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A REMARK ON A REMARK BY MACAULAY OR ENHANCING LAZARD STRUCTURAL THEOREM

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In [10] (also cf.[6]) Macaulay gave a construction which, to each monomial ideal $\mathsf{J} \subset k[X_1, \dots, X_n]$, associates a set of points $\mathsf{X} \subset k^n$ whose associated radical ideal I has, using modern lingo, J as the monomial ideal associated to its Gröbner basis — $\mathsf{J} = \mathbf{T}(\mathsf{I})$; moreover Macaulay explicitly stated a direct correspondence between the points X and the monomials $\tau \notin \mathsf{J}$ under the "Gröbner escalier" $\mathbf{N}(\mathsf{I})$.

Partial converse of the Macaulay's result appeared in the earlier research on Gröbner Technology:

- In 1981 Möller [1] introduced Duality in Computer Algebra proposing an algorithm which, for each finite set of points $\mathsf{X} \subset k^n$, computes the Gröbner basis and the "Gröbner escalier" of its associted radical ideal I.
- In 1985 Lazard [9] gave a characterization of the Gröbner basis of any ideal $\mathsf{I} \subset k[X_1, X_2]$ and such characterization is a refinement of Macaulay's result.
- In 1990 Cerlienco–Mureddu [2] gave an algorithm which, for each finite set of points $\mathsf{X} \subset k^n$, computes the "Gröbner escalier" $\mathbf{N}(\mathsf{I})$ of its associated radical ideal I and a direct correspondence between X and $\mathbf{N}(\mathsf{I})$.

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What started our investigation was the realization that

- in the common setting (simple points in $k[X_1, X_2]$) Lazard and Cerlienco–Mureddu together state exactly the converse of Macaulay's result, and
- the most elementary proof of this converse consists of direct application of Möller's Algorithm.

This gave us a reasonable strategy in order to investigate how it could be possible to generalize Lazard's Structural Theorem to more than two variables: the application of Cerlienco–Mureddu Correspondence and Möller's Algorithm together, on the same line as in the case of two variables, should automatically produce the required result.

This paper is a report on our investigation.

In Section 1 we introduce the notation and we recall the Gröebnerian results we need; Section 2 is devoted to Möller's Algorithm, Section 3 to Macaulay's Trick, Section 4 to Lazard's Structural Theorem and Section 5 to Cerlienco–Mureddu Correspondence. We are then able to state (in Section 6) the Gröebnerian Structural description of configurations of points in a plane and prove it in Section 7.

Out next step is to discuss (in Section 8) some illustrating examples in 3 variables; this is sufficient, after a deeper analysis of Cerlienco–Mureddu Correspondence (in Section 9) to state (in Section 10) and prove (in Section 11) our enhanced Lazard Structural Theorem.

Our investigation is not yet satisfactory as we will explain in Section 12.

1. **Notation and Recalls**

Let $\mathcal{P} := k[X_1, \dots, X_n]$ and $\mathcal{T}$ the semigroup generated by $\{X_1, \dots, X_n\}$,

$$\mathcal{T} := \{X_1^{a_1} \cdots X_n^{a_n} : (a_1, \dots, a_n) \in \mathbb{N}^n\};$$

if $<$ is a semigroup ordering on $\mathcal{T}$, i.e. an ordering such that

$$t_1 < t_2 \Rightarrow tt_1 < tt_2 \textit{ for each } t, t_1, t_2 \in \mathcal{T},$$

each element $f \in \mathcal{P}$ can be uniquely expressed either as

$$f = \sum_{i=0}^{\deg(f)} g_i X_n^i \in k[X_1, \dots, X_{n-1}][X_n],$$

$$g_i \in k[X_1, \dots, X_{n-1}], g_{\deg(f)} \neq 0,$$

or as a linear combination

$$f = \sum_{t \in \mathcal{T}} c(f,t)t = \sum_{i=1}^{s} c(f,t_i)t_i : c(f,t_i) \neq 0, t_i \in \mathcal{T}, t_1 > \dots > t_s$$

of terms $t \in \mathcal{T}$ with coefficients $c(f,t)$ in k; and we will denote

$\mathrm{Lp}(f) := g_{\deg(f)}$ the *leading polynomial* of f,
$\mathbf{T}(f) := t_1$ its *maximal term*,
$\mathrm{lc}(f) := c(f,t_1)$ its *leading cofficient* and
$\mathbf{M}(f) := c(f,t_1)t_1$ its *maximal monomial*.

For each set $G \subset \mathcal{P}$, $\mathbf{T}\{G\}$ denotes the set $\{\mathbf{T}(g) : g \in G\}$, and $\mathbf{T}(G)$ denotes the monomial ideal $\{\tau \mathbf{T}(g) : \tau \in \mathcal{T}, g \in G\}$ it generates. For each ideal $\mathsf{I} \subset \mathcal{P}$, we will consider not only the monomial ideal $\mathbf{T}(\mathsf{I}) = \mathbf{T}\{\mathsf{I}\}$ but also $\mathbf{G}(\mathsf{I})$, the minimal basis of $\mathbf{T}(\mathsf{I})$, and the sets

$$\mathbf{N}(\mathsf{I}) := \mathcal{T} \setminus \mathbf{T}(\mathsf{I})$$

and

$$\begin{aligned} \mathbf{B}(\mathsf{I}) &:= \{X_h t : 1 \leq h \leq n, t \in \mathbf{N}(\mathsf{I})\} \setminus \mathbf{N}(\mathsf{I}) \\ &= \mathbf{T}(\mathsf{I}) \cap (\{1\} \cup \{X_h t : 1 \leq h \leq n, t \in \mathbf{N}(\mathsf{I})\}) \end{aligned}$$

and we set $k[\mathbf{N}(\mathsf{I})] := \mathrm{Span}_k(\mathbf{N}(\mathsf{I}))$.

All these notations (and the definitions below) depend on the term-ordering $<$: all over the paper, we will choose as $<$ the *lexicographical ordering induced by* $X_1 < \dots < X_n$ defined by

$$X_1^{a_1} \dots X_n^{a_n} < X_1^{b_1} \dots X_n^{b_n} \iff \text{exists } j : a_j < b_j \text{ and } a_i = b_i \text{ for } i > j.$$

Lemma 1.1 (Buchberger). *With this notation, it holds:*

(1) $\mathcal{P} \cong \mathsf{I} \oplus k[\mathbf{N}(\mathsf{I})]$;
(2) $\mathcal{P}/\mathsf{I} \cong k[\mathbf{N}(\mathsf{I})]$;
(3) *for each $f \in \mathcal{P}$, there is a unique*

$$g := \mathrm{Can}(f, \mathsf{I}) = \sum_{t \in \mathbf{N}(\mathsf{I})} \gamma(f, t, <)t \in k[\mathbf{N}(\mathsf{I})]$$

such that $f - g \in \mathsf{I}$.
Moreover:

(a) $\mathrm{Can}(f_1, \mathsf{I}) = \mathrm{Can}(f_2, \mathsf{I}) \iff f_1 - f_2 \in \mathsf{I}$;
(b) $\mathrm{Can}(f, \mathsf{I}) = 0 \iff f \in \mathsf{I}$.

Definition 1.2. With this notation

- a *Gröbner basis* of I is any set $G \subset \mathsf{I}$ such that $\mathbf{T}(G) = \mathbf{T}\{\mathsf{I}\}$, i.e. $\mathbf{T}\{G\}$ generates the monomial ideal $\mathbf{T}(\mathsf{I}) = \mathbf{T}\{\mathsf{I}\}$;
- the *reduced Gröbner basis* of I is the set
$$\mathcal{G}(\mathsf{I}) := \{\tau - \mathrm{Can}(\tau, \mathsf{I}) : \tau \in \mathbf{G}(\mathsf{I})\};$$
- the *border basis* of I is the set
$$\mathcal{B}(\mathsf{I}) := \{\tau - \mathrm{Can}(\tau, \mathsf{I}) : \tau \in \mathbf{B}(\mathsf{I})\}.$$

Proposition 1.3 (Buchberger). *For each $f \in \mathcal{P}, f - \mathrm{Can}(f, \mathsf{I})$ has a* Gröbner representation

$$f = \sum_{i=1}^{m} p_i g_i, p_i \in \mathcal{P}, g_i \in G, \mathbf{T}(p_i)\mathbf{T}(g_i) \leq \mathbf{T}(f) \text{ for each } i,$$

in terms of any Gröbner basis $G = \{g_1, \dots, g_m\}$.

Definition 1.4. Let

$$\mathbb{L} := \{\ell_1, \dots, \ell_s\} \subset \mathcal{P}^* := \mathrm{Hom}_k(\mathcal{P}, k), \quad \mathbf{q} = \{q_1, \dots, q_s\} \subset \mathcal{P}.$$

$\mathbb{L}$ and $\mathbf{q}$ are said to be

- triangular if $\ell_i(q_j) = 0$, for each $i < j$;

- biorthogonal if $\ell_i(q_j) = \delta_{ij} = \begin{cases} 1 & \text{i}f\ i = j \\ 0 & \text{i}f\ i \neq j. \end{cases}$

Let $\mathsf{X} := \{\mathsf{a}_1, \dots, \mathsf{a}_s\} \subset k^n$ be a finite set of points, $\mathsf{a}_i := (a_{i1}, \dots, a_{in})$. We will denote by ℓ_i, for each i, the linear functional $\ell_i \in \mathcal{P}^*$ defined by

$$\ell_i(f) = f(a_{i1}, \dots, a_{in}) \textit{ for each } f(X_1, \dots, X_n) \in \mathcal{P}.$$

Moreover we will denote

$$\mathbb{L}(\mathsf{X}) := \mathrm{Span}_k(\{\ell_i, 1 \leq i \leq s\}) \subset \mathcal{P}^*,$$
$$\mathsf{I}(\mathsf{X}) := \{f \in \mathcal{P} : f(\mathsf{a}_i) = 0, \textit{ for each } i\},$$

so that $\mathbb{L}(\mathsf{X})$ and $\mathsf{I}(\mathsf{X})$ are dual.

In particular

Lemma 1.5 (Lagrange Interpolation Formula)**.** [1]

There exists (and it is possible to compute) a set $\mathbf{q} = \{q_1, \dots, q_s\} \subset \mathcal{P}$ *such that*

(1) $q_i = \mathrm{Can}(q_i, \mathsf{I}(\mathsf{X})) \in \mathrm{Span}_k(\mathbf{N}(\mathsf{I}(\mathsf{X})))$;
(2) $\mathbb{L}(\mathsf{X})$ *and* $\mathbf{q}(\mathsf{X})$ *are triangular;*
(3) $\mathcal{P}/\mathsf{I}(\mathsf{X}) \cong \mathrm{Span}_k(\mathbf{q}(\mathsf{X}))$.

This allows to compute a set $\mathbf{q}'(\mathsf{X}) = \{q'_1, \dots, q'_s\} \subset \mathcal{P}$ *such that*

(1) $q'_i = \mathrm{Can}(q'_i, \mathsf{I}(\mathsf{X})) \in \mathrm{Span}_k(\mathbf{N}(\mathsf{I}(\mathsf{X})))$;
(2) $\mathbb{L}(\mathsf{X})$ *and* $\mathbf{q}'(\mathsf{X})$ *are biorthogonal;*
(3) $\mathcal{P}/\mathsf{I}(\mathsf{X}) \cong \mathrm{Span}_k(\mathbf{q}'(\mathsf{X}))$.

Let $c_1, \dots, c_s \in k$ *and let* $q := \sum_i c_i q'_i \in \mathcal{P}$*. Then, if* $\{g_1, \dots, g_t\}$ *denotes a Gröbner basis of* $\mathsf{I}(\mathsf{X})$*, one has*

(1) q *is the unique polynomial in* $\mathrm{Span}_k(\mathbf{N}(\mathsf{I}))$ *such that* $q(\mathsf{a}_i) = c_i$*, for each* i;
(2) *for each* $p \in \mathcal{P}$ *the following are equivalent*
   (a) $p(\mathsf{a}_i) = c_i$*, for each* i,
   (b) $q = \mathrm{Can}(p, \mathsf{I}(\mathsf{X}))$,
   (c) *exist* $h_j \in \mathcal{P}$ *such that*

$$p = q + \sum_{j=1}^{t} h_j g_j, \mathbf{T}(h_j)\mathbf{T}(g_j) \leq \mathbf{T}(p - q).$$

$\square$

2. **Möller Algorithm**

Duality for 0-dimension ideals was introduced in Computer Algebra by Möller in [1] and [11] (cf. also [4]) where he introduced Lemma 1.5 and solved the related computational problem:

Problem 2.1. *Given a finite set of points* $\mathsf{X} := \{\mathsf{a}_1, \dots, \mathsf{a}_s\} \subset k^n$ *compute* $\mathbf{q}(\mathsf{X})$ *and* $\mathcal{B}(\mathsf{I}(\mathsf{X}))$.

As it is usual with elementary linear algebra algorithms, there are many ways to perform Möller Algorithm; the version we describe here is a restatement of *Alg. 2 (variant)* presented in [11] pp. 127-128.

Algorithm 2.2 (Möller)**.** After having remarked that, if $s = \#(\mathsf{X}) = 1$, the problem has the obvious solution

$$\mathbf{q}(\mathsf{X}) = \{1\} \text{ and } \mathcal{B}(\mathsf{I}(\mathsf{X})) = \{X_h - a_{h1} : 1 \le h \le n\},$$

the algorithm can be described by induction on $\#(\mathsf{X}) = s$.

