Bulletin of the Iranian Mathematical Society Vol. 29 *No. 1 (2003), pp 55-64.*

COMPACTIFICATION OF κ-FRAMES

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Abstract. In this paper we show that the category $\mathbf{KR}\kappa\mathbf{Frm}$, of all compact regular κ-frames and κ-frame homomorphisms, is a coreflective subcategory of the category $\kappa\mathbf{Frm}$, of all κ-frames and κ-frame homomorphisms. Then, a compactification for any completely regular κ-frame and any proximal κ-frame is given. The theory of κ-frames was introduced by Madden [3].

1. Background

Here we recall some notions and notations from [2], [4].

1.1 Let κ be any regular cardinal. A *κ-set* is a set of cardinality strictly less than κ. A *κ-frame* is a bounded lattice L which has joins of κ-subsets and satisfies the distributive law:

$$x \wedge \bigvee S = \bigvee \{x \wedge s : s \in S\}$$

for $x \in L$ and S a κ-subset of L. A *κ-frame homomorphism* $h : L \to M$ is a lattice homomorphism preserving joins of κ-subsets. The resulting category is denoted by $\kappa\mathbf{Frm}$. The theory of κ-frames was introduced by Madden [3].

MSC(2000): 06B10, 06D22, 18A40, 54D35
Keywords: Compact regular κ-frame, Completely regular ideal, Strongly regular ideal, Compactification, Proximal κ-frame
Received: 16 October 2002 , Accepted: 4 August 2003

If $\kappa = w_1$, the smallest uncountable cardinal, a κ-frame is called a σ-frame. An example of a σ-frame is the lattice of cozero-sets of a topological space X.

1.2 An element a of a bounded lattice L is said to be *rather below* b, written $a \prec b$, if there exists $s \in L$, called the *separating element*, such that $a \wedge s = 0$ and $b \vee s = e$.

An element a of a bounded lattice L is said to be *completely below* b, written $a \prec\!\prec b$, if there exists an interpolating sequence (c_{nk}), $k = 0, 1, \cdots, 2^n$ and $n = 0, 1, \cdots$, between a and b, where $c_{00} = a$, $c_{01} = b$, $c_{nk} = c_{n+1,2k}$, $c_{nk} \prec c_{n,k+1}$.

A frame L is called *regular* (*completely regular*) if each $a \in L$ is a join of elements rather below (completely below) it.

1.3 A κ-frame L is called *compact* if whenever $e = \bigvee S$, for $S \subseteq L$, then $e = \bigvee F$ for some finite subset F of S. The category **KR**κ**Frm**, of all compact regular κ-frames, is a full subcategory of κ**Frm**. A κ-frame homomorphism $h : M \to L$ is called a *compactification* of L if M is compact regular and h is surjective and dense, that is $h(x) = 0$ implies $x = 0$.

1.4 Let L be a κ-frame. We call an ideal $J \subseteq L$ *regular* (*completely regular*) if for each $a \in J$ there exists $b \in J$ such that $a \prec b$ ($a \prec\!\prec b$).

2. Compact regular κ-frames

Our main aim in this section is to show that the category **KR**κ**Frm** is a coreflective subcategory of the category κ**Frm**. To prove this, we need to find a right adjoint to the inclusion functor $\mathcal{I}$: **KR**κ**Frm** $\to$ κ**Frm**.

Let L be a κ-frame and $\mathcal{K}L$ be the set of all completely regular ideals generated by κ-subsets of L. In the following we show that $\mathcal{K}L$ is a compact regular κ-frame.

