d(T(u_n), u_n) = 0$. Then T has a fixed point.*

Proof. Suppose $\{u_n\}$ is a bounded sequence and $\liminf_{n\to\infty} d(T(u_n), u_n) = 0$. Then there exists a subsequence $\{u_{n_j}\}$ of $\{u_n\}$ such that

$$\lim_{j\to\infty} d(T(u_{n_j}), u_{n_j}) = 0.$$

By Lemma 3.10, we have $u_1 \preceq u_{n_j} \preceq u_{n_{j+1}}$. Define $\mathcal{K}_j = \{p \in \mathcal{K} : u_{n_j} \preceq p\}$ for all $j \in \mathbb{N}$. Clearly for each $j \in \mathbb{N}$, $\mathcal{K}_j$ is closed convex and $v \in \mathcal{K}_j$ so $\mathcal{K}_j$ is nonempty. Set

$$\mathcal{K}_\infty = \bigcap_{j=1}^{\infty} \mathcal{K}_j \neq \emptyset.$$

Then $\mathcal{K}_\infty$ is a closed convex subset of $\mathcal{K}$. Let $u \in \mathcal{K}_\infty$; then $u_{n_j} \preceq u$ for all $j \in \mathbb{N}$. Since T is monotone, for all $j \in \mathbb{N}$

$$u_{n_j} \preceq T(u_{n_j}) \preceq T(u).$$

This implies that $T(\mathcal{K}_\infty) \subset \mathcal{K}_\infty$. Let $\sigma : \mathcal{K}_\infty \to [0, \infty)$ be a type function generated by $\{u_{n_j}\}$, that is,

$$\sigma(u) = \limsup_{j\to\infty} d(u_{n_j}, u).$$

From Lemma 3.5, there exists a unique element $w \in \mathcal{K}_\infty$ such that

$$\sigma(w) = \inf\{\sigma(u); u \in \mathcal{K}_\infty\}.$$

By the definition of type function,

$$\sigma(T(w)) = \limsup_{j\to\infty} d(u_{n_j}, T(w)).$$

Using Lemma 3.11, we get

$$\begin{aligned}\tau(T(w)) &= \limsup_{j\to\infty} d(u_{n_j}, T(w))\\ &\le 3\limsup_{j\to\infty} d(u_{n_j}, T(u_{n_j})) + \limsup_{j\to\infty} d(u_{n_j}, w)\\ &= \tau(w).\end{aligned}$$

By the uniqueness of minimum point this implies that $T(w) = w$. □

Now, we present some illustrative examples.

Example 3.13. Let $\mathbb{R}$ (the set of reals) be equipped with the usual ordering and standard norm $\|u\| = |u|$. Let $\mathcal{K} = [0,1] \subset \mathbb{R}$ and $T : \mathcal{K} \to \mathcal{K}$ be a mapping defined by

$$T(u) = \begin{cases} 1-u, & \text{if } u \in [0, \frac{1}{4}) \\ \frac{u+3}{4}, & \text{if } u \in [\frac{1}{4}, 1]. \end{cases}$$

Then

- T is not a nonexpansive mapping.
- T satisfies condition (C).

For $u = \frac{24}{100}$ and $v = \frac{25}{100}$, we have $\|T(u) - T(v)\| = \frac{21}{400} > \frac{1}{100} = \|u - v\|$. Therefore T is not a nonexpansive mapping.

Now we show that T satisfies condition (C). For this, we consider the following two cases:

Case (i): $u, v \in [0, \frac{1}{4})$ or $u, v \in [\frac{1}{4}, 1]$. Then we have $\|T(u) - T(v)\| \le \|u - v\|$.

Case (ii): $u \in [0, \frac{1}{4})$ and $v \in [\frac{1}{4}, 1]$.