Let us therefore denote $\mathsf{X}' := \{\mathsf{a}_1, \dots, \mathsf{a}_{s-1}\}$ and let us assume, by induction, to already know $\mathbf{q}(\mathsf{X}') = \{q_1, \dots, q_{s-1}\}$ and $\mathcal{B}(\mathsf{I}(\mathsf{X}'))$; to simplify the notation, for each $\tau \in \mathbf{B}(\mathsf{I}(\mathsf{X}'))$ we will denote $b_\tau := \tau - \mathrm{Can}(\tau, \mathsf{I}(\mathsf{X}'))$ so that

$$\mathcal{B}(\mathsf{I}(\mathsf{X}')) = \{b_\tau : \tau \in \mathbf{B}(\mathsf{I}(\mathsf{X}'))\}.$$

The algorithm then performs the computations sketched in Figure 1

FIGURE 1. Möller Algorithm for point evaluation

$r := 1, \mathsf{B} := \emptyset$
$t_1 := 1, \mathbf{N} := \{t_1\}, q_1 := t_1,\ \mathbf{q} := \{q_1\},$
For $h = 1..n$ **do**
  $t := X_h, b_t := X_h - a_{h1}, \mathbf{B} := \mathbf{B} \cup \{t\}$
While $r \leq s$ **do**
  Let $t := \min_<\{t \in \mathbf{B} : b_t(\mathsf{a}_{r+1}) \neq 0\}$
  $\mathbf{B} := \mathbf{B} \setminus \{t\},$
  $r := r + 1$
  $t_r := t, \mathbf{N} := \mathbf{N} \cup \{t_r\},\ q_r := b_t(\mathsf{a}_r)^{-1} b_t, \mathbf{q} := \mathbf{q} \cup \{q_r\},$
  %% $\ell_i(q_j) = \delta_{ij}, \forall i \leq j \leq r$
  For each $\tau \in \mathbf{B}$ **do**
    $b_\tau := b_\tau - b_\tau(\mathsf{a}_r) q_r,$
    %% $\ell_r(b_\tau) = 0, b_\tau \in \mathsf{I}(\mathsf{X})$
  %% $\mathbf{B}(\mathsf{I}(\mathsf{X})) = \mathbf{B}(\mathsf{I}(\mathsf{X}')) \setminus \{t_r\} \cup \{X_h t_r, 1 \leq h \leq n\}$
  For $h = 1..n$ **do**
    If $X_h t_r \notin \mathbf{B}$ **then**
      $t := X_h t_r,$
      $f := X_h b_{t_r} - \sum_{\substack{\tau \in \mathbf{N} \\ X_h \tau \in \mathbf{B}}} c(b_{t_r}, \tau) b_{X_h \tau}$
      $b_t := f - f(\mathsf{a}_r) q_r$
      %% $b_t \in \mathsf{I}(\mathsf{X}), b_t - t \in \mathrm{Span}_k(\mathbf{N}(\mathsf{I}(\mathsf{X}))), b_t = t - \mathrm{Can}(t, \mathsf{I}(\mathsf{X}))$
    $\mathbf{B} := \mathbf{B} \cup \{X_h t_r, h = 1..n\}$
  %% $\mathbf{B} = \mathbf{B}(\mathsf{I}(\mathsf{X})),\ \mathcal{B}(\mathsf{I}(\mathsf{X})) = \{b_\tau : \tau \in \mathbf{B}(\mathsf{I}(\mathsf{X}))\}$
$\mathbf{q}, \{b_\tau : \tau \in \mathbf{B}\}$

Example 2.3. Let us consider the set $\mathsf{Y} := \{\mathsf{a}_i, 1 \le i \le 6\}$ where

$$\begin{array}{lllllllll} \mathsf{a}_1 & = & (0,0) & \mathsf{a}_2 & = & (0,1) & \mathsf{a}_3 & = & (2,0) \\ \mathsf{a}_4 & = & (0,2) & \mathsf{a}_5 & = & (1,0) & \mathsf{a}_6 & = & (1,1) \end{array}$$

If we denote, for $j \le 6$,

$$\mathsf{Y}_j := \{\mathsf{a}_i, 1 \le i \le j\}, \quad \mathcal{B}_j := \mathcal{B}(\mathsf{I}(\mathsf{Y}_j), \quad \mathcal{G}_j := \mathcal{B}(\mathsf{I}(\mathsf{Y}_j).$$

Möller Algorithm returns in each iteration:

(0,0) $t_1 := 1$, $q_1 := 1$, and
$\mathcal{B}_1 := \{X_1, X_2\}$;
(0,1) $t_2 = X_2$, $b_{t_2} = X_2$, $q_2 := X_2$,
$X_1 b_{t_2} - X_1X_2 = 0, X_2 b_{t_2} - X_2^2 = 0,$
$\ell_2(X_1) = 0, \ell_2(X_1X_2) = 0, \ell_2(X_2^2) = 1,$
$\mathcal{B}_2 = \{X_1, \mathtt{X_1X_2}, X_2^2 - X_2\}$;
(2,0) $t_3 := X_1$, $b_{t_3} = X_1$, $q_3 := \frac{1}{2}X_1$,
$X_1 b_{t_3} - X_1^2 = 0,$
$\ell_3(X_1^2) = 4$, $\ell_3(b_{X_1X_2}) = 0$, $\ell_3(b_{X_2^2}) = 0$,
$\mathcal{B}_3 = \{X_1^2 - 2X_1, X_1X_2, X_2^2 - X_2\}$;
(0,2) $t_4 := X_2^2$, $b_{t_4} := X_2^2 - X_2$, $q_4 := \frac{1}{2}X_2^2 - \frac{1}{2}X_2$,
$X_1 b_{t_4} - X_1X_2^2 = -X_1X_2 \equiv 0, X_2 b_{t_4} - X_2^3 = -X_2^2 \equiv -X_2 \bmod \mathsf{I}(\mathsf{Y}_3)$
$\ell_4(b_{X_1^2}) = 0$, $\ell_4(b_{X_1X_2}) = 0$, $\ell_4(X_1X_2^2) = 0$, $\ell_4(X_2^3 - X_2) = 6$,
$\mathcal{B}_4 = \{X_1^2 - 2X_1, X_1X_2, \mathtt{X_1X_2^2}, X_2^3 - 3X_2^2 + 2X_2\}$;
(1,0) $t_5 = X_1^2$, $b_{t_5} := X_1^2 - 2X_1$, $q_5 = -X_1^2 + 2X_1$,
$X_1 b_{t_5} - X_1^3 = -2X_1^2 \equiv -4X_1, X_2 b_{t_5} - X_1^2X_2 = -2X_1X_2 \equiv 0 \bmod \mathsf{I}(\mathsf{Y}_4)$
$\ell_5(X_1^3 - 4X_1) = -3$, $\ell_5(b_{X_1X_2}) = 0$, $\ell_5(X_1^2X_2) = 0$,
$\ell_5(b_{X_1X_2^2}) = 0$, $\ell_5(b_{X_2^3}) = 0$,
$\mathcal{B}_5 = \{X_1^3 - 3X_1^2 + 2X_1, X_1X_2, \mathtt{X_1^2X_2}, \mathtt{X_1X_2^2}, X_2^3 - 3X_2^2 + 2X_2\}$;
(1,1) $t_6 = X_1X_2$, $b_{t_6} := X_1X_2$, $q_6 := X_1X_2$,
$\ell_6(b_{X_1^3}) = 0$, $\ell_6(b_{X_1^2X_2}) = 1$, $\ell_6(b_{X_1X_2^2}) = 1$, $\ell_6(b_{X_2^3}) = 0$,
$\mathcal{B}_6 = \{X_1^3 - 3X_1^2 + 2X_1, X_1^2X_2 - X_1X_2, X_1X_2^2 - X_1X_2, X_2^3 - 3X_2^2 + 2X_2\}$

where the `typewriter` polynomials are members of $\mathcal{B}_i \setminus \mathcal{G}_i$.

□

3. **Macaulay's Trick**

As it was recalled in [12], Macaulay ([10]) solved the following problem:

Problem 3.1. *Given a finite order ideal*[1]*,* $\mathbf{N} \subset \mathcal{T}$ *of the semigroup* $\mathcal{T}$*, to produce an ideal* $\mathsf{I} \subset \mathcal{P}$ *which has as many separate roots as there are elements in* $\mathbf{N}$ *and which satisfies* $\mathbf{N}(\mathsf{I}) = \mathbf{N}$*.*

We can state Macaulay's Trick, to solve this problem in the following way.

Let $\mathbf{G} := \{m_1, \dots , m_r\}$ be the minimal basis of the monomial ideal $\mathcal{T} \setminus \mathbf{N}$, where

$$m_l = X_1^{e_{1l}} \cdots X_n^{e_{nl}}, \text{ for each } l.$$

Since $\mathbf{N}$ is finite,

$$\textit{for each } i, \text{ exists } d_i : X_i^{d_i} \in \mathbf{G} \text{ and } e_{il} \le d_i, \textit{ for each } l.$$

Let us then take, for each $i, j, k, j \neq k$, elements

$$a_{ij} \in k, 1 \le i \le n, 0 \le j < d_i : a_{ij} \neq a_{ik},$$

and let us define, for each $l, 1 \le l \le r$,

$$g_l := \prod_{i=1}^{n} \prod_{j=0}^{e_{il}-1} (X_i - a_{ij}),$$

which is such that $\mathbf{T}(g_l) = m_l$.

Moreover, to each term $t = X_1^{e_1} \cdots X_n^{e_n} \in \mathbf{N}$ let us associate the affine point

$$\mathsf{a}(t) := (a_{1e_1}, \dots a_{ne_n}) \in k^n,$$

and let $\mathsf{X} := \{\mathsf{a}(t) : t \in \mathbf{N}\}$. Then:

Theorem 3.2 (Macaulay)**.** [10] *Under this notation, for any degree-compatible term-ordering, it holds*

(1) $\mathbf{N} = \mathbf{N}(\mathsf{I}(\mathsf{X}))$,

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[1] A subset $\mathbf{N} \subset \mathcal{T}$ of a semigroup $\mathcal{T}$ is called an order ideal if it satisfies

$$st \in \mathbf{N} \Longrightarrow t \in \mathbf{N}, \textit{ for each } s, t \in \mathcal{T}.$$

(2) $\mathcal{G}(\mathsf{I}(\mathsf{X})) = \{g_1, \dots, g_r\}$.

□

Since

$$e_i \le d_i, \text{ for each } t = X_1^{e_1} \cdots X_n^{e_n} \in \{X_j\tau : 1 \le j \le n, \tau \in \mathbf{N}\} \text{ and each } i,$$

it is natural to consider also the polynomials

$$g_t := \prod_{i=1}^{n} \prod_{j=0}^{e_i-1} (X_i - a_{ij}), \quad t = X_1^{e_1} \cdots X_n^{e_n} \in \{X_j\tau : 1 \le j \le n, \tau \in \mathbf{N}\}$$

and investigate their relation with the notions already introduced.

Let us order $\mathbf{N} := \{t_1, \dots, t_s\}$ in such a way that $t_1 < t_2 < \cdots < t_s$, where $<$ is the lexicographical ordering induced by $X_1 < \cdots < X_n$, and let us write $\mathsf{a}_i := \mathsf{a}(t_i)$ in order to fix a suitable enumeration of X and $\mathbb{L}(\mathsf{X})$. Moreover let us define $q_i := g_{t_i}$, for each $i, 1 \le i \le s$. Then

Lemma 3.3. *It holds*

(1) $\mathcal{B}(\mathsf{I}(\mathsf{X})) = \{g_t : t \in \mathbf{B}(\mathsf{I}(\mathsf{X}))\}$,
(2) $\mathcal{G}(\mathsf{I}(\mathsf{X})) = \{g_t : t \in \mathbf{G}(\mathsf{I}(\mathsf{X}))\}$,
(3) $\mathbf{q}(\mathsf{X}) = \{q_i : 1 \le i \le s\}$.

□

Example 3.4. Let us give an example considering $\mathcal{P} = k[X_1, X_2, X_3]$, $\mathcal{T}$ ordered by the lexicographical ordering $<$ induced by $X_1 < X_2 < X_3$ and

$$\begin{aligned} \mathbf{N} \ = \ \{&1, X_1, X_1^2, X_2, X_1X_2, X_1^2X_2, X_2^2, X_1X_2^2, X_2^3, X_1X_2^3, X_3, \\ &X_1X_3, X_1^2X_3, X_2X_3, X_2^2X_3, X_2^3X_3, X_3^2\}. \end{aligned}$$

Choosing $a_{ij} = j$, for each i, j, here is the ordered list of the elements

$$t \in (\{1\} \cup \{X_j\tau : 1 \le j \le n, \tau \in \mathbf{N}\}) :$$

1: $t_1 = 1 \in \mathbf{N}, \mathsf{a}_1 = (0,0,0),\ 1 = g_1 = q_1$;
X_1: $t_2 = X_1 \in \mathbf{N}, \mathsf{a}_2 = (1,0,0), X_1 = g_{X_1} = q_2$;
X_1^2: $t_3 = X_1^2 \in \mathbf{N}, \mathsf{a}_3 = (2,0,0), X_1(X_1 - 1) = g_{X_1^2} = q_3$;

X_1^3: $X_1^3 \in \mathbf{G}, X_1(X_1-1)(X_1-2) = g_{X_1^3} \in \mathcal{G}(\mathsf{I})$;

X_2: $t_4 = X_2 \in \mathbf{N}, \mathsf{a}_4 = (0,1,0), X_2 = g_{X_2} = q_4$;

X_1X_2: $t_5 = X_1X_2 \in \mathbf{N}, \mathsf{a}_5 = (1,1,0), X_1X_2 = g_{X_1X_2} = q_5$;

$X_1^2X_2$: $t_6 = X_1^2X_2 \in \mathbf{N}, \mathsf{a}_6 = (2,1,0), X_1(X_1-1)X_2 = g_{X_1^2X_2} = q_6$;

$X_1^3X_2$: $X_1^3X_2 \in \mathbf{B}, X_1(X_1-1)(X_1-2)X_2 = g_{X_1^3X_2} \in \mathcal{B}(\mathsf{I})$;

X_2^2: $t_7 = X_2^2 \in \mathbf{N}, \mathsf{a}_7 = (0,2,0), X_2(X_2-1) = g_{X_2^2} = q_7$;

$X_1X_2^2$: $t_8 = X_1X_2^2 \in \mathbf{N}, \mathsf{a}_8 = (1,2,0), X_1X_2(X_2-1) = g_{X_1X_2^2} = q_8$;

$X_1^2X_2^2$: $X_1^2X_2^2 \in \mathbf{G}, X_1(X_1-1)X_2(X_2-1) = g_{X_1^2X_2^2} \in \mathcal{G}(\mathsf{I})$;

X_2^3: $t_9 = X_2^3 \in \mathbf{N}, \mathsf{a}_9 = (0,3,0), X_2(X_2-1)(X_2-2) = g_{X_2^3} = q_9$;

$X_1X_2^3$: $t_{10} = X_1X_2^3 \in \mathbf{N}, \mathsf{a}_{10} = (1,3,0), X_1X_2(X_2-1)(X_2-2) = g_{X_1X_2^3} = q_{10}$;

$X_1^2X_2^3$: $X_1^2X_2^3 \in \mathbf{B}, X_1(X_1-1)X_2(X_2-1)(X_2-2) = g_{X_1^2X_2^3} \in \mathcal{B}(\mathsf{I})$;

X_2^4: $X_2^4 \in \mathbf{G}, X_2(X_2-1)(X_2-2)(X_2-3) = g_{X_2^4} \in \mathcal{G}(\mathsf{I})$;

$X_1X_2^4$: $X_1X_2^4 \in \mathbf{B}, X_1X_2(X_2-1)(X_2-2)(X_2-3) = g_{X_1X_2^4} \in \mathcal{B}(\mathsf{I})$;

X_3: $t_{11} = X_3 \in \mathbf{N}, \mathsf{a}_{11} = (0,0,1), X_3 = g_{X_3} = q_{11}$;

X_1X_3: $t_{12} = X_1X_3 \in \mathbf{N}, \mathsf{a}_{12} = (1,0,1), X_1X_3 = g_{X_1X_3} = q_{12}$;

$X_1^2X_3$: $t_{13} = X_1^2X_3 \in \mathbf{N}, \mathsf{a}_{13} = (2,0,1), X_1(X_1-1)X_3 = g_{X_1^2X_3} = q_{13}$;

$X_1^3X_3$: $X_1^3X_3 \in \mathbf{B}, X_1(X_1-1)(X_1-2)X_3 = g_{X_1^3X_3} \in \mathcal{B}(\mathsf{I})$;

X_2X_3: $t_{14} = X_2X_3 \in \mathbf{N}, \mathsf{a}_{14} = (0,1,1), X_2X_3 = g_{X_2X_3} = q_{14}$;

$X_1X_2X_3$: $X_1X_2X_3 \in \mathbf{G}, X_1X_2X_3 = g_{X_1X_2X_3X_2} \in \mathcal{G}(\mathsf{I})$;

$X_1^2X_2X_3$: $X_1^2X_2X_3 \in \mathbf{B}, X_1(X_1-1)X_2X_3 = g_{X_1^2X_2X_3} \in \mathcal{B}(\mathsf{I})$;

$X_2^2X_3$: $t_{15} = X_2^2X_3 \in \mathbf{N}, \mathsf{a}_{15} = (0,2,1), X_2(X_2-1)X_3 = g_{X_2^2X_3} = q_{15}$;

