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AN EXTENSION OF THE WEDDERBURN-ARTIN THEOREM

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Abstract. In this paper we give conditions under which a ring is isomorphic to a structural matrix ring over a division ring.
Keywords: Duo, artinian, distributive, uniserial, structural matrix.
MSC(2010): Primary: 16D60; Secondary: 16G10, 16P20, 16S50.

1. **Introduction**

Let D a ring, $\mathcal{A}$ a nonempty set and $\mathcal{H} \subseteq \mathcal{A} \times \mathcal{A}$ a quasi-ordering on $\mathcal{A}$. Set

$$\mathrm{M}_{\mathcal{H}}(D) = \{A \in \mathrm{M}_{\mathcal{A}\times\mathcal{A}}(D) \mid \forall (i,j) \notin \mathcal{H}, A_{ij} = 0\},$$

which is called a structural matrix ring. By the Wedderburn-Artin Theorem, every primitive Artinian ring is isomorphic to $\mathrm{M}_{\mathcal{H}}(D)$ for a division ring D, a finite set $\mathcal{A}$ and $\mathcal{H} = \mathcal{A} \times \mathcal{A}$. In this paper we generalize this theorem to every quasi-ordering $\mathcal{H}$ and weaken the primitiveness condition. Also we determine the relation between the structure of $\mathcal{H}$ and certain ideals of the ring. Structural matrix rings have been investigated since they provide examples and counterexamples in ring theory, and for their connection to **PI** algebra. Structural matrix rings include the ring of triangular matrices and the ring of blocked triangular matrices, as well as the complete matrix rings when $\mathcal{H}$ is chosen appropriately. $\mathrm{M}_{\mathcal{H}}(D)$ has been studied in [2,3,5] and [6].

In this paper, for any additive groups U, V and W, any $X \subseteq U$, $Z \subseteq W$ and multiplication $U \times V \longrightarrow W$, we set $(Z : X) = \{v \in V \mid Xv \subseteq Z\}$ and $\mathrm{ann}_V(X) = \{v \in V \mid Xv = 0\}$. For the case $V \times V \longrightarrow W$, we set $(Z : X)_{\mathrm{r}} = \{v \in V \mid Xv \subseteq Z\}$ and $\mathrm{ann}_{\mathrm{r}}(X) = \{v \in V \mid Xv = 0\}$. For any family S of subsets of a set, we set $\mathrm{Int}(S) = \bigcap_{I \in S} I$ and $\mathrm{Un}(S) = \bigcup_{I \in S} I$.

Also, for a class $\mathcal{C}$ of subgroups and a subgroup L of an additive group, the sum of $\mathcal{C}$-subgroups not containing L is denoted by $\mathrm{Nov}_{\mathcal{C}}(L)$ and the sum of

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$\mathcal{C}$-subgroups properly contained in L is denoted by $\mathrm{Tp}_{\mathcal{C}}(L)$. If the group is a module, and $\mathcal{C}$ is the class of submdules, then we simply use the notations $\mathrm{Nov}(L)$ and $\mathrm{Tp}(L)$, respectively.

2. Preliminaries

Definition 2.1. Let $\mathcal{A}$ be a nonempty set and let $\mathcal{H} \subseteq \mathcal{A} \times \mathcal{A}$ be a quasi-ordering.

(1) For every $b \in \mathcal{A}$ we set $\mathrm{l}(b) = \{c \in \mathcal{A} \mid (c, b) \in \mathcal{H}\}$.
(2) For every $a \in \mathcal{A}$ we set $\mathrm{r}(a) = \{c \in \mathcal{A} \mid (a, c) \in \mathcal{H}\}$.
(3) We set ${}_{\Box}\mathcal{H} = \{\mathcal{B} \subseteq \mathcal{A} \mid \forall x \in \mathcal{A}(x \in \mathcal{B} \Rightarrow \mathrm{l}(x) \subseteq \mathcal{B})\}$.
(4) We set $\mathcal{H}_{\Box} = \{\mathcal{B} \subseteq \mathcal{A} \mid \forall x \in \mathcal{A}(x \in \mathcal{B} \Rightarrow \mathrm{r}(x) \subseteq \mathcal{B})\}$.
(5) $\mathcal{H}$ is called **indecomposable** if ${}_{\Box}\mathcal{H} \cap \mathcal{H}_{\Box} = \{\emptyset, \mathcal{A}\}$.
(6) $\mathcal{H}$ is called **triangular** if for every $a, b \in \mathcal{A}$, either $(a, b) \in \mathcal{H}$ or $(b, a) \in \mathcal{H}$.