Lemma 2.1. *The set $\mathcal{K}L$ is a κ-frame.*

Proof. Since $0 \prec\prec 0$ and $e \prec\prec e$, $\{0\}$ and L belong to $\mathcal{K}L$. Let $I, J \in \mathcal{K}L$ be generated by κ-subsets X, Y of L, respectively. Then $I \wedge J = I \cap J$ is regular, since $a \prec\prec b$, $c \prec\prec d$ imply $a \wedge c \prec\prec b \wedge d$, and it is generated by $X \wedge Y = \{x \wedge y : x \in X, y \in Y\}$. Thus $I \wedge J \in \mathcal{K}L$. Let $\{I_\lambda : \lambda \in \Lambda\}$ be a κ-subset of $\mathcal{K}L$ and for each λ, I_λ be generated by the κ-subset X_λ. Take J to be the ideal generated by the set $\bigcup X_\lambda$. We show that $J \in \mathcal{K}L$ and $\bigvee I_\lambda = J$. For each $a \in J$ there exists a finite set $\{x_1, ..., x_n\} \subseteq \bigcup X_\lambda$ such that $a \leq x_1 \vee ... \vee x_n$. Let $x_i \in X_{\lambda_i}$ for each i. By complete regularity of I_{λ_i} there exists $y_i \in I_{\lambda_i}$ such that $x_i \prec\prec y_i$. Hence $a \leq x_1 \vee ... \vee x_n \prec\prec y_1 \vee ... \vee y_n \in J$. Therefore $J \in \mathcal{K}L$, since it is a completely regular ideal and it is generated by a κ-set. Clearly $I_\lambda \subseteq J$ for each $\lambda \in \Lambda$. Let $I_\lambda \subseteq I \in \mathcal{K}L$ for each $\lambda \in \Lambda$. Take $a \in J$. Then $a \leq x_1 \vee ... \vee x_n$, where $x_i \in X_{\lambda_i}$ for each i. Since $X_\lambda \subseteq I_\lambda \subseteq I$ for each $\lambda \in \Lambda$, we have $a \in I$. Thus $J \subseteq I$. Hence $\bigvee I_\lambda = J$. Also we have $I \wedge \bigvee I_\lambda = \bigvee\{I \wedge I_\lambda : \lambda \in \Lambda\}$ for each $I \in \mathcal{K}L$ and κ-set $\{I_\lambda\}$ of $\mathcal{K}L$. Therefore $\mathcal{K}L$ is a κ-frame. $\Box$

Note 2.2. *Let $x \prec y \prec z$ be in a κ-frame L and s, t be the separating elements of $x \prec y$ and $y \prec z$, respectively. Then $y \wedge t = 0$ and $y \vee s = e$ imply that $t \prec s$.*

Lemma 2.3. *Let $x_1 \prec\prec x_2 \prec\prec x_3$ be in a κ-frame L. Then, there exist t, u in L such that $u \vee x_3 = e, t \wedge x_1 = 0$ and $u \prec\prec t$.*

Proof. By the definition of $\prec\prec$ there exists x_4 such that $x_1 \prec\prec x_4 \prec\prec x_2 \prec\prec x_3$. Let t, a, u be the separating elements, respectively. By the above note $u \prec a \prec t$, also $u \vee x_3 = e$ and $t \wedge x_1 = 0$. It is enough to show that $u \prec\prec t$. Take x_5, x_6 such that $x_4 \prec x_5 \prec x_6 \prec x_2$. Let b, c, d be the separating elements, respectively. By the above note $d \prec c \prec b$. Since

$x_1 \prec x_4 \prec x_5$ and $x_6 \prec x_2 \prec x_3$ and t, b and d, u are separating elements, respectively, we have $u \prec d$ and $b \prec t$. Thus $u \prec c \prec t$. Take x_7, x_8, x_9, x_{10} such that $x_4 \prec x_7 \prec x_8 \prec x_5$ and $x_6 \prec x_9 \prec x_{10} \prec x_2$. Similar to the above discussion we can easily show that if f, g, h, i, j, k are the separating elements, respectively, then $u \prec k \prec j \prec i \prec c \prec h \prec g \prec f \prec t$. Thus $u \prec j \prec c \prec g \prec t$. Continuing this procces, it shows that we can find an element between each two elements of this series. Hence $u \prec\prec t$.