$X_1X_2^2X_3$: $X_1X_2^2X_3 \in \mathbf{B}, X_1X_2(X_2-1)X_3 = g_{X_1X_2^2X_3} \in \mathcal{B}(\mathsf{I})$;

$X_2^3X_3$: $t_{16} = X_2^3X_3 \in \mathbf{N}, \mathsf{a}_{16} = (0,3,1), X_2(X_2-1)(X_2-2)X_3 = g_{X_2^3X_3} = q_{16}$;

$X_1X_2^3X_3$: $X_1X_2^3X_3 \in \mathbf{B}, X_1X_2(X_2-1)(X_2-2)X_3 = g_{X_1X_2^3X_3} \in \mathcal{B}(\mathsf{I})$;

$X_2^4X_3$: $X_2^4X_3 \in \mathbf{B}, X_2(X_2-1)(X_2-2)(X_2-3)X_3 = g_{X_2^4X_3} \in \mathcal{B}(\mathsf{I})$;

X_3^2: $t_{17} = X_3^2 \in \mathbf{N}, \mathsf{a}_{17} = (0,0,2), X_3(X_3-1) = g_{X_3^2} = q_{17}$;

$X_1X_3^2$: $X_1X_3^2 \in \mathbf{G}, X_1X_3(X_3-1) = g_{X_1X_3^2} \in \mathcal{G}(\mathsf{I})$;

$X_1^2X_3^2$: $X_1^2X_3^2 \in \mathbf{B}, X_1(X_1-1)X_3(X_3-1) = g_{X_1^2X_3^2} \in \mathcal{B}(\mathsf{I})$;

$X_2X_3^2$: $X_2X_3^2 \in \mathbf{G}, X_2X_3(X_3-1) = g_{X_2X_3^2} \in \mathcal{G}(\mathsf{I})$;

$X_2^2X_3^2$: $X_2^2X_3^2 \in \mathbf{B}, X_2(X_2-1)X_3(X_3-1) = g_{X_2^2X_3^2} \in \mathcal{B}(\mathsf{I})$;

$X_2^3X_3^2$: $X_2^3X_3^2 \in \mathbf{B}, X_2(X_2-1)(X_2-2)X_3(X_3-1) = g_{X_2^3X_3^2} \in \mathcal{B}(\mathsf{I})$;
X_3^3: $X_3^3 \in \mathbf{G}, X_3(X_3-1)(X_3-2) = g_{X_3^3} \in \mathcal{G}(\mathsf{I})$.

□

4. **Lazard Structural Theorem**

In the case $n = 2$, the structure of the Gröbner basis constructed by Macaulay for the ideal $\mathsf{I}(\mathsf{X})$ gives an easy example of the structural result proved by Lazard [9]:

Theorem 4.1 (Lazard)**.** *Let $\mathcal{P} := k[X_1, X_2]$ and let $<$ be the lexicographical ordering induced by $X_1 < X_2$.*

Let $\mathsf{I} \subset \mathcal{P}$ be an ideal and let $\{f_0, f_1, \dots, f_k\}$ be a Gröbner basis of I ordered so that

$$\mathbf{T}(f_0) < \mathbf{T}(f_1) < \cdots < \mathbf{T}(f_k).$$

Then

- $f_0 = PG_1 \cdots G_{k+1}$,
- $f_j = PH_jG_{j+1} \cdots G_{k+1}, 1 \le j < k$,
- $f_k = PH_kG_{k+1}$,

where

P is the primitive part of $f_0 \in k[X_1][X_2]$;
$G_i \in k[X_1], 1 \le i \le k+1$;

$H_i \in k[X_1][X_2]$ is a monic polynomial of degree $d(i)$, for each i;

$d(1) < d(2) < \cdots < d(k)$;

$H_{i+1} \in (G_1 \cdots G_i, H_1G_2 \cdots G_i, \dots, H_jG_{j+1} \cdots G_i, \dots, H_{i-1}G_i, H_i)$ for each i.

□

Example 4.2. Let us consider in Example 3.4 the ideal $\mathsf{I}\cap k[X_1, X_2]$ whose Gröbner basis is $\{g_{X_1^3}, g_{X_1^2X_2^2}, g_{X_2^4}\}$ and for which we have

$$\begin{array}{rcl} f_0 &=& X_1(X_1-1)(X_1-2) &=& G_1G_2,\\ f_1 &=& X_1(X_1-1)X_2(X_2-1) &=& H_1G_2,\\ f_2 &=& X_2(X_2-1)(X_2-2)(X_2-3) &=& H_2, \end{array}$$

where

$$\begin{array}{rclrcl} G_1 &=& (X_1-2), & G_2 &=& X_1(X_1-1),\\ H_1 &=& X_2(X_2-1), & H_2 &=& X_2(X_2-1)(X_2-2)(X_2-3) \end{array}$$

and $G_3 = P = 1$.

□

5. Cerlienco–Mureddu Correspondence

Cerlienco and Mureddu [2] gave a partial converse of Macaulay's result:

Problem 5.1. *Given a finite set of points,*

$$\mathsf{X} := \{\mathsf{a}_1, \ldots, \mathsf{a}_s\} \subset k^n, \quad \mathsf{a}_i := (a_{i1}, \ldots, a_{in}),$$

compute $\mathbf{N}(\mathsf{I}(\mathsf{X}))$ *w.r.t. the lexicographical ordering* $<$ *induced by* $X_1 < \cdots < X_n$.

More precisely the algorithm proposed by them, to each *ordered* finite set of points

$$\mathsf{X} := \{\mathsf{a}_1, \ldots, \mathsf{a}_s\} \subset k^n, \quad \mathsf{a}_i := (a_{i1}, \ldots, a_{in}),$$

associates

- an order ideal $\mathbf{N} := \mathbf{N}(\mathsf{X})$ and
- a bijection $\Phi := \Phi(\mathsf{X}) : \mathsf{X} \mapsto \mathbf{N}$;

which, as we will prove later, satisfies

Fact 5.2. $\mathbf{N}(\mathsf{I}(\mathsf{X})) = \mathbf{N}(\mathsf{X})$ *holds for each finite set of points* $\mathsf{X} \subset k^n$.

□

Since they do so by induction on $s = \#(\mathsf{X})$ let us consider the subset $\mathsf{X}' := \{\mathsf{a}_1, \ldots, \mathsf{a}_{s-1}\}$, and the corresponding order ideal $\mathbf{N}' := \mathbf{N}(\mathsf{X}')$ and bijection $\Phi' := \Phi(\mathsf{X}')$[2].

We need also to consider, for each $m < n$, the set

$$\mathcal{T}[1, m] := \mathcal{T} \cap k[X_1, \ldots, X_m] = \{X_1^{a_1} \cdots X_m^{a_m} : (a_1, \ldots, a_m) \in \mathbb{N}^m\},$$

and the projection

$$\pi_m : k^n \mapsto k^m, \quad \pi_m(x_1, \ldots, x_n) = (x_1, \ldots, x_m),$$

which we freely use to denote also the projection

$$\pi_m : \mathcal{T} \cong \mathbb{N}^n \mapsto \mathbb{N}^m \cong \mathcal{T}[1, m], \quad \pi_m(X_1^{a_1} \cdots X_n^{a_n}) = X_1^{a_1} \cdots X_m^{a_m}.$$

With this notation, let us set

$m := \max (j : \text{ exists } i < s : \pi_j(\mathsf{a}_i) = \pi_j(\mathsf{a}_s));$
$d := \#\{\mathsf{a}_i, i < s : \pi_m(\mathsf{a}_i) = \pi_m(\mathsf{a}_s), \Phi'(\mathsf{a}_i) \in \mathcal{T}[1, m+1]\};$
$\mathsf{W} := \{\mathsf{a}_i : \Phi'(\mathsf{a}_i) = \tau_i X_{m+1}^d, \tau_i \in \mathcal{T}[1, m]\} \cup \{\mathsf{a}_s\};$
$\mathsf{Z} := \pi_m(\mathsf{W});$
$\tau := \Phi(\mathsf{Z})(\pi_m(\mathsf{a}_s));$
$t_s := \tau X_{m+1}^d;$

where $\mathbf{N}(\mathsf{Z})$ and $\Phi(\mathsf{Z})$ are the result of the application of the present algorithm to Z, which can be inductively applied since $\#(\mathsf{Z}) \leq s-1$.

We then define

- $\mathbf{N} := \mathbf{N}' \cup \{t_s\}$,
- $\Phi(\mathsf{a}_i) := \begin{cases} \Phi'(\mathsf{a}_i) & i < s \\ t_s & i = s \end{cases}$

Example 5.3. Let us consider the same set $\mathsf{X} := \{\mathsf{a}_i, 1 \leq i \leq 6\}$ as in Example 2.3; Cerlienco–Mureddu Algorithms returns:

(0,0) $\mathsf{a}_1 := (0,0), \Phi(\mathsf{a}_1) := t_1 := 1;$
(0,1) $\mathsf{a}_2 := (0,1), m = 1, d = \#\{(0,0)\} = 1, \mathsf{W} = \{(0,1)\},$
$\tau = 1, \Phi(\mathsf{a}_2) := t_2 := X_2,$
(2,0) $\mathsf{a}_3 := (2,0), m = 0, d = \#\{(0,0)\} = 1, \mathsf{W} = \{(2,0)\},$
$\tau = 1, \Phi(\mathsf{a}_3) := t_3 := X_1,$
(0,2) $\mathsf{a}_4 := (0,2), m = 1, d = \#\{(0,0), (0,1)\} = 2, \mathsf{W} = \{(0,2)\},$

[2] If $s = 1$ the only possible solution is $\mathbf{N} = \{1\}, \Phi(\mathsf{a}_1) = 1$.

$\tau = 1, \Phi(\mathsf{a}_4) := t_4 := X_2^2$,
(1,0) $\mathsf{a}_5 := (1,0), m = 0, d = \#\{(0,0),(2,0)\} = 2, \mathsf{W} = \{(1,0)\}$,
$\tau = 1, \Phi(\mathsf{a}_5) := t_5 := X_1^2$,
(1,1) $\mathsf{a}_6 := (1,1), m = 1, d = \#\{(1,0)\} = 1, \mathsf{W} = \{(0,1),(1,1)\}$,
$\tau = X_1, \Phi(\mathsf{a}_6) := t_6 := X_1X_2$.

The fact, that Möller and Cerlienco–Mureddu algorithms give the same solution is not an accident and it suggests a proof strategy.

□

Remark 5.4. In the case $n = 2$, Cerlienco–Mureddu result can be simplified and described as follows. Let

$\{a_0, \dots, a_{r-1}\} := \pi_1(\mathsf{X})$;
$d(i) := \#\{(x_1, x_2) \in \mathsf{X} : x_1 = a_i\}$;

after renumbering the a_is, we can assume $d(0) \geq d(1) \geq \cdots \geq d(r-1)$. Then there are values $b_{il}, 0 \leq i < r, 0 \leq l < d(i)$, such that

$$\mathsf{X} = \{(a_i, b_{il}) : 0 \leq i < r, 0 \leq l < d(i)\}.$$

Then:

(1) $\mathbf{N}(\mathsf{I}(\mathsf{X})) = \{X_1^i X_2^l : 0 \leq i < r, 0 \leq l < d(i)\}$,
(2) $\Phi(a_i, b_{il}) = X_1^i X_2^l$.

6. **Configuration of points in a plane**

If we restrict ourselves to a radical 0-dimensional ideal in 2 variables, one can merge the corresponding restrictions of Lazard Theorem (which considers only ideals in 2 variables) and Cerlienco–Mureddu Correspondence (which is available for a radical 0-dimensional ideal), obtaining in this way a complete description of the structure of the ideal $\mathsf{I}(\mathsf{X})$ of any finite set of points $\mathsf{X} \subset k^2$.

Let, therefore, $\mathsf{X} \subset k^2$ be a finite set of points. Following Remark 5.4, we can assume[3] that there are

[3]This assumption, which simplifies both the description and the proof of the structural results, cannot be performed in the general case $n > 2$ requiring a more involved argument.

r different values $a_0, \dots, a_{r-1} \in k$,

r ordered integer values $d(0) \geq d(1) \geq \cdots \geq d(r-1) > 0 =: d(r)$,

for each $i < r, d(i)$ different values $b_{il}, 0 \leq l < d(i)$,

so that $\mathsf{X} = \{(a_i, b_{il}) : 0 \leq i < r, 0 \leq l < d(i)\}$.

Then we define

$\mathbf{N} := \{X_1^i X_2^l : 0 \leq i < r, 0 \leq l < d(i)\} \subset k[X_1, X_2]$;

$\Phi : \mathsf{X}_1 \mapsto \mathbf{N}$ the bijection such that $\Phi(a_i, b_{il}) = X_1^i X_2^l$;

$i_0 := r > i_1 > \ldots > i_j > i_{j+1} > \ldots > i_{h-1} > 0 =: i_h$ all the indices in which there is a jump $d(i_j - 1) > d(i_j)$;

$\mathsf{t}_0 := X_1^r, \mathsf{t}_j := X_1^{i_j} X_2^{d(i_j)}$, for each $j, 0 < j \leq h$;

$\mathbf{G} := \{\mathsf{t}_j, 0 \leq j \leq h\}$;

$G_j(X_1) := \prod_{i_j \leq \alpha < i_{j-1}} (X_1 - a_\alpha)$;

$\mathbf{B} := (\{1\} \cup \{X_1\tau, X_2\tau : \tau \in \mathbf{N}\}) \setminus \mathbf{N}$;

$\prec$ any ordering on $\mathcal{T}$ such that, for each $\tau \in \mathcal{T}$, $\{\omega \in \mathcal{T} : \omega \preceq \tau\}$ is an order ideal;

we enumerate $\mathbf{N} = \{\tau_1, \dots, \tau_s\}$ and $\mathsf{X}_1 = \{\mathsf{a}_1, \dots, \mathsf{a}_s\}$ so that

for each $\alpha, \beta, \tau_\alpha \prec \tau_\beta \iff \alpha < \beta$;
for each $\sigma, \tau_\sigma = \Phi(\mathsf{a}_\sigma)$,

and we define
$\mathfrak{B}(\tau) := \{X_1^a X_2^b \in \mathbf{N} : a \geq i, b < l\},$ *for each* $\tau := X_1^i X_2^l \in \mathbf{B} \cup \mathbf{N}$
$\mathfrak{N}(\tau) := \{\omega \in \mathfrak{B}(\tau) : \omega \prec \tau\},$ *for each* $\tau := X_1^i X_2^l \in \mathbf{N}$.