It is clear that for every $a \in \mathcal{A}$ we have $a \in \mathrm{l}(a) \cap \mathrm{r}(a)$, $\mathrm{l}(a) \in {}_{\Box}\mathcal{H}$ and $\mathrm{r}(a) \in \mathcal{H}_{\Box}$.

Definition 2.2. Let D be a division ring, ${}_{D}U$ a vector space and $\mathcal{A}$ a basis for ${}_{D}U$.

(1) For every $X \subseteq U$, the subspace generated by X is denoted by $\langle X \rangle$.
(2) For every $X \subseteq U$ we set $X^* = \mathrm{Int}\{\mathcal{B} \subseteq \mathcal{A} \mid X \subseteq \langle \mathcal{B} \rangle\}$.

It is easy to see that for every $\mathcal{B}, \mathcal{C} \subseteq \mathcal{A}$, $\langle \mathcal{B} \rangle \subseteq \langle \mathcal{C} \rangle$ if and only if $\mathcal{B} \subseteq \mathcal{C}$.

Lemma 2.3. *Let $\mathcal{A}$ be a nonempty set, $\mathcal{H} \subseteq \mathcal{A} \times \mathcal{A}$ a quasi-ordering and $a \in \mathcal{A}$. Then,*

(1) $\mathrm{l}(a) - \mathrm{r}(a) \in {}_{\Box}\mathcal{H}$.
(2) *For every $\mathcal{B} \in {}_{\Box}\mathcal{H}$, $\mathcal{B} \subset \mathrm{l}(a)$ implies $\mathcal{B} \subseteq \mathrm{l}(a) - \mathrm{r}(a)$.*
(3) $\mathrm{l}(a) - \mathrm{r}(a) \subset \mathrm{l}(a)$.

Proof. Straightforward. □

Lemma 2.4. *Let $\mathcal{A}$ be a nonempty set and $\mathcal{H} \subseteq \mathcal{A} \times \mathcal{A}$ a quasi-ordering.*

(1) *${}_{\Box}\mathcal{H}$ and $\mathcal{H}_{\Box}$ are closed under intersection and union.*
(2) *For each $a \in \mathcal{A}$, $\mathrm{l}(a) = \mathrm{Int}\{\mathcal{B} \in {}_{\Box}\mathcal{H} \mid a \in \mathcal{B}\}$.*

Proof. Straightforward. □

Lemma 2.5. *Let $\mathcal{A}$ be a nonempty set, $\mathcal{H} \subseteq \mathcal{A} \times \mathcal{A}$ a quasi-ordering and $\mathcal{B} \in {}_{\Box}\mathcal{H}$. The following conditions are equivalent*

(1) *There exists $b \in \mathcal{A}$ with $\mathcal{B} = \mathrm{l}(b)$.*
(2) $\mathrm{Un}\{\mathcal{C} \in {}_{\Box}\mathcal{H} \mid \mathcal{C} \subset \mathcal{B}\} \subset \mathcal{B}$.
(3) $\mathcal{B} \nsubseteq \mathrm{Un}\{\mathcal{C} \in {}_{\Box}\mathcal{H} \mid \mathcal{B} \nsubseteq \mathcal{C}\}$.

Proof. (1) $\Rightarrow$ (3) For every $\mathcal{C} \in {}_{\square}\mathcal{H}$ with $\mathrm{l}(b) \nsubseteq \mathcal{C}$ we have $b \notin \mathcal{C}$. So $b \notin \mathrm{Un}\{\mathcal{C} \in {}_{\square}\mathcal{H} \mid \mathcal{B} \nsubseteq \mathcal{C}\}$. Thus $\mathcal{B} \nsubseteq \mathrm{Un}\{\mathcal{C} \in {}_{\square}\mathcal{H} \mid \mathcal{B} \nsubseteq \mathcal{C}\}$.