If $\frac{1}{2}\|u - T(u)\| \le \|u - v\|$, we must have $\frac{(1-2u)}{2} \le (v - u)$, or $\frac{1}{2} \le v$. Thus

$$\|T(u) - T(v)\| = \left|\frac{v + 4u - 1}{4}\right| < \frac{1}{4} < \|u - v\|.$$

Again, if $\frac{1}{2}\|v - T(v)\| \le \|u - v\|$, we must have

$$\frac{3-3v}{8} \le \|u - v\|, \tag{3.14}$$

which implies that $\frac{3}{11} + \frac{8}{11}u \le v$, so $v \in [\frac{3}{11}, 1]$. Thus, from (3.14)

$$\|T(u) - T(v)\| = \left|\frac{v + 4u - 1}{4}\right| < \frac{2}{11} < \frac{3}{16} \le \|u - v\|.$$

Therefore, T satisfies the condition (C). Note that T has a (unique) fixed point 1.

Example 3.14. Let $\mathcal{M} = \{(u_1, u_2) \in \mathbb{R}^2 : u_1, u_2 > 0\}$. Define $d : \mathcal{M} \times \mathcal{M} \to [0, \infty)$ by

$$d(u, v) = |u_1 - v_1| + |u_1 u_2 - v_1 v_2|$$

for all $u = (u_1, u_2)$ and $v = (v_1, v_2)$ in $\mathcal{M}$. Then it can be easily seen that d is a metric on $\mathcal{M}$ and $(\mathcal{M}, d)$ is a metric space. Now for $\beta \in [0,1]$, define a function $W : \mathcal{M} \times \mathcal{M} \times [0,1] \to \mathcal{M}$ by

$$W(u, v, \beta) = \left((1-\beta)u_1 + \beta v_1, \frac{(1-\beta)u_1u_2 + \beta v_1v_2}{(1-\beta)u_1 + \beta v_1} \right).$$

We show that $(\mathcal{M}, d, W)$ is a hyperbolic metric space. For $u = (u_1, u_2)$, $v = (v_1, v_2)$, $z = (z_1, z_2)$ and $w = (w_1, w_2)$ in $\mathcal{M}$:

(H1) $$\begin{aligned} d(z, W(u, v, \beta)) &= |z_1 - (1-\beta)u_1 - \beta v_1| \\ &+ |z_1z_2 - (1-\beta)u_1u_2 - \beta v_1v_2| \\ &\leq (1-\beta)|z_1 - u_1| + \beta|z_1 - v_1| \\ &+ (1-\beta)|z_1z_2 - u_1u_2| + \beta|z_1z_2 - v_1v_2| \\ &= (1-\beta)d(z, u) + \beta d(z, v). \end{aligned}$$

(H2) $$\begin{aligned} d(W(u, v, \beta), W(u, v, \gamma)) &= |(1-\beta)u_1 + \beta v_1 - (1-\gamma)u_1 - \gamma v_1| \\ &+ |(1-\beta)u_1u_2 + \beta v_1v_2 - (1-\gamma)u_1u_2 \\ &- \gamma v_1v_2| \\ &= |\beta - \gamma|(|u_1 - v_1| + |u_1u_2 - v_1v_2|) \\ &= |\beta - \gamma| d(u, v). \end{aligned}$$

(H3) $$\begin{aligned} W(u, v, \beta) &= \left((1-\beta)u_1 + \beta v_1, \frac{(1-\beta)u_1u_2 + \beta v_1v_2}{(1-\beta)u_1 + \beta v_1} \right) \\ &= \left(\beta v_1 + (1-\beta)u_1, \frac{\beta v_1v_2 + (1-\beta)u_1u_2}{\beta v_1 + (1-\beta)u_1} \right) \\ &= W(v, u, 1-\beta). \end{aligned}$$

(H4) $$\begin{aligned} d(W(u, z, \beta), W(v, w, \beta)) &= |(1-\beta)u_1 + \beta z_1 - (1-\beta)v_1 - \beta w_1| \\ &+ |(1-\beta)u_1u_2 + \beta z_1z_2 - (1-\beta)v_1v_2 \\ &- \beta w_1w_2| \\ &\leq (1-\beta)(|u_1 - v_1| + |u_1u_2 - v_1v_2|) \\ &+ \beta(|z_1 - w_1| + |z_1z_2 - w_1w_2|) \\ &= (1-\beta)d(u, v) + \beta d(z, w). \end{aligned}$$

Therefore $(\mathcal{M}, d, W)$ is a hyperbolic metric space but not a normed linear space. Now we define an order on $\mathcal{M}$ as follows: for $u = (u_1, u_2)$ and $v = (v_1, v_2)$, $u \preceq v$ if and only if $u_1 < v_1$ or $u_1 = v_1$ and $u_2 \leq v_2$. Thus $(\mathcal{M}, d, \preceq)$ is an ordered hyperbolic metric space.