Theorem 6.1. *With these notations, it holds:*

1: $\mathbf{N}(\mathsf{I}(\mathsf{X})) = \mathbf{N}$;
2: $\mathbf{B}(\mathsf{I}(\mathsf{X})) = \mathbf{B}$;
3: $\mathbf{G}(\mathsf{I}(\mathsf{X})) = \mathbf{G}$;
4: *for each* $\tau := X_1^i X_2^l \in \mathbf{N}$, *there is*

$$g_\tau := X_2^l + \sum_{\omega \in \mathfrak{B}(\tau)} c(g_\tau, \frac{\omega}{X_1^i}) \frac{\omega}{X_1^i}$$

such that

$$g_\tau(\mathsf{a}) = 0, \text{ for each } \mathsf{a} \in \mathsf{X} : \Phi(\mathsf{a}) \in \mathfrak{N}(\tau);$$

5: *for each* $\tau := X_1^i X_2^l \in \mathbf{B}$, *there is*

$$g_\tau := X_2^l + \sum_{\omega \in \mathfrak{B}(\tau)} c(g_\tau, \frac{\omega}{X_1^i}) \frac{\omega}{X_1^i}$$

such that

$$g_\tau(\mathsf{a}) = 0, \text{ for each } \mathsf{a} \in \mathsf{X} : \Phi(\mathsf{a}) \in \mathfrak{B}(\tau).$$

Denoting

$$f_\tau := g_\tau \prod_{\alpha < i} (X_1 - a_\alpha), \text{ for each } \tau := X_1^i X_2^l \in \mathbf{B} \cup \mathbf{N},$$

it moreover holds

6: $\mathcal{G}(\mathsf{I}(\mathsf{X})) = \{f_{\mathsf{t}_j} : 0 \leq j \leq h\}$;
7: *for each* $j, f_{\mathsf{t}_j} = g_{\mathsf{t}_j} G_{j+1} \cdots G_k$;
8: $H_j := g_{\mathsf{t}_j} \in k[X_1][X_2]$ *is a monic polynomial of degree* $d(i_j)$, *for each* j;

9: $H_{j+1} \in (G_1 \cdots G_j, H_1 G_2 \cdots G_j, \ldots, H_{j-1}G_j, H_j)$, *for each* j*;*
10: $\mathbf{B} = \{X_1^i X_2^{d(i)}, 0 \le i \le r\} \cup \{X_1^{i_j} X_2^l, 0 \le j < h, d(i_j) < l < d(i_{j+1})\}$;
11: $\mathcal{B}(\mathsf{I}(\mathsf{X})) = \{f_\tau : \tau \in \mathbf{B}\}$*;*
12: *for each* $\tau_\alpha \in \mathbf{N}, \mathsf{a}_\beta \in \mathsf{X}, \alpha > \beta \Longrightarrow f_{\tau_\alpha}(\mathsf{a}_\beta) = 0$;
13: *for each* $\alpha, f_{\tau_\alpha}(\mathsf{a}_\alpha) \neq 0$.

$\square$

7. An algorithmic proof

What is fascinating us is that all one needs in order to prove these structural results[4] is just a direct application of Möller Algorithm.

Let us begin the proof with few remarks.

Lemma 7.1. *The theorem holds if conditions* **1**-**5***,* **9***,* **13** *hold.*

Proof.

6: For each $\tau = X_1^i X_2^l \in \mathbf{G}$ and for each $\mathsf{a} := (a_\alpha, b) \in \mathsf{X}$ we have that either $\Phi(\mathsf{a}) \in \mathfrak{B}(\tau)$ and so $g_\tau(\mathsf{a}) = 0$, or $\alpha < i$. Also, for each $\upsilon \in \mathcal{T}$ such that $c(f_\tau, \upsilon) \neq 0$ there is $\omega \in \mathfrak{B}(\tau)$ and $j \le i$ such that $c(f_\tau, \upsilon) = c(g_\tau, \frac{\omega}{X_1^i})$ and

$$\upsilon = X_1^j \frac{\omega}{X_1^i} = \frac{\omega}{X_1^{i-j}} \mid \omega \in \mathbf{N} = \mathbf{N}(\mathsf{I}(\mathsf{X})),$$

implying $f_\tau - \tau \in \mathrm{Span}_k(\mathbf{N}(\mathsf{I}(\mathsf{X}))$ and f_τ is reduced.
7: Follows immediately by the definition of f_τ.
8: Follows immediately by the definition of g_τ.
10: Obvious.
11: Follows by the same considerations of the proof of **6**.
12: Setting $\tau_\alpha := X_1^{a_\alpha} X_2^{b_\alpha}, \tau_\beta := X_1^{a_\beta} X_2^{b_\beta}$, it holds $\alpha > \beta \Longrightarrow \tau_\alpha \succ \tau_\beta$; this either implies $\tau_\beta \in \mathfrak{N}(\tau_\alpha)$ or,

[4] Which include (the restrictions of) Lazard Structural Theorem and Cerlienco–Mureddu Correspondence.

$\tau_\beta \notin \mathfrak{B}(\tau_\alpha)$ since $a_\beta < a_\alpha$;

in both cases $f_{\tau_\alpha}(a_\beta) = 0$. $\square$

Lemma 7.2. *If* X *satisfies the conditions of Theorem 6.1, let* $\tau := X_1^I X_2^{d(I)} \in \mathbf{B} = \mathbf{B}(\mathsf{I}(\mathsf{X}))$*, and*
$\mathsf{Z} := \{\mathsf{a}_i : \Phi(\mathsf{a}_i) \in \mathfrak{B}(\tau)\} = \{\Phi^{-1}(X_1^a X_2^b) : a \geq I, b < d(I)\}$.
Then Z *satisfies the conditions of Theorem 6.1.*

Proof. Denoting

$\mathbf{N}' := \{X_1^i X_2^l : 0 \leq i < r - I, 0 \leq l < d(i + I)\}$,

$\Phi' : \mathsf{Z} \mapsto \mathbf{N}'$ the bijection defined by $\Phi'(a_i, b_{il}) = X_1^{i-I} X_2^l$,

J the value such that $d(i_J) = d(I)$,

$\mathbf{G}' := \{\frac{\mathsf{t}_j}{X_1^I}, 0 \leq j < J\} \cup \{X_2^{d(I)}\}$,

$\mathbf{B}' := (\{1\} \cup \{X_1\tau, X_2\tau : \tau \in \mathbf{N}'\}) \setminus \mathbf{N}'$,

for each $\sigma := X_1^i X_2^l \in \mathbf{B}' \cup \mathbf{N}', \mathfrak{B}'(\sigma) := \{X_1^a X_2^b \in \mathbf{N}' : a \geq i, b < l\}$,

for each $\sigma := X_1^i X_2^l \in \mathbf{N}', \mathfrak{N}'(\sigma) := \{\omega \in \mathfrak{B}'(\sigma) : X_1^I \omega \prec X_1^I \sigma\}$,

for each $\sigma := X_1^i X_2^l \in \mathbf{B}' \cup \mathbf{N}'$, $\phi_\sigma := g_{\sigma X_1^I} \prod_{I \leq \alpha < i+I} (X_1 - a_\alpha)$,

$\mathsf{J} := (\phi_\sigma : \sigma \in \mathbf{B}')$,

we have obviously:

J is an ideal,

$\mathcal{B}(\mathsf{J}) = \{\phi_\sigma : \sigma \in \mathbf{B}'\}$,

$\mathbf{N}' = \mathbf{N}(\mathsf{J})$,

$\mathbf{B}' = \mathbf{B}(\mathsf{J})$,

$\mathbf{G}' = \mathbf{G}(\mathsf{J})$,

for each $\mathsf{a} \in \mathsf{Z}$, and each $\sigma \in \mathbf{B}', \phi_\sigma(\mathsf{a}) = 0$,

$\mathsf{J} \subset \mathsf{I}(\mathsf{Z})$,

$\mathrm{mult}(\mathsf{J}) = \#\mathbf{N}' = \#\mathsf{Z} = \mathrm{mult}(\mathsf{I}(\mathsf{Z}))$, so that

$\mathsf{J} = \mathsf{I}(\mathsf{Z})$,

for each $\sigma \in \mathbf{N}'$, $\tau := X_1^I \sigma \in \mathbf{N}$, and

$\mathfrak{B}(\tau) = \{X_1^I \omega : \omega \in \mathfrak{B}'(\sigma)\}$,

$\mathfrak{N}(\tau) = \{X_1^I \omega : \omega \in \mathfrak{N}'(\sigma)\}$,

from which we can deduce that

1: $\mathbf{N}(\mathsf{I}(\mathsf{Z})) = \mathbf{N}(\mathsf{J}) = \mathbf{N}'$;

2: $\mathbf{B}(\mathsf{I}(\mathsf{Z})) = \mathbf{B}(\mathsf{J}) = \mathbf{B}'$;

3: $\mathbf{G}(\mathsf{I}(\mathsf{Z})) = \mathbf{G}(\mathsf{J}) = \mathbf{G}'$;

4: for each $\sigma := X_1^i X_2^l \in \mathbf{N}'$, setting $\tau := X_1^I \sigma$, it holds

$$g'_\sigma := g_\tau = X_2^l + \sum_{\omega \in \mathfrak{B}(\tau)} c(g_\tau, \frac{\omega}{X_1^{i+I}}) \frac{\omega}{X_1^{i+I}} = X_2^l + \sum_{\omega \in \mathfrak{B}'(\sigma)} c(g_\tau, \frac{\omega}{X_1^i}) \frac{\omega}{X_1^i},$$

and $g'_\sigma(\mathsf{a}) = g_\tau(\mathsf{a}) = 0$, for each $\mathsf{a} \in \mathsf{X}$ such that $\Phi(\mathsf{a}) \in \mathfrak{N}(\tau)$, i.e. for each $\mathsf{a} \in \mathsf{Z}$ such that $\Phi'(\mathsf{a}) \in \mathfrak{N}'(\sigma)$;

5: as in the proof of **4**, for each $\sigma \in \mathbf{B}'$ it is sufficient to take $g'_\sigma := g_\tau$ where $\tau := X_1^I \sigma$;

9: follows directly from the same result for X;

13: for each $\mathsf{a} \in \mathsf{Z}$ and each $\sigma \in \mathbf{N}'$ such that $\Phi'(\mathsf{a}) = \sigma$, i.e. $\Phi(\mathsf{a}) = \tau := X_1^I \sigma$, we have

$$f_\tau = \phi_\sigma \prod_{\alpha < I} (X_1 - a_\alpha)$$

so that $f_\tau(\mathsf{a}) \neq 0 \Longrightarrow \phi_\sigma(\mathsf{a}) \neq 0$.

□

The proof being by induction, we begin with

Lemma 7.3. *If* $\#\mathsf{X} = 1$ *conditions* **1**-**5***,* **9***,* **13** *hold.*

Proof. When we have a single point $(a, b) \in k^2$, we have

$$\mathbf{N} = \{1\}, \mathbf{B} = \mathbf{G} = \{X_1, X_2\}, g_1 = 1,$$

$$\mathbf{G}(\mathsf{I}(\mathsf{X})) = \mathbf{B}(\mathsf{I}(\mathsf{X})) = \{X_1 - a, X_2 - b\},$$

and the properties are obviously satisfied. □

This giving a starting point for induction, let us assume we have a set

$$\mathsf{X} := \{\mathsf{a}_1, \ldots, \mathsf{a}_s\} \subset k^2, s > 0,$$

and let us denote $\mathsf{X}' := \{\mathsf{a}_1, \ldots, \mathsf{a}_{s-1}\}$ for which we use the notation of Section 6 and for which Theorem 6.1 holds. With a slight abuse of notation, the notions of Section 6 for X' and X will be distinguished by accenting the former with $'$.

Since X is ordered so that $\Phi(\mathsf{X})$ is an order ideal, necessarily $\Phi(\mathsf{a}_s) \in \mathbf{G}'$ and

$$\text{exists } J, 0 \leq J \leq h, \mathsf{b} \in k, \mathsf{b} \neq b_{i_J l} \textit{ for each } l : \mathsf{a}_s = (a_{i_J}, \mathsf{b}),$$
$$\Phi(\mathsf{a}_s) = \mathsf{t}_J.$$

Lemma 7.4. $f'_{\mathsf{t}_j}(\mathsf{a}_s) = 0,$ *for each* $j < J$*, and* $f'_{\mathsf{t}_J}(\mathsf{a}_s) \neq 0.$

Proof. Let $\mathsf{Z} := \{\mathsf{a}_i : \Phi'(\mathsf{a}_i) \in \mathfrak{B}(\mathsf{t}_J)\}$; by Lemma 7.2 we know that $\mathsf{I}(\mathsf{Z})$ is generated by

$$g'_{\mathsf{t}_j} \prod_{i_J \leq \alpha < i_j} (X - a_\alpha) \text{ for } j \leq J$$

thus implying that $f'_{\mathsf{t}_j}(\mathsf{a}_s) = 0$, for each $j < J$.

Therefore either $g'_{\mathsf{t}_J}(\mathsf{a}_s) \neq 0$ (and $f'_{\mathsf{t}_J}(\mathsf{a}_s) \neq 0$ too) or

$$\mathsf{I}(\mathsf{Z}) = \mathsf{I}(\mathsf{Z} \cup \{\mathsf{a}_s\}),$$

which is impossible by a multiplicity argument. $\square$

As a consequence, to apply Möller Algorithm to $\mathsf{X} = \mathsf{X}' \cup \{\mathsf{a}_s\}$ produces

$q_s := c^{-1} f'_{\mathsf{t}_J}$, with $c = f'_{\mathsf{t}_J}(\mathsf{a}_s)$;

$\mathbf{N} := \mathbf{N}' \cup \{X_1^{i_J} X_2^{d(i_J)}\}$;

$\mathbf{B} := (\mathbf{B}' \setminus \{X_1^{i_J} X_2^{d(i_J)}\}) \cup \{X_1^{i_J} X_2^{d(i_J)+1}, X_1^{i_J+1} X_2^{d(i_J)}\}$;

$f_\tau := f'_\tau - f'_\tau(\mathsf{a}_s) q_s$ for each $\tau := X_1^i X_2^l \in \mathbf{B}'$ such that $i < i_J$ and

$f_\tau := f'_\tau$, for each $\tau := X_1^i X_2^l \in \mathbf{B}'$ such that $i > i_J$, since $f'_\tau(\mathsf{a}_s) = 0$;

$f_\tau := X_2 q_s - \mathsf{b} q_s - \sum_{X_2\omega \in \mathbf{B}} c^{-1} c(g'_{\mathsf{t}_J}, \frac{\omega}{X^{i_J}}) f'_{X_2\omega}$ for $\tau := X_1^{i_J} X_2^{d(i_J)+1}$ where

$$g'_{\mathsf{t}_J} := X_2^{d(i_J)} + \sum_{\omega \in \mathfrak{B}(\mathsf{t}_J)} c(g'_{\mathsf{t}_J}, \frac{\omega}{X^{i_J}}) \frac{\omega}{X^{i_J}};$$

$f_\tau := X_2 f'_\sigma - \sum_{\substack{\omega \in \mathfrak{B}(\sigma) \\ X_2\omega \in \mathbf{B}}} c(g'_\sigma, \frac{\omega}{X^{i_J+1}}) f'_{X_2\omega}$ for $\tau := X_1^{i_J+1} X_2^{d(i_J)}$, if $\tau \notin \mathbf{B}$, where $\sigma := X_1^{i_J+1} X_2^{d(i_J)-1} \in \mathbf{B}$ and

$$g'_\sigma := X_2^{d(i_J)-1} + \sum_{\omega \in \mathfrak{B}(\sigma)} c(g'_\sigma, \frac{\omega}{X^{i_J+1}}) \frac{\omega}{X^{i_J+1}}.$$

Proposition 7.5. *If* X' *satisfies the conditions of Theorem 6.1, then* X *satisfies them too.*

Proof.

1: Trivial.

2: Trivial.

3: Trivial.

4: The thesis follows defining $g_{\mathsf{t}_J} := g'_{\mathsf{t}_J}$ and, for each $\tau \in \mathbf{N}'$, $g_\tau := g'_\tau$, since $\Phi(\mathsf{a}_s) \notin \mathcal{B}(\tau)$.

5: Defining $g_\tau := \frac{f_\tau}{\prod\limits_{\alpha<i}(X-a_\alpha)}$ the property holds.