(3) $\Rightarrow$ (1) Consider $b \in \mathcal{B}$ such that $b \notin \mathrm{Un}\{C \in {}_{\square}\mathcal{H} \mid \mathcal{B} \nsubseteq \mathcal{C}\}$. If $\mathcal{B} \nsubseteq \mathrm{l}(b)$, then $\mathrm{l}(b) \in \{\mathcal{C} \in {}_{\square}\mathcal{H} \mid \mathcal{B} \nsubseteq \mathcal{C}\}$, so $b \in \mathrm{l}(b) \subseteq \mathrm{Un}\{\mathcal{C} \in {}_{\square}\mathcal{H} \mid \mathcal{B} \nsubseteq \mathcal{C}\}$ which is a contradiction. Thus $\mathcal{B} \subseteq \mathrm{l}(b)$, consequently, $\mathcal{B} = \mathrm{l}(b)$. □

Lemma 2.6. *Let $\mathcal{A}$ be a nonempty set and let $\mathcal{H}$ and $\mathcal{E}$ be quasi-orderings on $\mathcal{A}$. If ${}_{\square}\mathcal{H} = {}_{\square}\mathcal{E}$, then $\mathcal{H} = \mathcal{E}$.*

Proof. Follows from Lemma 2.5. □

Lemma 2.7. *Let $\mathcal{A}$ be a nonempty set and $Q \subseteq \mathbb{P}(\mathcal{A})$. If $\mathcal{A} \in Q$ and Q is closed under intersection and union, then there exists one and only one quasi-ordering $\mathcal{H} \subseteq \mathcal{A} \times \mathcal{A}$ such that ${}_{\square}\mathcal{H} = Q$.*

Proof. For every $a \in \mathcal{A}$ we set $G_a = \mathrm{Int}(\{N \in Q \mid a \in N\})$. Clearly for every $a \in \mathcal{A}$ and $N \in Q$, $a \in G_a \in Q$. Also $a \in N$ implies $G_a \subseteq N$. Now we set $\mathcal{H} = \{(c, a) \mid a \in \mathcal{A}, c \in G_a\}$. It is easy to see that $\mathcal{H}$ is a quasi-ordering on $\mathcal{A}$ and for every $a \in \mathcal{A}$, $\mathrm{l}(a) = G_a$ implies $G_a \in {}_{\square}\mathcal{H}$ and $\mathrm{l}(a) \in Q$. Let $N \in Q$. We have $N = \mathrm{Un}\{G_a \mid a \in \mathcal{A}\}$, so $N \in {}_{\square}\mathcal{H}$. Finally let $\mathcal{B} \in {}_{\square}\mathcal{H}$. We have $\mathcal{B} = \mathrm{Un}\{\mathrm{l}(a) \mid a \in \mathcal{B}\}$ by Lemma 2.5, implying $\mathcal{B} \in Q$. Thus ${}_{\square}\mathcal{H} = Q$. The uniqueness follows from Lemma 2.6. □

Proposition 2.8. *Let D be a division ring and ${}_DU$ a vector space. If Q is a set of subspaces such that*

1. *For every chain $P \subseteq Q$, $\mathrm{Un}(P) \in Q$.*
2. *For every $K, L \in Q$, $K \cap L \in Q$ and $K + L \in Q$.*
3. *For every $K, L, N \in Q$, $K \subseteq L + N$ and $K \cap L \subseteq N$ implies $K \subseteq N$.*
4. *Every nonempty subset of Q has a minimal element.*
5. *$0, U \in Q$.*

Then there exists a basis $\mathcal{A}$ for ${}_DU$ such that $Q \subseteq \{\langle \mathcal{C} \rangle \mid \mathcal{C} \subseteq A\}$. Also for every $K, L, N \in Q$ we have $K \cap (L + N) = K \cap L + K \cap N$.