Let $\mathcal{K} := [1,4] \times [1,4] \subset \mathcal{M}$ and $T : \mathcal{K} \to \mathcal{K}$ be a mapping defined by

$$T(u) = \begin{cases} (1,1), & \text{if } (u_1, u_2) \neq (4,4) \\ (2,2), & \text{if } (u_1, u_2) = (4,4). \end{cases}$$

First we show that T is not a nonexpansive mapping on $\mathcal{K}$. Let $u = (\frac{39}{10}, \frac{39}{10})$ and $v = (4, 4)$. Then

$$d(T(u), T(v)) = 4 > \frac{89}{100} = d(u, v).$$

To show that T satisfies condition (C), we consider the following cases:
Case 1: $u = (u_1, u_2), v = (v_1, v_2) \neq (4, 4)$, then

$$d(T(u), T(v)) = 0 \leq d(u, v).$$

Case 2: Here we consider two subcases: if $u = (u_1, u_2) \in \mathcal{A} := [1, 3.2] \times [1, 4] \cup [3.2, 3.5] \times [1, 3.5] \cup [3.5, 4] \times [1, 3]$ and $v = (4, 4)$, then

$$d(T(u), T(v)) = 4 \leq d(u, v).$$

If $u = (u_1, u_2) \in \mathcal{B} := (3.5, 4] \times (3, 3.5] \cup (3.2, 4] \times (3.5, 4] \setminus \{(4, 4)\}$ and $v = (4, 4)$, then

$$\frac{1}{2} d(u, T(u)) > 6 > d(u, v) \text{ and } \frac{1}{2} d(v, T(v)) = 7 > 6 > d(u, v).$$

Therefore T satisfies the condition (C).

In the following example we consider the well-known *river metric* d. A river metric space $(\mathbb{R}^2, d)$ is a $\mathbb{R}$-tree. Further, $\mathbb{R}$-trees are CAT(0) spaces and these spaces are particular cases of hyperbolic spaces (c.f. [6]).

Example 3.15. Let $\mathbb{R}^2$ be equipped with the *river metric* defined by

$$d(u, v) = \begin{cases} |v_2 - u_2|, & \text{if } v_1 = u_1 \\ |u_2| + |v_2| + |v_1 - u_1|, & \text{if } v_1 \neq u_1. \end{cases}$$

for all $u = (u_1, u_2)$, and $v = (v_1, v_2)$ in $\mathbb{R}^2$. Let $\mathcal{K} := [0, 4] \times [0, 4] \subset \mathbb{R}^2$ and $T : \mathcal{K} \to \mathcal{K}$ a mapping defined by

$$T(u) = \begin{cases} (\frac{u_1}{4}, \frac{u_2}{2}), & \text{if } (u_1, u_2) \neq (4, 4) \\ (0, \frac{1}{2}), & \text{if } (u_1, u_2) = (4, 4). \end{cases}$$

If $u = (u_1, u_2), v = (v_1, v_2) \neq (4, 4)$, we easily see that $d(T(u), T(v)) \leq d(u, v)$. For $u = (u_1, u_2) \neq (4, 4)$ with $u_1 \neq 4$, and $v = (v_1, v_2) = (4, 4)$, we have

$$d(T(u), T(v)) \leq d(u, v).$$

Further, if $u = (4, u_2)$ with $0 \leq u_2 \leq \frac{5}{3}$, and $v = (4, 4)$, we have

$$d(T(u), T(v)) = \frac{3 + u_2}{2} \leq |4 - u_2| = d(u, v).$$

Finally, if $u = (4, u_2)$ with $\frac{5}{3} < u_2 < 4$, and $v = (4, 4)$, we obtain

$$\begin{aligned} \frac{1}{2} d(u, T(u)) &= \frac{1}{2} \left(u_2 + \frac{u_2}{2} + |4 - 1| \right) \\ &> \frac{11}{4} > \frac{7}{3} > d(u, v). \end{aligned}$$