9: The only g'_τs modified are those such that $\tau = X_1^i X_2^l$ with $i \le i_J$ and they are modified by elements which are in

$$(G_1 \cdots G_i, H_1 G_2 \cdots G_{i-1}, \dots, H_l G_{l+1} \cdots G_{i-1}, \dots, H_{i-2} G_{i-1}, H_{i-1}).$$

13: Since $f'_{\mathsf{t}_J}(\mathsf{a}_s) \neq 0$.

□

8. What happens in dimension 3?

Example 8.1. Let us consider the set $\mathsf{X} := \{\mathsf{b}_i, 1 \le i \le 6\}$ where

$$\begin{array}{lllllllll} \mathsf{b}_1 & = & (0,0,1) & \mathsf{b}_2 & = & (0,1,-2) & \mathsf{b}_3 & = & (2,0,2) \\ \mathsf{b}_4 & = & (0,2,-2) & \mathsf{b}_5 & = & (1,0,3) & \mathsf{b}_6 & = & (1,1,3) \end{array}$$

and let us set $\mathsf{a}_i := \pi_2(\mathsf{b}_i)$, for each i, so that $\pi_2(\mathsf{X}) = \mathsf{Y}$, where Y is the set of points discussed in Example 2.3.

Clearly Cerlienco-Mureddu Correspondence returns $\Phi(\mathsf{b}_i) = \Phi(\mathsf{a}_i)$ for each i and the Gröbner basis $\mathcal{G}(\mathsf{I}(\mathsf{X}))$ of $\mathsf{I}(\mathsf{X})$ is

$$\mathcal{G}(\mathsf{I}(\mathsf{X})) = \mathcal{G}_6 \cup \{X_3 - \frac{3}{2}X_2^2 - 3X_1X_2 + \frac{9}{2}X_2 + \frac{3}{2}X_1^2 - \frac{7}{2}X_1 - 1\}$$

where
$\mathcal{G}_6 = \{X_1^3 - 3X_1^2 + 2X_1, X_1^2X_2 - X_1X_2, X_1X_2^2 - X_1X_2, X_2^3 - 3X_2^2 + 2X_2\}$ is the result returned by Möller Algorithm in our computation performed in Example 2.3. □

Example 8.2. If we now add a new point $\mathsf{b}_7 := (1,1,1)$ both Cerlienco–Mureddu and Möller return $t_7 := X_3$. The Gröbner basis is

$$\mathcal{G}_6 \cup \{f_1, f_2, f_3\}$$

where

$$\begin{aligned} f_1 &= X_3X_1 - X_3 + \frac{3}{2}X_2^2 + 3X_1X_2 - \frac{9}{2}X_2 - \frac{1}{2}X_1^2 - \frac{1}{2}X_1 + 1, \\ f_2 &= X_3X_2 - X_3 + \frac{3}{2}X_2^2 - 2X_2X_1 - \frac{5}{2}X_2 - \frac{3}{2}X_1^2 + \frac{7}{2}X_1 + 1, \\ f_3 &= X_3^2 - 4X_3 + \frac{15}{2}X_2^2 + 15X_2X_1 - \frac{45}{2}X_2 + \frac{1}{2}X_1^2 - \frac{1}{2}X_1 + 3 \end{aligned}$$

and (modulo $\mathsf{I}(\mathsf{Y})$)

$$\begin{aligned} f_1 &= (X_1 - 1)(X_3 - \frac{3}{2}X_2^2 + \frac{9}{2}X_2 - \frac{1}{2}X_1 - 1), \\ f_2 &= (X_2 - 1)(X_3 + \frac{3}{2}X_2 + \frac{3}{2}X_1^2 - \frac{7}{2}X_1 - 1). \end{aligned}$$

The interesting aspect is that

- $\{\mathsf{b} \in \mathsf{X} : (X_1 - 1)(\mathsf{b}) \neq 0\} = \{\mathsf{b}_1, \mathsf{b}_2, \mathsf{b}_3, \mathsf{b}_4\}$ to which Cerlienco–Mureddu Correspondence associates $\{1, X_1, X_2, X_2^2\}$
- $\{\mathsf{b} \in \mathsf{X} : (X_2 - 1)(\mathsf{b}) \neq 0\} = \{\mathsf{b}_1, \mathsf{b}_3, \mathsf{b}_4, \mathsf{b}_5\}$ to which Cerlienco–Mureddu Correspondence associates $\{1, X_1, X_1^2, X_2\}$.

□

Example 8.3. The same pattern appears if we add a new point $\mathsf{b}_8 := (2,0,1)$ to which both Cerlienco–Mureddu and Möller return $t_8 := X_1X_3$. The corresponding Gröbner basis is

$$\mathcal{G}_6 \cup \{f_1, f_2, f_3\}$$

where

$$\begin{aligned}
f_1 &= X_3X_1^2 - 3X_3X_1 + 2X_3 - 3X_2^2 - 6X_2X_1 + 9X_2 - X_1^2 \\
&\quad +3X_1 - 2, \\
f_2 &= X_3X_2 + X_3X_1 - 2X_3 + 3X_2^2 + X_2X_1 - 7X_2 - 2X_1^2 \\
&\quad +3X_1 + 2, \\
f_3 &= X_3^2 + X_3X_1 - 5X_3 + 9X_2^2 + 18X_2X_1 - 27X_2 - X_1 + 4,
\end{aligned}$$

and (modulo $\mathsf{I}(\mathsf{Y})$)

$$\begin{aligned}
f_1 &= (X_1^2 - 3X_1 + 2)(X_3 - \frac{3}{2}X_2^2 + \frac{9}{2}X_2 - 1) \\
f_2 &= (X_2 + X_1 - 2)(X_3 + 3X_2 - 2X_1 - 1)
\end{aligned}$$

where $(X_1^2 - 3X_1 + 2, X_2 + X_1 - 2)$ is the Gröbner basis of the ideal whose roots are $\{\pi_2(\mathsf{b}_7), \pi_2(\mathsf{b}_8)\}$ and

- $\{\mathsf{b} \in \mathsf{X} : (X_1^2 - 3X_1 + 2)(\mathsf{b}) \neq 0\} = \{\mathsf{b}_1, \mathsf{b}_2, \mathsf{b}_4\}$ to which Cerlienco–Mureddu Correspondence associates $\{1, X_2, X_2^2\}$
- $\{\mathsf{b} \in \mathsf{X} : (X_2 + X_1 - 2)(\mathsf{b}) \neq 0\} = \{\mathsf{b}_1, \mathsf{b}_2, \mathsf{b}_5\}$ to which Cerlienco–Mureddu Correspondence associates $\{1, X_1, X_2\}$

i.e. exactly the terms which appear in the corresponding cofactor of f_i.

□

Example 8.4. If we add the further point $\mathsf{b}_9 := (2, 0, 0)$ to which both Cerlienco–Mureddu and Möller associate $t_9 := X_3^2$, the corresponding Gröbner basis is

$$\mathcal{G}(\mathsf{I}(\mathsf{X})) \cup \{f_1, f_2, f_3, f_4\}$$

where

$$
\begin{aligned}
f_1 &= X_3X_1^2 - 3X_3X_1 + 2X_3 - 3X_2^2 - 6X_2X_1 + 9X_2 - X_1^2\\
&\quad +3X_1 - 2,\\
f_2 &= X_3X_2 + X_3X_1 - 2X_3 + 3X_2^2 + X_2X_1 - 7X_2 - 2X_1^2\\
&\quad +3X_1 + 2,\\
f_3 &= X_3^2X_1 - 2X_3^2 - 4X_3X_1 + 8X_3 - 15X_2^2 - 30X_2X_1 + 45X_2\\
&\quad +3X_1 - 6,\\
f_4 &= X_3^3 - 3X_3^2 + 3X_3X_1 - 4X_3 - 3X_2^2 - 6X_2X_1 + 9X_2\\
&\quad -3X_1 + 6,
\end{aligned}
$$

and (modulo $\mathsf{I}(\mathsf{Y})$)

$$
\begin{aligned}
f_1 &= (X_1^2 - 3X_1 + 2)(X_3 - \frac{3}{2}X_2^2 + \frac{9}{2}X_2 - 1)\\
f_2 &= (X_2 + X_1 - 2)(X_3 + 3X_2 - 2X_1 - 1)\\
f_3 &= (X_1 - 2)(X_3^2 - 4X_3 + 15/2X_2^2 + 15X_2X_1 - 45/2X_2 + 3)
\end{aligned}
$$

where

- $\{\mathsf{b} \in \mathsf{X} : (X_1 - 2)(\mathsf{b}) \neq 0\} = \{\mathsf{b}_1, \mathsf{b}_2, \mathsf{b}_4, \mathsf{b}_5, \mathsf{b}_6, \mathsf{b}_7\}$ to which Cerlienco–Mureddu Correspondence associates $\{1, X_1, X_2, X_1X_2, X_2^2, X_3\}$.

$\square$

9. **Cerlienco–Mureddu Correspondence (2)**

The aim of this section is to remark some easy-to-prove properties satisfied by Cerlienco–Mureddu Correspondence (for which see [2]):

Let therefore

$$\mathsf{X} := \{\mathsf{a}_1, \ldots, \mathsf{a}_s\} \subset k^n, \quad \mathsf{a}_i := (a_{i1}, \ldots, a_{in}),$$

be an ordered set of points and let us denote $\mathbf{N} := \mathbf{N}(\mathsf{X})$ and $\Phi := \Phi(\mathsf{X})$ the result of Cerlienco–Mureddu Correspondence.

Lemma 9.1. *If* $\mathsf{Y} = \{\mathsf{a}_1, \ldots, \mathsf{a}_r\} \subset \mathsf{X}$ *is an initial segment of* X *then*

- $\mathbf{N}(\mathsf{Y}) \subset \mathbf{N}(\mathsf{X})$,
- *for each* $j \le r < s, \Phi(\mathsf{Y})(\mathsf{a}_j) = \Phi(\mathsf{X})(\mathsf{a}_j)$.

□

Let $\tau := X_1^{d_1} \cdots X_n^{d_n} \in \mathcal{T} \setminus \mathbf{N}(\mathsf{X})$ be any term such that $\mathbf{N} \cup \{\tau\}$ is an order ideal and, let us define, for each $m, 1 \le m \le n$:

$\mathsf{N}_m(\tau) := \mathsf{N}_m(\mathsf{X}, \tau) := \{\omega \in \mathcal{T}[1, m] : \tau \ge \omega X_{m+1}^{d_{m+1}} \cdots X_n^{d_n} \in \mathbf{N}\}$,

$\mathsf{A}_m(\tau) := \mathsf{A}_m(\mathsf{X}, \tau) := \{\Phi^{-1}(\omega X_{m+1}^{d_{m+1}} \cdots X_n^{d_n}) : \omega \in \mathsf{N}_m(\tau)\} \subset \mathsf{X} \subset k^n$,

$\mathsf{B}_m(\tau) := \mathsf{B}_m(\mathsf{X}, \tau) := \pi_m(\mathsf{A}_m(\tau)) \subset k^m$,

$\mathsf{C}_m(\tau) := \mathsf{C}_m(\mathsf{X}, \tau) := \{\pi_m(\mathsf{a}) \in \mathsf{B}_m(\tau) : \pi_{m-1}(\mathsf{a}) \notin \mathsf{B}_{m-1}(\tau)\} \subset k^m$,

$\mathsf{D}_m(\tau) := \mathsf{D}_m(\mathsf{X}, \tau) := \{\mathsf{a} \in \mathsf{X} : \pi_m(\mathsf{a}) \in \mathsf{C}_m(\tau)\} \subset \mathsf{X}$;

$\mathsf{M}_m(\tau) := \mathsf{M}_m(\mathsf{X}, \tau) := \{\omega \in \mathcal{T}[1, m] : \omega < X_m^{d_m}, \omega X_{m+1}^{d_{m+1}} \cdots X_n^{d_n} \in \mathbf{N}\}$,
where, with slight abuse of notation, we have

$$\mathsf{N}_n(\tau) := \{\omega \in \mathcal{T} : \omega < \tau\}, \mathsf{A}_n(\tau) := \mathsf{B}_n(\tau) := \{\mathsf{a} : \Phi(\mathsf{a}) < \tau\},$$

$$\mathsf{C}_1(\tau) := \mathsf{B}_1(\tau).$$

Example 9.2. *With respect to the previous examples, if we choose* $\tau := X_2 X_3$ *we have*

$$\mathsf{N}_1 = \mathsf{A}_1 = \mathsf{B}_1 = \mathsf{C}_1 = \mathsf{D}_1 = \mathsf{M}_1 = \emptyset,$$

and

$$\begin{array}{lcllcl}
\mathsf{N}_2 & = & \{1, X_1\}, & \mathsf{N}_3 & = & \mathbf{N} \setminus \{X_3^2\},\\
\mathsf{A}_2 & = & \{\mathsf{b}_7, \mathsf{b}_8\}, & \mathsf{A}_3 & = & \{\mathsf{b}_i, 1 \le i \le 8\},\\
\mathsf{B}_2 & = & \{(1,1),(2,0)\}, & \mathsf{B}_3 & = & \{\mathsf{b}_i, 1 \le i \le 8\},\\
\mathsf{C}_2 & = & \{(1,1),(2,0)\}, & \mathsf{C}_3 & = & \{\mathsf{b}_1, \mathsf{b}_2, \mathsf{b}_4, \mathsf{b}_5\},\\
\mathsf{D}_2 & = & \{\mathsf{b}_3, \mathsf{b}_6, \mathsf{b}_7, \mathsf{b}_8, \mathsf{b}_9\}, & \mathsf{D}_3 & = & \{\mathsf{b}_1, \mathsf{b}_2, \mathsf{b}_4, \mathsf{b}_5\},\\
\mathsf{M}_2 & = & \{1, X_1\}, & \mathsf{M}_3 & = & \{1, X_1, X_1^2, X_2, X_1X_2, X_2^2\}.
\end{array}$$

Example 9.3. If we instead choose $\tau := X_1X_3^2$ we have

$$\begin{array}{lcllcllcl}
\mathsf{N}_1 & = & \{1\}, & \mathsf{N}_2 & = & \{1\}, & \mathsf{N}_3 & = & \mathbf{N},\\
\mathsf{A}_1 & = & \{\mathsf{b}_9\}, & \mathsf{A}_2 & = & \{\mathsf{b}_9\}, & \mathsf{A}_3 & = & \{\mathsf{b}_i, 1 \le i \le 9\},\\
\mathsf{B}_1 & = & \{2\}, & \mathsf{B}_2 & = & \{(2,0)\}, & \mathsf{B}_3 & = & \{\mathsf{b}_i, 1 \le i \le 9\},\\
\mathsf{C}_1 & = & \{2\}, & \mathsf{C}_2 & = & \emptyset, & \mathsf{C}_3 & = & \{\mathsf{b}_1, \mathsf{b}_2, \mathsf{b}_4, \mathsf{b}_5\},\\
\mathsf{D}_1 & = & \{\mathsf{b}_3, \mathsf{b}_8, \mathsf{b}_9\}, & \mathsf{D}_2 & = & \emptyset, & \mathsf{D}_3 & = & \{\mathsf{b}_1, \mathsf{b}_2, \mathsf{b}_4, \mathsf{b}_5, \mathsf{b}_6, \mathsf{b}_7\},\\
\mathsf{M}_1 & = & \{1\}, & \mathsf{M}_2 & = & \emptyset, & \mathsf{M}_3 & = & \mathbf{N} \setminus \{X_3^2\}.
\end{array}$$

Lemma 9.4. *With the notation above it holds*

1. $\#(\mathsf{B}_m(\tau)) = \#(\mathsf{A}_m(\tau)) = \#(\mathsf{N}_m(\tau))$*;*
2. *Cerlienco–Mureddu Correspondence associates to* $\mathsf{B}_m(\tau)$ *the order ideal*
$$\mathbf{N}(\mathsf{B}_m(\tau)) = \mathsf{N}_m(\tau)$$
and the bijection $\Phi(\mathsf{B}_m(\tau))$ *defined by*
$$\Phi(\mathsf{B}_m(\tau))(\pi_m(\mathsf{a}))X_{m+1}^{d_{m+1}} \cdots X_n^{d_n} = \Phi(\mathsf{a}), \text{ for each } \mathsf{a} \in \mathsf{A}_m;$$
3. $\#(\mathsf{C}_m(\tau)) \le \#(\mathsf{M}_m(\tau))$*;*
4. *Under Cerlienco–Mureddu Correspondence one has*
$$\mathbf{N}(\mathsf{C}_m(\tau)) \subset \{\omega \in \mathcal{T}[1,m] : \omega < X_m^{d_m}\}.$$
5. $\mathsf{X} = \cup_m \mathsf{D}_m(\tau)$.