Proof. Let T be the set of linearly independent sets $\mathcal{B} \subseteq U$ such that $\langle \mathcal{B} \rangle \in Q$ and for every $L \in Q$, $L \subseteq \langle \mathcal{B} \rangle$ implies $\langle L \cap \mathcal{B} \rangle = L$. Clearly $\emptyset \in T$, so $T \neq \emptyset$. Also for every $\mathcal{B} \in T$ and $L \in Q$ we have $L \cap \langle B \rangle \in Q$ and $L \cap \langle \mathcal{B} \rangle \subseteq \langle \mathcal{B} \rangle$, so $L \cap \langle \mathcal{B} \rangle = \langle L \cap \langle \mathcal{B} \rangle \cap \mathcal{B} \rangle = \langle L \cap \mathcal{B} \rangle$ by the nature of T.

First we show that T has a maximal member $\mathcal{A}$ by applying the Zorn Lemma. Let $P \subseteq T$ be a chain. Set $\mathcal{C} = \mathrm{Un}(P)$. We show that $\mathcal{C} \in T$. Clearly $\mathcal{C}$ is linearly idependent. Also $\langle \mathcal{C} \rangle = \mathrm{Un}\{\langle \mathcal{B} \rangle \mid \mathcal{B} \in P\}$. Let $L \in Q$ with $L \subseteq \langle \mathcal{C} \rangle$. It is enough to show that $L \subseteq \langle L \cap \mathcal{C} \rangle$. Suppose $l \in L \cap \langle \mathcal{C} \rangle$. There exists $\mathcal{B} \in P$ such that $l \in \langle \mathcal{B} \rangle$. Then $l \in L \cap \langle \mathcal{B} \rangle = \langle L \cap \mathcal{B} \rangle \subseteq \langle L \cap \mathcal{C} \rangle$.

Now we show that $\langle \mathcal{A} \rangle = U$. Assume that it is not so. The set $\{N \in Q \mid N \nsubseteq \langle \mathcal{A} \rangle\}$ has a minimal member J. There exists $\mathcal{C} \subseteq J$ such that $(J \cap \mathcal{A}) \cap \mathcal{C} = \emptyset$ and $(J \cap \mathcal{A}) \cup \mathcal{C}$ is a basis for J. It is clear that $J = \langle J \cap \mathcal{A} \rangle \oplus \langle \mathcal{C} \rangle$, $\mathcal{A} \cap \mathcal{C} = \emptyset$

and $\langle \mathcal{A} \cup \mathcal{C} \rangle = \langle \mathcal{A} \rangle + J \in Q$. Showing $\mathcal{A} \cup \mathcal{C} \in T$ completes the proof. Let $L \in Q$ with $L \subseteq \langle \mathcal{A} \cup \mathcal{C} \rangle$. We may assume that $L \nsubseteq \langle \mathcal{A} \rangle$. Then $L \cap J \nsubseteq \langle \mathcal{A} \rangle$ by the nature of Q, so $L \cap J = J$, thus $J \subseteq L$, implying $\mathcal{C} \subseteq L$. Consequently,

$$L = (L\langle L \cap \mathcal{A} \rangle \cap \langle \mathcal{A} \rangle) + J = \langle L \cap \mathcal{A} \rangle + \langle J \cap \mathcal{A} \rangle \langle \mathcal{C} \rangle =$$

$$\langle L \cap \mathcal{A} \rangle + \langle \mathcal{C} \rangle = \langle (L \cap \mathcal{A}) \cup \mathcal{C} \rangle = \langle L \cap (\mathcal{A} \cup \mathcal{C}) \rangle.$$

□

3. **Main Results**

Definition 3.1. Let D and S be rings. A bimodule ${}_D U_S$ is called *left stable* if for every $f \in \mathrm{End}(U_S)$ there exists $d \in D$ such that $f(x) = dx$ for all $x \in U$.