Also $\frac{1}{2}d(v,T(v)) = \frac{11}{4} > \frac{7}{3} > d(u,v)$. Hence T satisfies the condition (C).

On the other hand at $u = (4,2)$ and $v = (4,4)$, we have

$$d(T(u),T(v)) = \frac{5}{2} > 2 = d(u,v)$$

and T is not nonexpansive.

4. **Convergence results**

In this section, we discuss some convergence results in a partially ordered hyperbolic metric space for S-iteration process. The following proposition is analogous to [31, Proposition 2].

Proposition 4.1. *Let $(\mathcal{M},d,\preceq)$ be a partially ordered hyperbolic metric space and $\mathcal{K}$ a nonempty subset of $\mathcal{M}$. Let $T:\mathcal{K}\to\mathcal{K}$ be a monotone mapping satisfying the condition (C) with a fixed point $w\in\mathcal{K}$. Then T is quasi-nonexpansive, that is, $d(T(u),w) \le d(u,w)$ for all $u\in\mathcal{K}$ and $w\in F(T)$ such that u is comparable with $T(u)$ and w.*

We also need the following lemma to prove our next theorem.

Lemma 4.2 ([16])**.** *Let $(\mathcal{M},d)$ be a uniformly convex hyperbolic metric space with monotone modulus of uniform convexity δ. Let $z\in\mathcal{M}$ and $\{\alpha_n\}$ be a sequence such that $0<a\le\alpha_n\le b<1$. If $\{u_n\}$ and $\{v_n\}$ are sequences in $\mathcal{M}$ such that $\limsup_{n\to\infty} d(u_n,z)\le r$, $\limsup_{n\to\infty} d(v_n,z)\le r$ and $\lim_{n\to\infty} d(\alpha_n v_n \oplus (1-\alpha_n)u_n, z) = r$ for some $r\ge 0$, then we have $\lim_{n\to\infty} d(v_n,u_n)=0$.*

Theorem 4.3. *Let $(\mathcal{M},d,\preceq)$ be a uniformly convex partially ordered hyperbolic metric space and $\mathcal{K}$ a nonempty closed convex subset of $\mathcal{M}$. Let $T:\mathcal{K}\to\mathcal{K}$ be a monotone mapping satisfying the condition (C). Assume that there exists $u_1\in\mathcal{K}$ such that u_1 and $T(u_1)$ are comparable. Suppose $F(T)$ is nonempty and u_1 and z are comparable for every $z\in F(T)$. Let $\{u_n\}$ be a sequence defined by (2.1). Then the following assertions hold:*

- (i) *the sequence $\{u_n\}$ is bounded;*
- (ii) $\max\{d(u_{n+1},z), d(v_n,z)\}\le d(u_n,z)$ *for all* $n\in\mathbb{N}$;
- (iii) $\lim_{n\to\infty} d(u_n,z)$ *exists and* $\lim_{n\to\infty} \operatorname{dist}(u_n,F(T))$ *exists;*
- (iv) $\lim_{n\to\infty} d(T(u_n),u_n)=0$.