Proof.

1. is trivial;
2. Cerlienco–Mureddu Algorithm when applied to the ordered set X associates each element $\mathsf{a} \in \mathsf{A}_m(\tau)$ to the term
$$\Phi(\mathsf{a}) = \Phi(\pi_m(\mathsf{A}_m(\tau)))(\pi_m(\mathsf{a}))X_{m+1}^{d_{m+1}} \cdots X_n^{d_n};$$

(3) in order to obtain $\mathsf{M}_m(\tau)$ one has to remove from $\mathsf{N}_m(\tau)$ the subset

$$\{\omega X_m^{d_m} \in \mathsf{N}_m(\tau) : \omega \in \mathcal{T}[1, m-1]\} = \{\omega X_m^{d_m} : \omega \in \mathsf{N}_{m-1}(\tau)\}$$

while for each $\omega \in \mathbf{N}_{m-1}(\tau)$ there are $d_m + 1$ points $\mathsf{a} \in \mathsf{B}_m(\tau)$ such that

$$\Phi(\mathsf{B}_{m-1}(\tau))(\pi_{m-1}(\mathsf{a})) = \omega.$$

(4) In order that there is $\omega \in \mathbf{N}(\mathsf{C}_m(\tau))$ such that $\omega \geq X_m^{d_m}$, Cerlienco–Mureddu Algorithm requires the existence of at least $d_m + 1$ points $\mathsf{b}_0, \dots, \mathsf{b}_{d_m}$ such that

$$\pi_m(\mathsf{b}_0) = \cdots = \pi_m(\mathsf{b}_i) = \cdots = \pi_m(\mathsf{b}_{d_m}),$$

so that $\pi_{m-1}(\mathsf{b}_0) \in \mathsf{B}_{m-1}(\tau)$.

(5) If $\mathsf{a} \in \mathsf{X}$ is such that $\Phi(\mathsf{a}) \leq \tau$, then there is a minimal value $m \leq n$ for which $\mathsf{a} \in \mathsf{A}_m(\tau)$, $\pi_m(\mathsf{a}) \in \mathsf{B}_m(\tau)$, $\pi_m(\mathsf{a}) \in \mathsf{C}_m(\tau)$, $\mathsf{a} \in \mathsf{D}_m(\tau)$.
If $\mathsf{a} \in \mathsf{X}$ is such that $\Phi(\mathsf{a}) = X_1^{e_1} \cdots X_n^{e_n} > \tau$, there is $m \leq n$ such that $e_m > d_m$, while $e_i = d_i$, for each $i > m$; this implies that there is $\mathsf{b} \in \mathsf{A}_m(\tau)$ such that $\pi_m(\mathsf{b}) = \pi_m(\mathsf{a})$ so that $\mathsf{a} \in \mathsf{D}_m(\tau)$.

□

Remark 9.5. Let us denote, for each $\nu, 1 \leq \nu < n$, and each $\mathsf{b} \in \pi_\nu(\mathsf{X})$,

$$\mu(\mathsf{b}) := \#(\mathsf{a} \in \mathsf{X} : \mathsf{b} = \pi_\nu(\mathsf{a}))\},$$

and for each $\nu, 1 \leq \nu < n$, and each $\delta \in \mathbb{N}$,

$$\mathsf{Y}_{\nu\delta} := \{\pi_\nu(\mathsf{a}) : \text{ exists } \omega \in \mathcal{T}[1, \nu] : \Phi(\mathsf{a}) = \omega X_{\nu+1}^\delta\}.$$

Then

- $\mathsf{Y}_{\nu\delta} = \{\mathsf{b} \in \pi_\nu(\mathsf{X}) : \delta < \mu(\mathsf{b})\}$,
- $\pi_\nu(\mathsf{X}) = \mathsf{Y}_{\nu 0} \supset \mathsf{Y}_{\nu 1} \supset \cdots \supset \mathsf{Y}_{\nu\delta} \supset \mathsf{Y}_{\nu\delta+1} \supset \cdots$,
- $\mathsf{I}(\pi_\nu(\mathsf{X})) = \mathsf{I}(\mathsf{Y}_{\nu 0}) \subset \mathsf{I}(\mathsf{Y}_{\nu 1}) \subset \cdots \subset \mathsf{I}(\mathsf{Y}_{\nu\delta}) \subset \mathsf{I}(\mathsf{Y}_{\nu\delta+1}) \subset \cdots$.

The result is, essentially a specialization of Kalkbrener's Theorem [8] □

10. **Enhancing Lazard Structural Theorem**

Let us now consider an ordered set of points

$$\mathsf{X} := \{\mathsf{a}_1, \dots, \mathsf{a}_s\} \subset k^n, \quad \mathsf{a}_i := (a_{i1}, \dots, a_{in}),$$

and let us denote $\mathbf{N} := \mathbf{N}(\mathsf{X})$ and $\Phi := \Phi(\mathsf{X})$ the result of Cerlienco–Mureddu Correspondence which satisfies

Fact 10.1. *It holds*

(A): $\mathbf{N} := \mathbf{N}(\mathsf{I}(\mathsf{X}))$.

$\square$

Since $\mathbf{N}(\mathsf{X})$ is an order ideal,

$$\mathbf{T}(\mathsf{X}) := \mathcal{T} \setminus \mathbf{N}(\mathsf{X})$$

is a monomial ideal whose minimal basis

$$\mathbf{G} := \{\mathsf{t}_1, \dots, \mathsf{t}_r\}$$

will be ordered so that $\mathsf{t}_1 < \mathsf{t}_2 < \dots < \mathsf{t}_r$.

Denoting further

$$\mathbf{B} := (\{1\} \cup \{X_i\tau : \tau \in \mathbf{N}\}) \setminus \mathbf{N}$$

we obviously obtain

Corollary 10.2. *It holds*

(B) $\mathbf{G}(\mathsf{I}(\mathsf{X})) = \mathbf{G} = \{\mathsf{t}_1, \dots, \mathsf{t}_r\}, \mathsf{t}_1 < \mathsf{t}_2 < \dots < \mathsf{t}_r$;

(C) $\mathbf{B}(\mathsf{I}(\mathsf{X})) = \mathbf{B}$.

$\square$

Let us extend the ordering of X to $\mathbf{N} = \{\tau_1, \dots, \tau_s\}$ enumerating it so that

$$\textit{for each } \sigma, \tau_\sigma = \Phi(\mathsf{a}_\sigma),$$

and let us denote the ordering of X and $\mathbf{N}$ by $\prec$ so that

$$\textit{for each } \alpha, \beta, \tau_\alpha \prec \tau_\beta, \mathsf{a}_\alpha \prec \mathsf{a}_\beta \iff \alpha < \beta.$$

Denote for each $\tau \in \mathbf{N}$

$$\mathfrak{X}(\tau) := \{\mathsf{a} \in \mathsf{X} : \mathsf{a} \prec \Phi^{-1}(\tau)\} = \{\mathsf{a} \in \mathsf{X} : \Phi(\mathsf{a}) \prec \tau\},$$

and, for each $\tau \in \mathbf{N} \cup \mathbf{B}$:

- $\mathfrak{N}(\tau) := \{\omega \in \mathbf{N} : \omega \prec \tau\}$,
- $\mathfrak{M}_m(\tau) := \{\omega \in \mathsf{M}_m : \omega \prec \tau\}$,

so that

Corollary 10.3. *It holds*

(D): *For each $\tau \in \mathbf{N}$ there is a unique polynomial*

$$f_\tau := \tau - \sum_{\omega \in \mathfrak{N}(\tau)} c(f_\tau, \omega)\omega$$

such that $f_\tau(\mathsf{a}) = 0$, for each $\mathsf{a} \in \mathfrak{X}(\tau)$.

(E): *For each $\tau \in \mathbf{G}$ there is a unique polynomial*

$$f_\tau := \tau - \sum_{\omega \in \mathbf{N}} c(f_\tau, \omega)\omega$$

such that $f_\tau(\mathsf{a}) = 0$, for each $\mathsf{a} \in \mathsf{X}$.

Proof. Since $\#\mathfrak{X}(\tau) = \#\mathfrak{N}(\tau)$ and $\#\mathsf{X} = \#\mathbf{N}$, f_τ can be computed by interpolation. □

In the same mood, but interpolation is not sufficient to prove it, we can state

Fact 10.4. *It holds*

(F): *For each $\tau \in \mathbf{B}$ there is a polynomial*

$$f_\tau := \tau - \sum_{\omega \in \mathfrak{N}(\tau)} c(f_\tau, \omega)\omega$$

such that $f_\tau(\mathsf{a}) = 0$, for each $\mathsf{a} \in \mathsf{X}$.

□

Corollary 10.5. *It holds:*

(G): *The reduced Gröbner basis of $\mathsf{I}(\mathsf{X})$ is*

$$\mathcal{G}(\mathsf{I}(\mathsf{X})) := \{f_\tau : \tau \in \mathbf{G}\};$$

moreover, for each $\tau \in \mathbf{N}$, $\mathbf{T}(f_\tau) = \tau$.

(H): *The border basis of* $\mathsf{I}(\mathsf{X})$ *is*

$$\mathcal{B}(\mathsf{I}(\mathsf{X})) := \{f_\tau : \tau \in \mathbf{B}\}.$$

Proof. For each $\tau \in \mathbf{G} \cup \mathbf{B}$, τ is the only term in f_τ which is not a member of $\mathbf{N}$ so that $\mathbf{T}(f_\tau) = \tau$.

For any $\tau \in \mathbf{N}$, $\mathbf{T}(f_\tau) = \tau$ because Cerlienco–Mureddu Correspondence grants $\tau \in \mathbf{G}(\mathsf{I}(\mathfrak{x}(\tau)))$ and $\mathbf{N}(\mathsf{I}(\mathfrak{x}(\tau))) = \mathfrak{N}(\tau)$. □

Linear interpolation, again, is all one needs to prove

Proposition 10.6. *With the same notation as in Lemma 9.4 it holds*

(L): *for each* $\tau := X_1^{d_1} \cdots X_n^{d_n} \in \mathbf{G}$, *and each* $m, 1 \leq m \leq n$, *there are polynomials*

$$g_{m\tau} := X_m^{d_m} + \sum_{\omega \in \mathsf{M}_m(\tau)} c(g_{m\tau}, \omega)\omega$$

such that $g_{m\tau}(\mathsf{a}) = 0$, *for each* $\mathsf{a} \in \mathsf{D}_m(\tau)$*;*

(I): *for each* $\tau := X_1^{d_1} \cdots X_n^{d_n} \in \mathbf{N}$ *and each* $m, 1 \leq m \leq n$, *there are polynomials*

$$g_{m\tau} := X_m^{d_m} + \sum_{\omega \in \mathfrak{M}_m(\tau)} c(g_{m\tau}, \omega)\omega$$

such that $g_{m\tau}(\mathsf{a}) = 0$, *for each* $\mathsf{a} \in \mathsf{D}_m(\tau), \mathsf{a} \prec \Phi^{-1}(\tau)$;

Proof.

(L): Since $\#(\mathsf{C}_m(\tau)) \leq \mathsf{M}_m(\tau)$ interpolation allows to evaluate each $c(g_{m\tau}, \omega)$ so that $g_{m\tau}(\mathsf{b}) = 0, \forall \mathsf{b} \in \mathsf{C}_m(\tau)$ and $g_{m\tau}(\mathsf{a}) = g_{m\tau}(\pi_m(\mathsf{a})), \forall \mathsf{a} \in \mathsf{D}_m(\tau)$.

(I): One has just to apply **(L)** to the set $\mathfrak{x}(\tau)$.

□

For each $\tau := X_1^{d_1} \cdots X_n^{d_n} \in \mathbf{N}$, let us denote $\nu := \nu(\tau) \leq n$ the value such that $d_\nu \neq 0$ while $d_\mu = 0$ for each $\mu > \nu$ so that

$\tau \in \mathcal{T}[1,\nu]$, $g_{m\tau} = 1$ for $m > \nu$, and, denoting

$$\begin{aligned} h_\tau &:= \prod_{m=1}^{n} g_{m\tau} \in k[X_1,\ldots,X_{\nu-1}][X_\nu], \\ l_\tau &:= \prod_{m=1}^{\nu(\tau)-1} g_{m\tau} \in k[X_1,\ldots,X_{\nu-1}], \\ p_\tau &:= g_{\nu\tau} \in k[X_1,\ldots,X_{\nu-1}][X_\nu], \end{aligned}$$

it holds

$$h_\tau = l_\tau p_\tau = l_\tau X_\nu^{d_\nu} + \cdots$$

so that $l_\tau \in k[X_1,\ldots,X_{\nu-1}]$ is the leading polynomial and the content of h_τ while the monic polynomial p_τ is the primitive component of h_τ.

Therefore we have

Corollary 10.7. *With the notation above, it holds*

(M): *for each* $\tau = X_1^{d_1}\cdots X_\nu^{d_\nu} \in \mathbf{N}$, *there are*

$$l_\tau \in k[X_1,\ldots,X_{\nu-1}]$$

and a monic polynomial

$$p_\tau = X_\nu^{d_\nu} + \sum_{\omega\in\mathfrak{M}_\nu(\tau)} c(p_\tau,\omega)\omega \in k[X_1,\ldots,X_{\nu-1}][X_\nu]$$

so that $h_\tau := l_\tau p_\tau$ *are such that*

- $\mathbf{T}(h_\tau) = \tau$,
- $l_\tau(\pi_{\nu-1}(\mathsf{a})) = 0$, *for all* $\mathsf{a} \in \mathfrak{X}(\tau)$,
- $p_\tau(\mathsf{a}) = 0$, *for each* $\mathsf{a} \in \mathsf{D}_\nu(\tau)$,
- $h_\tau(\mathsf{a}) = 0$, *for each* $\mathsf{a} \in \mathsf{X}$ *such that* $\mathsf{a} \prec \Phi^{-1}(\tau)$.

(N): *for each* $i, 1 \le i \le r$ *there are*

$$l_i \in k[X_1,\ldots,X_{\nu-1}]$$

and a monic polynomial

$$p_i = X_\nu^{d_\nu} + \sum_{\omega\in\mathsf{M}_\nu(\mathsf{t}_i)} c(p_i,\omega)\omega \in k[X_1,\ldots,X_{\nu-1}][X_\nu]$$

so that $h_i := l_i p_i$ *are such that*

- $\mathbf{T}(h_i) = \mathsf{t}_i = X_1^{d_1}\cdots X_\nu^{d_\nu} \in \mathbf{G}\cap\mathcal{T}[1,\nu]$,
- $l_i(\pi_{\nu-1}(\mathsf{a})) = 0$, *for each* $\mathsf{a} \in \cup_{m=1}^{\nu-1}\mathsf{D}_m(\mathsf{t}_i)$,

- $p_i(\mathsf{a}) = 0$, *for each* $\mathsf{a} \in \mathsf{D}_\nu(\mathsf{t}_i)$,
- $h_i(\mathsf{a}) = 0$, *for each* $\mathsf{a} \in \mathsf{X}$.