Definition 3.2. Let D a division ring, ${}_D U$ a vector space, $\mathcal{A}$ a basis for ${}_D U$ and $\mathcal{H} \subseteq \mathcal{A} \times \mathcal{A}$ be a quasi-ordering. We set

$$\mathrm{End}_{\mathcal{H}}({}_D U) = \{ f \in \mathrm{End}({}_D U) \mid \forall a \in \mathcal{A}, f(a) \in \langle \mathrm{l}(a) \rangle \}.$$

Lemma 3.3. *Let D be a division ring, ${}_D U$ a vector space, $\mathcal{A}$ a basis for ${}_D U$ and let $\mathcal{H} \subseteq \mathcal{A} \times \mathcal{A}$ be a quasi-ordering. If S is a subring of $\mathrm{End}(U_S)^{\mathrm{op}}$ such that $\{\langle \mathcal{B} \rangle \mid \mathcal{B} \in {}_{\square}\mathcal{H}\}$ is the set of submodules of U_S and for every $v \in U$ we have $v \in vS$, then*

1. *For every $a \in \mathcal{A}$ we have $aS \in \langle \mathrm{l}(a) \rangle$.*
2. *For every $v \in U$, $v^* \subseteq vS$.*
3. *For every $a \in \mathcal{A}$ and every submodule N, $a \in \langle \mathcal{A} - \{a\} \rangle + N$ implies $a \in N$.*
4. *For every submodule L, $L \nsubseteq \mathrm{Nov}(L)$ if and only if $L = \langle \mathrm{l}(a) \rangle$ for some $a \in \mathcal{A}$.*
5. *$\mathcal{H}$ is indecomposable if and only if U_S is an indecomposable module.*
6. *$\mathcal{H}$ is triangular if and only if U_S is a uniserial module.*
7. *$\mathcal{H} = \mathcal{A} \times \mathcal{A}$ if and only if U_S is a simple module.*

Proof. It is easy to see that N is a submodule if and only if $N \cap \mathcal{A} \subseteq {}_{\square}\mathcal{H}$ and $N = \langle N \cap \mathcal{A} \rangle$.

(1) We have $a \in \langle \mathrm{l}(a) \rangle$ and $\langle \mathrm{l}(a) \rangle$ is a submodule, so $aS \subseteq \langle \mathrm{l}(a) \rangle$. On the other hand $aS \cap \mathcal{A} \in {}_{\square}\mathcal{H}$ and $a \in aS \cap \mathcal{A}$, so $\mathrm{l}(a) \subseteq aS \cap \mathcal{A}$. Consequently, $\langle \mathrm{l}(a) \rangle \subseteq \langle aS \cap \mathcal{A} \rangle = aS$.

(2) We have $v \in vS = \langle vS \cap \mathcal{A} \rangle$, so $v^* \subseteq vS \cap \mathcal{A} \subseteq vS$.

(3) There exist $a_i \in \mathcal{A} - \{a\}$, $0 \neq d_i \in D$ and $u \in N$ such that $a = \sum_{i=1}^n d_i a_i + u$, then $a - \sum_{i=1}^n d_i a_i = u$, implying $a \in u^* \subseteq uS \subseteq N$ by (2).

(4) Follows from Lemma 2.5.

(5⇒) Let N and K be submodules, $N \neq 0$ and $U = N \oplus K$. Set $\mathcal{B} = N \cap \mathcal{A}$ and $\mathcal{C} = K \cap \mathcal{A}$. We have $\mathcal{B}, \mathcal{C} \in {}_{\square}\mathcal{H}$, $\mathcal{B} \neq \emptyset$, $\mathcal{B} \cup \mathcal{C} = \mathcal{A}$ and $\mathcal{B} \cap \mathcal{C} = \emptyset$, so $\mathcal{B} = \mathcal{A} - \mathcal{C} \in \mathcal{H}_{\square}$, thus $\mathcal{B} = \mathcal{A}$, implying $N = U$.