Proof. Without loss of generality we may assume that $u_1\preceq z$. Since T is monotone, $T(u_1)\preceq T(z)=z$. By (2.1), (as in (3.12)), we have

$$u_1 \preceq v_1 \preceq T(u_1) \preceq u_2 \preceq T(v_1), \tag{4.1}$$

and $v_1\preceq z$. By monotonicity of T,

$$T(v_1)\preceq T(z) = z.$$

From (4.1), we have

$$u_1 \preceq v_1 \preceq T(u_1) \preceq u_2 \preceq z.$$

Since T is monotone, $T(u_2) \preceq T(z) = z$. Then again from (3.13), for $n=2$, we have

$$u_2 \preceq T(u_2) \preceq z.$$

Continuing in this way, we get

$$u_n \preceq T(u_n) \preceq z.$$

By (2.1) and Proposition 4.1, we have

$$\begin{aligned} d(v_n, z) &= d(\gamma_n T(u_n) \oplus (1-\gamma_n)u_n, z) \\ &\leq \gamma_n d(T(u_n), z) + (1-\gamma_n) d(u_n, z) \\ &\leq \gamma_n d(u_n, z) + (1-\gamma_n) d(u_n, z) \\ &= d(u_n, z). \end{aligned} \tag{4.2}$$

Further, by (2.1), (4.2) and Proposition 4.1, we have

$$\begin{aligned} d(u_{n+1}, z) &= d(\beta_n T(v_n) \oplus (1-\beta_n)T(u_n), z) \\ &\leq \beta_n d(T(v)_n, z) + (1-\beta_n) d(T(u_n), z) \\ &\leq \beta_n d(v_n, z) + (1-\beta_n) d(u_n, z) \\ &\leq \beta_n d(u_n, z) + (1-\beta_n) d(u_n, z) \\ &= d(u_n, z). \end{aligned}$$

Thus the sequence $\{d(u_n, z)\}$ is bounded and decreasing so $\lim_{n\to\infty} d(u_n, z)$ exists. For each $z \in F(T)$ and $n \in \mathbb{N}$ we have $d(u_{n+1}, z) \leq d(u_n, z)$. Taking infimum over all $z \in F(T)$, we get $\text{dist}(u_{n+1}, F(T)) \leq \text{dist}(u_n, F(T))$ for all $n \in \mathbb{N}$. So the sequence $\text{dist}(u_n, F(T))$ is bounded and decreasing. Therefore, $\lim_{n\to\infty} \text{dist}(u_n, F(T))$ exists. Suppose

$$\lim_{n\to\infty} d(u_n, z) = r. \tag{4.3}$$

From (4.3) and Proposition 4.1, we have

$$\limsup_{n\to\infty} d(T(u_n), z) \leq r. \tag{4.4}$$

By (4.2) and (4.3), we have

$$\limsup_{n\to\infty} d(v_n, z) \leq r. \tag{4.5}$$

Using (4.5) and Proposition 4.1, we get

$$\limsup_{n\to\infty} d(T(v_n), z) \leq r. \tag{4.6}$$

By (2.1), we have

$$r = \lim_{n\to\infty} d(u_{n+1}, z) = \lim_{n\to\infty} d((1-\beta_n)T(u_n) \oplus \beta_n T(v_n), z). \tag{4.7}$$

In view of (4.4), (4.6), (4.7) and Lemma 4.2, we get

$$\lim_{n\to\infty} d(T(v_n), T(u_n)) = 0. \tag{4.8}$$

Again by (2.1), we have

$$\begin{aligned} d(u_{n+1}, T(u_n)) &= d((1-\beta_n)T(u_n) \oplus \beta_n T(v_n), T(u_n)) \\ &\le \beta_n d(T(v_n), T(u_n)). \end{aligned}$$

By (4.8) and letting $n \to \infty$, we get

$$\lim_{n\to\infty} d(u_{n+1}, T(u_n)) = 0. \tag{4.9}$$

By the triangle inequality, we have

$$d(u_{n+1}, T(v_n)) \le d(u_{n+1}, T(u_n)) + d(T(v_n), T(u_n)).$$

By (4.8) and (4.9), we get $\lim_{n\to\infty} d(u_{n+1}, T(v_n)) = 0$. Now, we observe that

$$\begin{aligned} d(u_{n+1}, z) &\le d(u_{n+1}, T(v_n)) + d(T(v_n), z) \\ &\le d(u_{n+1}, T(v_n)) + d(v_n, z), \end{aligned}$$

which yields

$$r \le \liminf_{n\to\infty} d(v_n, z). \tag{4.10}$$

From (4.5) and (4.10), we get

$$r = \lim_{n\to\infty} d(v_n, z) = \lim_{n\to\infty} d((1-\gamma_n)u_n \oplus \gamma_n T(u_n), z). \tag{4.11}$$

Finally, from (4.3), (4.4), (4.11) and Lemma 4.2, we conclude that $\lim_{n\to\infty} d(T(u_n), u_n) = 0$. □

Now we present a result for Δ-convergence.