□

While $\#(\mathsf{C}_m(\tau)) \leq \mathsf{M}_m(\tau)$, in general equality does not hold and the polynomials $g_{m\tau}$ are not unique. However, uniqueness can be forced via Cerlienco–Mureddu Correspondence as follows: let us denote, for each $\tau \in \mathbf{N} \cup \mathbf{G}$ and each $m, 1 \leq m \leq n$:

$$\begin{aligned}
&\mathsf{R}_m(\tau) := \mathsf{R}_m(\mathsf{X}, \tau) := \{\mathsf{b} \in \mathsf{C}_m(\tau) : \textstyle\prod_{\nu=1}^{m-1} g_{\nu\tau}(\mathsf{b}) \neq 0\},\\
&\mathsf{E}_m(\tau) := \mathsf{E}_m(\mathsf{X}, \tau) := \mathbf{N}(\mathsf{R}_m(\tau)),\\
&\mathsf{S}_m(\tau) := \mathsf{S}_m(\mathsf{X}, \tau) := \{\pi_m(\mathsf{a}) \in \mathsf{R}_m(\tau) : \mathsf{a} \prec \Phi^{-1}(\tau)\},\\
&\mathsf{F}_m(\tau) := \mathsf{F}_m(\mathsf{X}, \tau) := \mathbf{N}(\mathsf{S}_m(\tau)).
\end{aligned}$$

Then:

Corollary 10.8. *With this notation it holds*

(P): *for each* $\tau := X_1^{d_1} \cdots X_n^{d_n} \in \mathbf{G}$, *and each* $m, 1 \leq m \leq n$ *there are unique polynomials*

$$\gamma_{m\tau} := X_m^{d_m} + \sum_{\omega \in \mathsf{E}_m(\tau)} c(\gamma_{m\tau}, \omega)\omega$$

such that $\gamma_{m\tau}(\mathsf{a}) = 0$, *for each* $\mathsf{a} \in \mathsf{D}_m(\tau)$*;*

(O): *for each* $\tau := X_1^{d_1} \cdots X_n^{d_n} \in \mathbf{N}$, *and each* $m, 1 \leq m \leq n$ *there are unique polynomials*

$$\gamma_{m\tau} := X_m^{d_m} + \sum_{\omega \in \mathsf{F}_m(\tau)} c(\gamma_{m\tau}, \omega)\omega$$

such that $\gamma_{m\tau}(\mathsf{a}) = 0$, *for each* $\mathsf{a} \in \mathsf{D}_m(\tau), \mathsf{a} \prec \Phi^{-1}(\tau)$;

(Q): *for each* $\tau = X_1^{d_1} \cdots X_\nu^{d_\nu} \in \mathbf{N}$, *there are*

$$\lambda_\tau \in k[X_1, \ldots, X_{\nu-1}]$$

and a unique monic polynomial

$$\rho_\tau = X_\nu^{d_\nu} + \sum_{\omega \in \mathsf{F}_\nu(\tau)} c(\rho_\tau, \omega)\omega \in k[X_1, \ldots, X_{\nu-1}][X_\nu]$$

so that $k_\tau := \lambda_\tau \rho_\tau$ *are such that*

- $\mathbf{T}(k_\tau) = \tau$,

- $\lambda_\tau(\pi_{\nu-1}(\mathsf{a})) = 0$, *for each* $\mathsf{a} \in \mathfrak{X}(\tau)$,
- $\rho_\tau(\mathsf{a}) = 0$, *for each* $\mathsf{a} \in \mathsf{D}_\nu(\tau)$,
- $k_\tau(\mathsf{a}) = 0$, *for each* $\mathsf{a} \in \mathsf{X} : \mathsf{a} \prec \Phi^{-1}(\tau)$.

(R): *for each* $i, 1 \leq i \leq r$ *there are*

$$\lambda_i \in k[X_1, \dots, X_{\nu-1}]$$

and a unique monic polynomial

$$\rho_i = X_\nu^{d_\nu} + \sum_{\omega \in \mathsf{E}_\nu(\mathsf{t}_i)} c(\omega)\omega \in k[X_1, \dots, X_{\nu-1}][X_\nu]$$

so that $k_i := \lambda_i \rho_i$ *are such that*

- $\mathbf{T}(k_i) = \mathsf{t}_i = X_1^{d_1} \cdots X_\nu^{d_\nu} \in \mathbf{G} \cap \mathcal{T}[1, \nu]$,
- $\lambda_i(\pi_{\nu-1}(\mathsf{a})) = 0$, *for each* $\mathsf{a} \in \cup_{m=1}^{\nu-1} \mathsf{D}_m(\mathsf{t}_i)$,
- $\rho_i(\mathsf{a}) = 0$, *for each* $\mathsf{a} \in \mathsf{D}_\nu(\mathsf{t}_i)$,
- $k_i(\mathsf{a}) = 0$, *for each* $\mathsf{a} \in \mathsf{X}$;

$\square$

Corollary 10.9. *It holds*

(S): $\{h_1, \dots, h_r\}$ *and* $\{k_1, \dots, k_r\}$ *are minimal Gröbner bases of* $\mathsf{I}(\mathsf{X})$*;*

(U): *For each* $\nu, 1 \leq \nu < n$, *and each* $\delta \in \mathbb{N}$ *let* $j(\nu\delta)$ *be the value such that* $\mathsf{t}_{j(\nu\delta)} < X_{\nu+1}^{\delta} \leq \mathsf{t}_{j(\nu\delta)+1}$; *then* $\{l_1, \dots, l_{j(\nu\delta)}\}$ *and* $\{\lambda_1, \dots, \lambda_{j(\nu\delta)}\}$ *are a Gröbner basis of* $\mathsf{I}(\mathsf{Y}_{\nu\delta})$*;*

(T): *For each* $\nu, 1 \leq \nu < n$ *let* j_ν *the value such that* $\mathsf{t}_{j_\nu} < X_{\nu+1} \leq \mathsf{t}_{j_\nu+1}$; *then* $\{h_1, \dots, h_{j_\nu}\}$ *and* $\{k_1, \dots, k_{j_\nu}\}$ *are minimal Gröbner bases of* $\mathsf{I}(\mathsf{X}) \cap k[X_1, \dots, X_\nu]$ *and of* $\mathsf{I}(\pi_\nu(\mathsf{X}))$.

$\square$

Proof.

(S): is obvious;

(U): is a direct application of **(S)** to the set of points $\mathsf{Y}_{\nu\delta}$ via Remark 9.5

(T): is a particular instance of **(U)**; minimality is trivial.

$\square$

Remark 10.10. *The only difference between the three bases*

$$\{f_1 : \dots, f_r\}, \{h_1, \dots, h_r\} \text{ and } \{k_1, \dots, k_r\}$$

is that the first is reduced unlike the others. On the other side, for each i, we have

$$\mathbf{T}(f_i) = \mathbf{T}(h_i) = \mathbf{T}(k_i) = \mathsf{t}_i.$$

Therefore we have

- $f_1 = h_1 = k_1$ *and*
- $f_i - h_i \in (h_1, \dots, h_{i-1}), f_i - k_i \in (k_1, \dots, k_{i-1})$ *for each* $i, 1 < i \leq r$,

whence

- $f_i \in (h_1, \dots, h_i), f_i \in (k_1, \dots, k_i)$ *for each* $i, 1 \leq i \leq r$.

Fact 10.11. *It holds*

(W): *For each* $i, 2 \leq i \leq r$, $p_i \in (h_j, j < i) : l_i$ *and* $\rho_i \in (k_j, j < i) : \lambda_i$.

$\square$

Fact 10.12. *It holds*

(X): *for each* $\tau \in \mathbf{N}$, $f_\tau(\Phi^{-1}(\tau)) \neq 0$, $h_\tau(\Phi^{-1}(\tau)) \neq 0$, $k_\tau(\Phi^{-1}(\tau)) \neq 0$.

$\square$

Corollary 10.13. *It holds*

(Z): $\mathbb{L}(\mathsf{X})$ *is triangular to* $\{f_\tau^{-1}(\Phi^{-1}(\tau))f_\tau, \tau \in \mathsf{X}\}$, $\{h_\tau^{-1}(\Phi^{-1}(\tau))h_\tau, \tau \in \mathsf{X}\}$ *and* $\{k_\tau^{-1}(\Phi^{-1}(\tau))k_\tau, \tau \in \mathsf{X}\}$.

$\square$

11. **Another algorithmic proof**

In order to complete the proof all we need is to directly apply Möller Algorithm.

The proof being by induction, we begin with

Lemma 11.1. *If* $\#\mathsf{X} = 1$ *conditions* **(A), (F), (W), (X)** *hold.*

Proof. When we have a single point $(a_1, \dots, a_n) \in k^n$, we have

- $\mathbf{N} = \{1\}$,
- $\mathbf{B} = \mathbf{G} = \{X_1, \dots, X_n\}$,
- $f_1 = 1$,
- $f_{X_i} = X_i - a_i$, for each i,

and the properties are obviously satisfied. $\square$

This giving a starting point for induction, let us assume we have a set

$$\mathsf{X} := \{\mathsf{a}_1, \dots, \mathsf{a}_s\} \subset k^n, s > 1,$$

and let us denote $\mathsf{X}' := \{\mathsf{a}_1, \dots, \mathsf{a}_{s-1}\}$ for which we assume conditions **(A-Z)** hold.

In particular:

$\Phi' := \mathbf{N}' \mapsto \mathsf{X}'$ is Cerlienco–Mureddu Correspondence,

$\mathbf{G}' := \mathbf{G}(\mathsf{I}(\mathsf{X}')) = \{\omega_1, \dots, \omega_r\}, \omega_1 < \omega_2 < \dots < \omega_r$,

$\mathbf{B}' := \mathbf{B}(\mathsf{I}(\mathsf{X}'))$,

$f'_\omega, \omega \in \mathbf{B}'$, are the polynomials whose existence is implied by **(F)**,

$F_i := f'_{\omega_i}$ are the polynomials whose existence is implied by **(E)**, so that

$\{F_i : 1 \leq i \leq r\}$ is the reduced Gröbner basis of $\mathsf{I}(\mathsf{X}')$;

l'_i, p'_i, h'_i are the polynomials whose existence is implied by **(N)**,

$\lambda'_i, q'_i, \rho'_i$ are the polynomials whose existence is implied by **(R)**.

Setting

$$I := \min_{<}\{j, 1 \leq j \leq r : F_j(\mathsf{a}_s) \neq 0\}$$

then it holds

Lemma 11.2. *If* X' *satisfies conditions* **(A-Z)** *then*

$$\Psi(\mathsf{X})(\mathsf{a}_s) = \omega_I.$$

Proof. Let $\omega_I = X_1^{d_1} \dots X_n^{d_n}$ and let $m+1 := \max(i : d_i \neq 0)$, so that

$$F_I \in k[X_1, \dots, X_{m+1}].$$

Since, by **(T)**, for each ν,

$$\mathsf{I}(\mathsf{X}') \cap k[X_1, \dots, X_\nu] = \mathsf{I}(\pi_\nu(\mathsf{X}')),$$

and

$$F_j \in k[X_1, \dots, X_\nu], \nu \leq m \Longrightarrow j < I$$

we deduce that

$F_j(\pi_\nu(\mathsf{a}_s)) = F_j(\mathsf{a}_s) = 0$, for each $F_j \in k[X_1, \dots, X_\nu], \nu \leq m$, while

$F_I(\pi_{m+1}(\mathsf{a}_s)) = F_I(\mathsf{a}_s) \neq 0.$
This allows to deduce that

$$m = \max\left(j : \text{ exists } i < s : \pi_j(\mathsf{a}_i) = \pi_j(\mathsf{a}_s)\right).$$

Therefore $\pi_{m+1}(\mathsf{a}_s) \notin \{\pi_{m+1}(\mathsf{a}), \mathsf{a} \in \mathsf{X}'\}$; also

$$d_m = \#\{\mathsf{a}_i, i < s : \pi_m(\mathsf{a}_i) = \pi_m(\mathsf{a}_s)\};$$

in fact, for each $\delta < d_m$, since

$$\mathbf{T}(F_j) = \omega_j < X_m^\delta < X_m^{d_m} \Longrightarrow j < I,$$

and $F_j(\pi_m(\mathsf{a}_s)) = 0$, **(U)** allows to deduce that

$$\pi_m(\mathsf{a}_s) \in \mathsf{Y}_{m\delta} := \left\{\mathsf{b} \in \pi_m(\mathsf{X}') : \delta < \#\left\{\mathsf{a} \in \mathsf{X}' : \mathsf{b} = \pi_m(\mathsf{a})\right\}\right\}$$

and $\pi_m(\mathsf{a}_s) \notin \mathsf{Y}_{md_m}$.

As a consequence we consider the sets of points

$$\mathsf{W} := \{\mathsf{a}_i : \Phi'(\mathsf{a}_i) = \tau_i X_{m+1}^{d_m}, \tau_i \in \mathcal{T}[1,m]\} \cup \{\mathsf{a}_s\} \text{ and } \mathsf{Z} := \pi_m(\mathsf{W});$$

in this setting Cerlienco–Mureddu Correspondence gives a relation between each point $\pi_m(\mathsf{a}_i)$ and the corresponding term τ_i; also, by

(U), the ideal $\mathsf{I}(\pi_{m+1}(\mathsf{W}))$ has the Gröbner basis $\{l'_1, \dots, l'_{j_{md_m}}\}$ where

$$l'_j(\pi_m(\mathsf{a}_s)) = 0, \forall j < I \text{ while } l'_I(\pi_m(\mathsf{a}_s)) \neq 0.$$

So the same argument grants that Cerlienco–Mureddu Correspondence returns $\Phi(\pi_m((\mathsf{a}_s)) = X_1^{d_1} \dots X_{m-1}^{d_{m-1}}$. □

As a consequence, the application of Möller Algorithm to $\mathsf{X} = \mathsf{X}' \cup \{\mathsf{a}_s\}$ produces

$q_s := c^{-1} F_I$, with $c = F_I(\mathsf{a}_s)$;
$\mathbf{N} := \mathbf{N}' \cup \{\omega_I\}$;
$\mathbf{B} := (\mathbf{B}' \setminus \{\omega_I\}) \cup \{X_i\omega_I, 1 \leq i \leq n\}$;
$f_\tau := f'_\tau - f_\tau(\mathsf{a}_s) q_s$ for each $\tau \in \mathbf{B}' \setminus \{\omega_I\}, \tau > \omega_I$ and
$f_\tau := f'_\tau$, for each $\tau \in \mathbf{B}' \setminus \{\omega_I\}, \tau < \omega_I$ since $f'_\tau(\mathsf{a}_s) = 0$;
for each $\tau := X_i\omega_I \notin \mathbf{B}'$

$$f_\tau := (X_i - a_{is}) F_I - \sum_{X_i\omega \in \mathbf{B}'} c(F_I, \omega) f_{X_i\omega}$$

where

$$F_I = \omega_I + \sum_{\omega \in \mathbf{N}'} c(F_I, \omega)\omega.$$

Proposition 11.3. *If* X' *satisfies conditions* **(A-Z)** *then* X *satisfies conditions* **(A)**, **(F)**, **(W)**, **(X)**.