(5⇐) Let $\emptyset \neq \mathcal{B} \in {}_{\square}\mathcal{H} \cap \mathcal{H}_{\square}$. Set $\mathcal{C} = \mathcal{A} - \mathcal{B}$. Then $\mathcal{C} \in {}_{\square}\mathcal{H}$, so $\langle \mathcal{B} \rangle$ and $\langle \mathcal{C} \rangle$ are submodules with $U = \langle \mathcal{B} \rangle \oplus \langle \mathcal{C} \rangle$, thus $\langle \mathcal{C} \rangle = 0$, implying $\mathcal{C} = \emptyset$. Thus $\mathcal{B} = \mathcal{A}$.

(6⇒) Let N and K be submodules and $N \nsubseteq K$. Set $\mathcal{B} = N \cap \mathcal{A}$ and $\mathcal{C} = K \cap \mathcal{A}$. We have $\mathcal{B}, \mathcal{C} \in {}_{\square}\mathcal{H}$ and $\mathcal{B} \nsubseteq \mathcal{C}$. Consider $b \in \mathcal{B}$ with $b \notin \mathcal{C}$. Then $\mathrm{l}(b) \nsubseteq \mathcal{C}$, so $\mathcal{C} \subseteq \mathrm{l}(b) \subseteq \mathcal{B}$, implying $K = \langle \mathcal{C} \rangle \subseteq \langle \mathcal{B} \rangle = N$.

(6⇐) Let $a, b \in \mathcal{A}$. We have either $\langle \mathrm{l}(a) \rangle \subseteq \langle \mathrm{l}(b) \rangle$ or $\langle \mathrm{l}(b) \rangle \subseteq \langle \mathrm{l}(a) \rangle$, then $\mathrm{l}(a) \subseteq \mathrm{l}(b)$ or $\mathrm{l}(b) \subseteq \mathrm{l}(a)$, so $a \in \mathrm{l}(b)$ or $b \in \mathrm{l}(a)$.

(7⇒) For every $a \in \mathcal{A}$ we have $\mathrm{l}(a) = \mathcal{A}$, so ${}_{\square}\mathcal{H} = \{\mathcal{A}, \emptyset\}$. Thus the only submodules of U are $\langle \mathcal{A} \rangle = U$ and $\langle \emptyset \rangle = 0$.

(7⇐) Let $a \in \mathcal{A}$. We have $\langle \mathrm{l}(a) \rangle = U$ so, $\mathrm{l}(a) = \mathcal{A}$. Thus $\mathcal{H} = \mathcal{A} \times \mathcal{A}$. □

Proposition 3.4. *Assume that the conditions of Lemma 3.3 are satisfied. If ${}_DU_S$ is left stable and for every S-submodule L with $L \nsubseteq \mathrm{Nov}(L)$, every S-submodule N and every S-module homomorphism $f : L \longrightarrow U/N$, there exists a S-module homomorphism $\overline{f} : U \longrightarrow U$ such that $f(x) = \overline{f}(x) + N$, then for every S-submodule N and every finite set $P \subseteq \mathcal{A}$ we have $(N : \mathrm{ann}_S(\langle P \rangle)) = \langle P \rangle + N$.*

Proof. We use induction on $n = |P|$. It is obvious for the case $n = 0$. Now let $n \geq 1$. Consider $a \in P$, set $W = \langle P - \{a\} \rangle$ and $I = \mathrm{ann}_S(W)$. We have

$$a \in (aI : I) = W + aI \subseteq \langle \mathcal{A} - \{a\} \rangle + aI$$

by the induction hypothesis, so $a \in aI$ by Lemma 3.3. Thus, $aI = aS = \langle \mathrm{l}(a) \rangle$ by Lemma 3.3. Let $v \in (N : \mathrm{ann}_S(\langle P \rangle))$. The map $\theta : aI \longrightarrow U/N$ given by $\theta(ax) = vx + N$ is a well defined S-module homomorphism, so there exists $d \in D$ such that $dax + N = vx + N$ for all $x \in I$. Thus, $(v - da)I \subseteq N$, implying $(v - da) \in (N : I) = W + N$ by the induction hypothesis. Consequently, $v \in W + Da + N = \langle P \rangle + N$. Therefore, $(N : \mathrm{ann}_S(\langle P \rangle)) \subseteq \langle P \rangle + N$. □