Theorem 4.4. *Let $(\mathcal{M}, d, \preceq)$ be a uniformly convex partially ordered hyperbolic metric space and $\mathcal{K}$ a nonempty closed convex subset of $\mathcal{M}$. Let $T : \mathcal{K} \to \mathcal{K}$ be a monotone mapping satisfying the condition (C). Assume that there exists $u_1 \in \mathcal{K}$ such that u_1 and $T(u_1)$ are comparable, $F(T)$ is nonempty and totally ordered. Then the sequence $\{u_n\}$ defined by (2.1) Δ-converges to a fixed point of T.*

Proof. By Theorem 4.3, $\{u_n\}$ is a bounded sequence. Therefore there exists a subsequence $\{u_{n_j}\}$ of $\{u_n\}$ such that $\{u_{n_j}\}$ Δ-converges to some $p \in \mathcal{K}$. By using Lemma 3.10, we have

$$u_1 \preceq u_{n_j} \preceq p \text{ for all } j \in \mathbb{N}.$$

Now we show that every Δ-convergent subsequence of $\{u_n\}$ has a unique Δ-limit in $F(T)$. Arguing by contradiction suppose $\{u_n\}$ has two subsequences

$\{u_{n_j}\}$ and $\{u_{n_k}\}$ Δ-converging to p and q, respectively. By Theorem 4.3, $\{u_{n_j}\}$ is bounded and

$$\lim_{j\to\infty} d(T(u_{n_j}), u_{n_j}) = 0. \tag{4.12}$$

We claim that $p \in F(T)$. Let $\tau : \mathcal{K} \to [0,\infty)$ be a type function generated by $\{u_{n_j}\}$, that is,

$$\tau(u) = \limsup_{j\to\infty} d(u_{n_j}, u).$$

By Lemma 3.11 and (4.12), we have

$$\begin{aligned}\tau(T(p)) &= \limsup_{j\to\infty} d(u_{n_j}, T(p)) \\ &\leq 3 \limsup_{j\to\infty} d(u_{n_j}, T(u_{n_j})) + \limsup_{j\to\infty} d(u_{n_j}, p) \\ &\leq \tau(p).\end{aligned}$$

By the uniqueness of element p and definition of Δ-convergence, $T(p) = p$. Similarly, $T(q) = q$. By the definition of Δ-convergence and Lemma 3.5, we have

$$\begin{aligned}\limsup_{n\to\infty} d(u_n, p) &= \limsup_{j\to\infty} d(u_{n_j}, p) < \limsup_{j\to\infty} d(u_{n_j}, q) \\ &= \limsup_{n\to\infty} d(u_n, q) = \limsup_{k\to\infty} d(u_{n_k}, q) \\ &< \limsup_{k\to\infty} d(u_{n_k}, p) = \limsup_{n\to\infty} d(u_n, p),\end{aligned}$$

which is a contradiction, unless $p = q$. □

Next we present a strong convergence theorem.

Theorem 4.5. *Let $(\mathcal{M}, d, \preceq)$ be a uniformly convex partially ordered hyperbolic metric space, $\mathcal{K}$, T and $\{u_n\}$ be the same as in Theorem 4.3 with $F(T) \neq \emptyset$. Then the sequence $\{u_n\}$ defined by (2.1) converges strongly to a fixed point of T if and only if $\liminf_{n\to\infty} \operatorname{dist}(u_n, F(T)) = 0$, provided $F(T)$ is a totally ordered set.*