Proof.

(A): is obvious;
(F): is obvious;
(W): on the basis of Remark 10.10 we know that $F_I \in (h'_1, \dots, h'_I)$; also all we need to prove is that, for each i,

$$h_i \in (h_1, \dots, h_{i-1}) = \{h_j, \mathbf{T}(h_j) < \mathbf{T}(h_i)\}.$$

Therefore

- if $\mathbf{T}(h_i) = \mathsf{t}_i \in \mathbf{G}', i < I$, we have
$$h_i = h'_i \in (h'_1, \dots, h'_{i-1}) = (h_1, \dots, h_{i-1});$$
- if $\mathbf{T}(h_i) = \mathsf{t}_i \in \mathbf{G}', i > I$, we have
$$h_i = h'_i - aF_I \in (h'_1, \dots, h'_{i-1}) = (h_1, \dots, h_{i-1})$$
so that, also $(h'_1, \dots, h'_i) = (h_1, \dots, h_i)$.

- Finally, for $\tau = X_i \mathbf{t}_I$ we have $l_\tau = l'_I$, and

$$l_\tau p_\tau = h_\tau \equiv f_\tau \equiv (X_i - a_{is})F_I \equiv (X_i - a_{is})l'_I p'_I \equiv 0$$

modulo $(h'_1, \dots, h'_I) = (h_1, \dots, h_I)$

The same argument proofs the claim for $\{k_1, \dots, k_r\}$.

(X): $f_{\omega_I}(\mathsf{a}_s) \neq 0$ for construction; $h_{\omega_I}(\mathsf{a}_s) \neq 0$ and $k_{\omega_I}(\mathsf{a}_s) \neq 0$ because both $h_{\omega_I} - f_{\omega_I}$ and $k_{\omega_I} - f_{\omega_I}$ have a representation in terms of $\{F_i, i < I\}$ and $F_i(\mathsf{a}_s) = 0$, for each $i < I$. □

In conclusion we have:

Theorem 11.4.
(A) $\mathbf{N} := \mathbf{N}(\mathsf{I}(\mathsf{X}))$.
(B) $\mathbf{G}(\mathsf{I}(\mathsf{X})) = \mathbf{G} = \{\mathsf{t}_1, \dots, \mathsf{t}_r\}, \mathsf{t}_1 < \mathsf{t}_2 < \dots < \mathsf{t}_r$;
(C) $\mathbf{B}(\mathsf{I}(\mathsf{X})) = \mathbf{B}$.
(D) *For each $\tau \in \mathbf{N}$ there is a unique polynomial*

$$f_\tau := \tau - \sum_{\omega \in \mathfrak{N}(\tau)} c(f_\tau, \omega)\omega$$

such that $f_\tau(\mathsf{a}) = 0$, for each $\mathsf{a} \in \mathfrak{X}(\tau)$.
(E) *For each $\tau \in \mathbf{G}$ there is a unique polynomial*

$$f_\tau := \tau - \sum_{\omega \in \mathbf{N}} c(f_\tau, \omega)\omega$$

such that $f_\tau(\mathsf{a}) = 0$, for each $\mathsf{a} \in \mathsf{X}$.
(F) *For each $\tau \in \mathbf{B}$ there is a polynomial*

$$f_\tau := \tau - \sum_{\omega \in \mathfrak{N}(\tau)} c(f_\tau, \omega)\omega$$

such that $f_\tau(\mathsf{a}) = 0$, for each $\mathsf{a} \in \mathsf{X}$.
(G) *The reduced Gröbner basis of $\mathsf{I}(\mathsf{X})$ is*

$$\mathcal{G}(\mathsf{I}(\mathsf{X})) := \{f_\tau : \tau \in \mathbf{G}\};$$

moreover, for each $\tau \in \mathbf{N}$, $\mathbf{T}(f_\tau) = \tau$.
(H) *The border basis of $\mathsf{I}(\mathsf{X})$ is*

$$\mathcal{B}(\mathsf{I}(\mathsf{X})) := \{f_\tau : \tau \in \mathbf{B}\}.$$

(I) *for each* $\tau := X_1^{d_1} \cdots X_n^{d_n} \in \mathbf{N}$ *and each* $m, 1 \leq m \leq n$, *there are polynomials*

$$g_{m\tau} := X_m^{d_m} + \sum_{\omega \in \mathfrak{M}_m(\tau)} c(g_{m\tau}, \omega)\omega$$

such that $g_{m\tau}(\mathsf{a}) = 0$, *for each* $\mathsf{a} \in \mathsf{D}_m(\tau), \mathsf{a} \prec \Phi^{-1}(\tau)$;

(L) *for each* $\tau := X_1^{d_1} \cdots X_n^{d_n} \in \mathbf{G}$, *and each* $m, 1 \leq m \leq n$, *there are polynomials*

$$g_{m\tau} := X_m^{d_m} + \sum_{\omega \in \mathsf{M}_m(\tau)} c(g_{m\tau}, \omega)\omega$$

such that $g_{m\tau}(\mathsf{a}) = 0$, *for each* $\mathsf{a} \in \mathsf{D}_m(\tau)$;

(M) *for each* $\tau = X_1^{d_1} \cdots X_\nu^{d_\nu} \in \mathbf{N}$, *there are*

$$l_\tau \in k[X_1, \dots, X_{\nu-1}]$$

and a monic polynomial

$$p_\tau = X_\nu^{d_\nu} + \sum_{\omega \in \mathfrak{M}_\nu(\tau)} c(p_\tau, \omega)\omega \in k[X_1, \dots, X_{\nu-1}][X_\nu]$$

so that $h_\tau := l_\tau p_\tau$ *are such that*

- $\mathbf{T}(h_\tau) = \tau$,
- $l_\tau(\pi_{\nu-1}(\mathsf{a})) = 0$, *for all* $\mathsf{a} \in \mathfrak{X}(\tau)$,
- $p_\tau(\mathsf{a}) = 0$, *for each* $\mathsf{a} \in \mathsf{D}_\nu(\tau)$,
- $h_\tau(\mathsf{a}) = 0$, *for each* $\mathsf{a} \in \mathsf{X}$ *such that* $\mathsf{a} \prec \Phi^{-1}(\tau)$.

(N) *for each* $i, 1 \leq i \leq r$ *there are*

$$l_i \in k[X_1, \dots, X_{\nu-1}]$$

and a monic polynomial

$$p_i = X_\nu^{d_\nu} + \sum_{\omega \in \mathsf{M}_\nu(\mathsf{t}_i)} c(p_i, \omega)\omega \in k[X_1, \dots, X_{\nu-1}][X_\nu]$$

so that $h_i := l_i p_i$ *are such that*

- $\mathbf{T}(h_i) = \mathsf{t}_i = X_1^{d_1} \cdots X_\nu^{d_\nu} \in \mathbf{G} \cap \mathcal{T}[1, \nu]$,
- $l_i(\pi_{\nu-1}(\mathsf{a})) = 0$, *for each* $\mathsf{a} \in \cup_{m=1}^{\nu-1} \mathsf{D}_m(\mathsf{t}_i)$,
- $p_i(\mathsf{a}) = 0$, *for each* $\mathsf{a} \in \mathsf{D}_\nu(\mathsf{t}_i)$,
- $h_i(\mathsf{a}) = 0$, *for each* $\mathsf{a} \in \mathsf{X}$.

(O) *for each* $\tau := X_1^{d_1} \cdots X_n^{d_n} \in \mathbf{N}$, *and each* $m, 1 \leq m \leq n$ *there are unique polynomials*

$$\gamma_{m\tau} := X_m^{d_m} + \sum_{\omega \in \mathsf{F}_m(\tau)} c(\gamma_{m\tau}, \omega)\omega$$

such that $\gamma_{m\tau}(\mathsf{a}) = 0$, *for each* $\mathsf{a} \in \mathsf{D}_m(\tau), \mathsf{a} \prec \Phi^{-1}(\tau)$;

(P) *for each* $\tau := X_1^{d_1} \cdots X_n^{d_n} \in \mathbf{G}$, *and each* $m, 1 \leq m \leq n$ *there are unique polynomials*

$$\gamma_{m\tau} := X_m^{d_m} + \sum_{\omega \in \mathsf{E}_m(\tau)} c(\gamma_{m\tau}, \omega)\omega$$

such that $\gamma_{m\tau}(\mathsf{a}) = 0$, *for each* $\mathsf{a} \in \mathsf{D}_m(\tau)$;

(Q) *for each* $\tau = X_1^{d_1} \cdots X_\nu^{d_\nu} \in \mathbf{N}$, *there are*

$$\lambda_\tau \in k[X_1, \ldots, X_{\nu-1}]$$

and a unique monic polynomial

$$\rho_\tau = X_\nu^{d_\nu} + \sum_{\omega \in \mathsf{F}_\nu(\tau)} c(\rho_\tau, \omega)\omega \in k[X_1, \ldots, X_{\nu-1}][X_\nu]$$

so that $k_\tau := \lambda_\tau \rho_\tau$ *are such that*

- $\mathbf{T}(k_\tau) = \tau$,
- $\lambda_\tau(\pi_{\nu-1}(\mathsf{a})) = 0$, *for each* $\mathsf{a} \in \mathfrak{X}(\tau)$,
- $\rho_\tau(\mathsf{a}) = 0$, *for each* $\mathsf{a} \in \mathsf{D}_\nu(\tau)$,
- $k_\tau(\mathsf{a}) = 0$, *for each* $\mathsf{a} \in \mathsf{X} : \mathsf{a} \prec \Phi^{-1}(\tau)$.

(R) *for each* $i, 1 \leq i \leq r$ *there are*

$$\lambda_i \in k[X_1, \ldots, X_{\nu-1}]$$

and a unique monic polynomial

$$\rho_i = X_\nu^{d_\nu} + \sum_{\omega \in \mathsf{E}_\nu(\mathsf{t}_i)} c(\omega)\omega \in k[X_1, \ldots, X_{\nu-1}][X_\nu]$$

so that $k_i := \lambda_i \rho_i$ *are such that*

- $\mathbf{T}(k_i) = \mathsf{t}_i = X_1^{d_1} \cdots X_\nu^{d_\nu} \in \mathbf{G} \cap \mathcal{T}[1, \nu]$,
- $\lambda_i(\pi_{\nu-1}(\mathsf{a})) = 0$, *for each* $\mathsf{a} \in \cup_{m=1}^{\nu-1} \mathsf{D}_m(\mathsf{t}_i)$,
- $\rho_i(\mathsf{a}) = 0$, *for each* $\mathsf{a} \in \mathsf{D}_\nu(\mathsf{t}_i)$,
- $k_i(\mathsf{a}) = 0$, *for each* $\mathsf{a} \in \mathsf{X}$;

(S) $\{h_1, \ldots, h_r\}$ *and* $\{k_1, \ldots, k_r\}$ *are minimal Gröbner bases of* $\mathsf{I}(\mathsf{X})$;

(T) *For each* $\nu, 1 \leq \nu < n$ *let* j_ν *be the value such that* $\mathsf{t}_{j_\nu} < X_{\nu+1} \leq \mathsf{t}_{j_\nu+1}$; *then* $\{h_1, \dots, h_{j_\nu}\}$ *and* $\{k_1, \dots, k_{j_\nu}\}$ *are minimal Gröbner bases of* $\mathsf{I}(\mathsf{X}) \cap k[X_1, \dots, X_\nu]$ *and of* $\mathsf{I}(\pi_\nu(\mathsf{X}))$.

(U) *For each* $\nu, 1 \leq \nu < n$, *and each* $\delta \in \mathbb{N}$ *let* $j(\nu\delta)$ *be the value such that* $\mathsf{t}_{j(\nu\delta)} < X_{\nu+1}^\delta \leq \mathsf{t}_{j(\nu\delta)+1}$; *then* $\{l_1, \dots, l_{j(\nu\delta)}\}$ *and* $\{\lambda_1, \dots, \lambda_{j_{\nu\delta})}\}$ *are Gröbner bases of* $\mathsf{I}(\mathsf{Y}_{\nu\delta})$;

(W) *For each* $i, 2 \leq i \leq r$, $p_i \in (h_j, j < i) : l_i$ *and* $\rho_i \in (k_j, j < i) : \lambda_i$.

(X) *for each* $\tau \in \mathbf{N}$, $f_\tau(\Phi^{-1}(\tau)) \neq 0$, $h_\tau(\Phi^{-1}(\tau)) \neq 0$, $k_\tau(\Phi^{-1}(\tau)) \neq 0$.

(Z) $\mathbb{L}(\mathsf{X})$ *is triangular to*
$\{f_\tau^{-1}(\Phi^{-1}(\tau))f_\tau, \tau \in \mathsf{X}\}$,
$\{h_\tau^{-1}(\Phi^{-1}(\tau))h_\tau, \tau \in \mathsf{X}\}$ *and* $\{k_\tau^{-1}(\Phi^{-1}(\tau))k_\tau, \tau \in \mathsf{X}\}$. □

12. **Congedo**

The reader can easily realize that (under the present assumptions, i.e. simple points forcing the ideal to be radical) Gianni-Kalkbrenner Theorem [5, 7] is a direct corollary of **(M)**.

However, the following trivial example

$$\mathsf{I} := \{X_2X_3 - X_1^2, X_1^3, X_2X_1^2, X_3^2, X_1^2X_3, X_2^2\} \subset k[X_1, X_2, X_3]$$

shows that a primary ideal does not necessarily satisfy conditions **(I-W)** and the relevant part of **(Z)**; and the example

$$\begin{aligned}
\mathsf{I} &:= (X_1^2, X_2 + X_1, X_3) \cap (X_1^2, X_2 - X_1, X_3 - 1) \\
&= \{X_1^2, X_1X_2, X_2^2, X_1X_3 - \frac{1}{2}X_1 - \frac{1}{2}X_2, X_2X_3 - \frac{1}{2}X_1 - \frac{1}{2}X_2, \\
&\quad X_3^2 - X_3\}
\end{aligned}$$

shows that Thereom 11.4 does not hold at all for zero-dimensional ideals, at least trivially.

On the other side we recently discovered that in 1995 Cerlienco and Mureddu [3] generalized their Correspondence to zero-dimensional ideals

$$\mathsf{I} := \bigcup_{i=1,n} \ell_{\mathsf{a}_i}(\mathbf{M}_i)$$

where, for each i

$$\mathsf{a}_i = (a_{i1}, \dots, a_{in}) \in k^n$$
$\ell_{\mathsf{a}_i} : \mathcal{P} \mapsto \mathcal{P}$ is the traslation

$$\ell_{\mathsf{a}_i}(f) = f(X_1 - a_{i1}, \dots, X_1 - a_{in}) \text{ for each } f(X_1, \dots, X_n) \in \mathcal{P}$$

$\mathbf{M}_i \subset \mathcal{T}$ is a monimal ideal.

We have the impression that in this setting Thereom 11.4 still holds, up to the elementary adaptations needed in order to properly define, in conditions **(I-R)**, the monomial sets and the linear functionals on which to perform linear interpolation, and that the proof presented here requires just Leibnitz Formula to grant correcteness of conditions **(M-N)** and **(Q-R)**.

Moreover Gianni-Kalkbrener Theorem is satisfied by configurations of multiple points but apparently is not provable by our approach and the example presented above gives us the feeling that a deeper analysis could suggest how to properly adapt Thereom 11.4.

So the description of the Gröbnerian Structure of configurations of multiple points still requires proper investigation.

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