Corollary 3.5. *Assume that the conditions of Proposition 3.4 are satisfied. For every finite set $P \subseteq \mathcal{A}$ and every $a \in \mathcal{A} - P$ we have $a \in \mathrm{aann}_S(\langle P \rangle)$.*

Proof. Set $N = \mathrm{aann}_S(\langle P \rangle)$. By Proposition 3.4 we have

$$a \in (N : \mathrm{ann}_S(\langle P \rangle)) = \langle P \rangle + N \subseteq \langle \mathcal{A} - \{a\} \rangle + N.$$

Thus, $a \in N$ by Lemma 3.3. □

Proposition 3.6. *Let D be a division ring, ${}_DU$ a vector space and S a right Artinian subring of $\mathrm{End}(U_S)^{\mathrm{op}}$. If*

(1) *For every $v \in U$ we have $v \in vS$.*
(2) *For every submodules K, L and N, $K \subseteq L + N$ and $K \cap L \subseteq N$ imply $K \subseteq N$.*
(3) *U_S is an Artinian Duo module.*
(4) *${}_DU_S$ is left stable .*

(5) *For every S-submodule L with $L \nsubseteq \mathrm{Nov}(L)$, every S-submodule N and every S-module homomorphism $f : L \longrightarrow U/N$, there exists a S-module homomorphism $\overline{f} : U \longrightarrow U$ such that $f(x) = \overline{f}(x) + N$.*

Then there exists a finite basis $\mathcal{A}$ for ${}_DU$ and an indecomposable quasi-ordering $\mathcal{H}$ on $\mathcal{A}$ such that $S = \mathrm{End}_{\mathcal{H}}({}_DU)^{\mathrm{op}}$ and $\{\langle \mathcal{B} \rangle \mid \mathcal{B} \in {}_\square\mathcal{H}\}$ is the set of submodules of U_S.

Proof. We denote the set of S-submodules by Q. Every S-submodule is a subspace, so there exists a basis $\mathcal{A}$ for ${}_DU$ such that $Q \subseteq \{\langle \mathcal{C} \rangle \mid \mathcal{C} \subseteq A\}$ by Proposition 2.8. Then there exists a quasi-ordering $\mathcal{H}$ on $\mathcal{A}$ such that $Q = \{\langle \mathcal{B} \rangle \mid \mathcal{B} \in {}_\square\mathcal{H}\}$ by Lemma 2.7. We claim that $\mathcal{A}$ is finite. Assume it is not so. Then, $\mathcal{A}$ contains an infinite subset $\{a_n \mid n \geq 1\}$. For every finite set $P \subseteq \mathcal{A}$, $\mathrm{ann}_U(\mathrm{ann}_S(P)) = \langle P \rangle$ by Proposition 3.4, so

$$\mathrm{ann}_S(a_1) \supset \mathrm{ann}_S(a_1, a_2) \supset \mathrm{ann}_S(a_1, a_2) \cdots ,$$

which is a contradiction. Let $a \in \mathcal{A}$. Set $I = \mathrm{ann}_S(\mathcal{A} - \{a\})$. Then, $a \in aI$ by Corollary 3.5, so there exists $r_a \in I$ with $a = ar_a$ and $(\mathcal{A} - \{a\})r_a = 0$. Now let $f \in \mathrm{End}_{\mathcal{H}}({}_DU)^{\mathrm{op}}$. Then, $f(a) \in \langle \mathrm{l}(a) \rangle = aS$ by Lemma 3.3, so there exists $s_a \in S$ with $f(a) = as_a$. Set $r = \sum_{a \in \mathcal{A}} r_a s_a$, then $f = r \in S$. Thus $\mathrm{End}_{\mathcal{H}}({}_DU)^{\mathrm{op}} \subseteq S$, consequently, $S = \mathrm{End}_{\mathcal{H}}({}_DU)^{\mathrm{op}}$. Finally, it is easy to see that $D \cong \mathrm{End}(U_S)$, so U_S is indecomposable and thus, $\mathcal{H}$ is indecomposable by Lemma 3.3. □