Proof. Suppose that $\liminf_{n\to\infty} \operatorname{dist}(u_n, F(T)) = 0$. From Theorem 4.3, $\lim_{n\to\infty} \operatorname{dist}(u_n, F(T))$ exists, so

$$\lim_{n\to\infty} \operatorname{dist}(u_n, F(T)) = 0. \tag{4.13}$$

First, we show that the set $F(T)$ is closed. For this, let $\{z_n\}$ be a sequence in $F(T)$ converging strongly to a point $w \in \mathcal{K}$. Since $\frac{1}{2} d(z_n, T(z_n)) = 0 \leq d(z_n, w)$ for all $n \in \mathbb{N}$, we have

$$\begin{aligned}\limsup_{n\to\infty} d(z_n, T(w)) &= \limsup_{n\to\infty} d(T(z_n), T(w)) \\ &\leq \limsup_{n\to\infty} d(z_n, w) = 0.\end{aligned}$$

Thus $\{z_n\}$ converges strongly to $T(w)$. This implies that $T(w) = w$. Therefore $F(T)$ is closed. In view of (4.13), let $\{u_{n_j}\}$ be a subsequence of sequence $\{u_n\}$ such that $d(u_{n_j}, z_j) \leq \frac{1}{2^j}$ for all $j \geq 1$, where $\{z_j\}$ is a sequence in $F(T)$. By Theorem 4.3, we have

$$d(u_{n_{j+1}}, z_j) \leq d(u_{n_j}, z_j) \leq \frac{1}{2^j}. \tag{4.14}$$

Now, by the triangle inequality and (4.14), we have

$$\begin{aligned} d(z_{j+1}, z_j) &\leq d(z_{j+1}, u_{n_{j+1}}) + d(u_{n_{j+1}}, z_j) \\ &\leq \frac{1}{2^{j+1}} + \frac{1}{2^j} < \frac{1}{2^{j-1}}. \end{aligned}$$

A standard argument shows that $\{z_j\}$ is a Cauchy sequence. Since $F(T)$ is closed, so $\{z_j\}$ converges to some point $z \in F(T)$. Now

$$d(u_{n_j}, z) \leq d(u_{n_j}, z_j) + d(z_j, z).$$

Letting $j \to \infty$ implies that $\{u_{n_j}\}$ converges strongly to z. By Lemma 4.3, $\lim_{n\to\infty} d(u_n, z)$ exists. Hence $\{u_n\}$ converges strongly to z. The converse part is obvious. □

Theorem 4.6. *Let $(\mathcal{M}, d, \preceq)$ be a uniformly convex partially ordered hyperbolic metric space, $\mathcal{K}$, T and $\{u_n\}$ be the same as in Theorem 4.3. Let T satisfy the condition (I) and $F(T) \neq \emptyset$. Then $\{u_n\}$ converges strongly to a fixed point of T.*

Proof. From Theorem 4.3, it follows that

$$\liminf_{n\to\infty} d(T(u_n), u_n) = 0. \tag{4.15}$$

Since T satisfies condition (I), we have $d(T(u_n), u_n) \geq f(\text{dist}(u_n, F(T)))$. From (4.15), we get

$$\liminf_{n\to\infty} f(\text{dist}(u_n, F(T))) = 0.$$

Since $f : [0, \infty) \to 0, \infty)$ is a nondecreasing function with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0, \infty)$, we have

$$\liminf_{n\to\infty} \text{dist}(u_n, F(T)) = 0.$$

Therefore all the assumptions of Theorem 4.5 are satisfied and $\{u_n\}$ converges strongly to a fixed point of T. □

Remark 4.7. The above theorem is also true when $\mathcal{K}$ is a compact subset of a hyperbolic metric space $\mathcal{M}$.

Acknowledgments

We are very much thankful to the reviewer and the editor for their constructive comments and suggestions which have been useful for the improvement of this paper.

The third author is thankful to the Ministry of Education, Science and Technological Development of Serbia, Grant No. 174002.
The fourth author is thankful to the USIEF, New Delhi, India and IIE/CIES, USA for Fulbright-Nehru PDF Award (No. 2052/FNPDR/2015).

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