Lemma 3.7. *Let D be a ring with unit and let ${}_DU$ be a free module with a basis $\mathcal{A}$. There exists an isomorphism $\Delta : \mathrm{End}({}_DU) \longrightarrow \mathrm{M}_{\mathcal{A}\times\mathcal{A}}(D^{\mathrm{op}})$ such that for every $f \in \mathrm{End}({}_DU)$ and $a \in \mathcal{A}$, $f(a) = \sum_{b \in \mathcal{A}} \Delta(f)_{ba} b$. In this case, for every quasi-ordering $\mathcal{H}$ on $\mathcal{A}$ we have $\Delta(\mathrm{End}_{\mathcal{H}}({}_DU)) = \mathrm{M}_{\mathcal{H}}(D^{\mathrm{op}})$.*

Proof. Straightforward. □

Theorem 3.8. *Let R be a left Artinian ring. If there exists a nonzero faithful Artinian Duo module U such that*

(1) *For every $v \in U$ we have $v \in Rv$.*
(2) *For every submodules K, L and N, $K \subseteq L + N$ and $K \cap L \subseteq N$ implies $K \subseteq N$.*
(3) $\mathrm{End}({}_RU)$ *is a division ring.*
(4) *For every submodule L with $L \nsubseteq \mathrm{Nov}(L)$, every submodule N and every homomorphism $f : L \longrightarrow U/N$, there exists a homomorphism $\overline{f} : U \longrightarrow U$ such that $f(x) = \overline{f}(x) + N$.*

Then R is isomorphic to $\mathrm{M}_{\mathcal{H}}(D)$ for a division ring D, a nonempty finite set $\mathcal{A}$ and an indecomposable quasi -ordering $\mathcal{H}$ on $\mathcal{A}$.

Proof. Set $D = \mathrm{End}(U_{R^{\mathrm{op}}})$ and consider the ring monomorphism $\beta : R \longrightarrow \mathrm{End}({}_DU)$ given by $\beta(r)(u) = ur$ and set $S = \beta(R^{\mathrm{op}})$. Then S is a right

Artinian subring of $\mathrm{End}({}_DU)^{\mathrm{op}}$, D is a division ring and ${}_DU_S$ is a left stable bimodule. Thus $S = \mathrm{End}_{\mathcal{H}}({}_DU)^{\mathrm{op}}$ by Proposition 3.6. Applying Lemma 3.7 completes the proof. □

Notice that distributive modules in [1] satisfy (2). For more information on (3), we refer to [4].

Theorem 3.9. *Let R be a left Artinian ring. If there exists a nonzero faithful Artinian uniserial Duo module U such that*

(1) *For every $v \in U$ we have $v \in Rv$.*
(2) $\mathrm{End}({}_RU)$ *is a division ring.*
(3) *For every submodule L with $L \not\subseteq \mathrm{Nov}(L)$, every submodule N and every homomorphism $f : L \longrightarrow U/N$, there exists a homomorphism $\overline{f} : U \longrightarrow U$ such that $f(x) = \overline{f}(x) + N$.*

Then R is isomorphic to a complete blocked triangular matrix over a division ring.

Proof. Set $D = \mathrm{End}(U_{R^{\mathrm{op}}})$ and consider the ring monomorphism $\beta : R \longrightarrow \mathrm{End}({}_DU)$ given by $\beta(r)(u) = ur$ and set $S = \beta(R^{\mathrm{op}})$. Then S is a right Artinian subring of $\mathrm{End}({}_DU)^{\mathrm{op}}$, D is a division ring and ${}_DU_S$ is a left stable bimodule. Thus $S = \mathrm{End}_{\mathcal{H}}({}_DU)^{\mathrm{op}}$ and $\{\langle \mathcal{B} \rangle \mid \mathcal{B} \in {}_{\square}\mathcal{H}\}$ is the set of submodules of U_S by Proposition 3.6. Consequently, $\mathcal{H}$ is triangular by Lemma 3.3. Applying Lemma 3.7 completes the proof. □

The Wedderburn-Artin Theorem can be derived from Theorem 3.8 and Lemma 3.3.

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