Proposition 2.4. *The κ-frame $\mathcal{K}L$ is compact regular.*

Proof. Trivially $\mathcal{K}L$ is compact, since the frame of all ideals of L is compact. To prove the regularity let $I \in \mathcal{K}L$ be generated by a κ-subset X of L. For each $x \in X$ there exists $y_x \in I$ such that $x \prec\prec y_x$. Since $\prec\prec$ interpolates, there exists a sequence $\{x_i : i \in \mathbb{N}\} \subseteq L$ such that $x = x_0 \prec\prec x_1 \prec\prec ... \prec\prec y_x$. Let J_x be the ideal generated by $\{x_i : i \in \mathbb{N}\}$. Then $J_x \in \mathcal{K}L$ for each $x \in X$. We show that $I = \bigvee\{J_x : x \in X\}$, and $J_x \prec I$ for each $x \in X$. Given $a \in I$ there exists a finite subset $\{a_1, ..., a_n\}$ of X such that $a \leq a_1 \vee ... \vee a_n$. Trivially $a \in J_{a_1} \vee ... \vee J_{a_n}$. Hence $I \subseteq \bigvee\{J_x : x \in X\}$. Also, for each $x \in X$, $J_x \subseteq I$. Thus $I = \bigvee\{J_x : x \in X\}$.

It is enough to show that $J_x \prec I$ for each $x \in X$. Take $z, w \in I$ such that $y_x \prec\prec z \prec\prec w$. By the above lemma, there exist t, u such that $w \vee u = e, y_x \wedge t = 0$, and $u \prec\prec t$. Let $u = u_0 \prec\prec u_1 \prec\prec \ \prec\prec t$ and K_x be the ideal generated by $\{u_i : i \in \mathbb{N}\}$. Then $K_x \in \mathcal{K}L$ and $e = u \vee w \in K_x \vee I$. Thus $K_x \vee I = L$. Also, if $a \in K_x \cap J_x$ then $a \leq u_m$ and $a \leq x_n$ for some $m, n \in \mathbb{N}$. Thus $a \leq t \wedge y_x = 0$. Hence $K_x \cap J_x = \{0\}$. Therefore K_x is a separating element of $J_x \prec I$. Hence $\mathcal{K}L$ is a regular κ-frame. □

Proposition 2.5. *The assignment $\mathcal{K} : \kappa\mathbf{Frm} \to \mathbf{KR}\kappa\mathbf{Frm}$ given by*

$$\mathcal{K}(f : L \to M) = \mathcal{K}(f) : \mathcal{K}L \to \mathcal{K}M$$

where $\mathcal{K}(f)(I) = <f[X]>$ *and* I *is generated by the* κ*-subset* X *of* L*, is a functor.*

Proof. By the above proposition $\mathcal{K}$ is well-defined on objects. Since $x \prec\prec y$ implies $f(x) \prec\prec f(y)$ it is easy to see that $\mathcal{K}$ is well-defined on morphisms. Also, $\mathcal{K}(f \circ g) = \mathcal{K}(f) \circ \mathcal{K}(g)$ and $\mathcal{K}(id_L) = id_{\mathcal{K}L}$. Hence $\mathcal{K}$ is a functor. $\square$

Now we show that $\mathcal{I} \dashv \mathcal{K}$. First we introduce the counit of this adjunction.

Lemma 2.6. *The map* $\kappa_L : \mathcal{K}L \to L$ *given by* $\kappa_L(I) = \bigvee X$, *where* I *is generated by the* κ*-set* X*, is a* κ*-frame homomorphism. Moreover,* $\varepsilon = (\kappa_L)_{L \in \kappa Frm}$ *is a natural transformation.*

Proof. The map κ_L is well-defined, since $<X> = <Y>$ implies that $\bigvee X = \bigvee Y$. Trivially $\kappa_L(0) = 0$, $\kappa_L(L) = \bigvee\{e\} = e$, and if $I = <X>, J = <Y> \in \mathcal{K}L$, then

$$\kappa_L(I \wedge J) = \bigvee(X \cap Y) = \bigvee X \wedge \bigvee Y = \kappa_L(I) \wedge \kappa_L(J).$$

Furthermore κ_L preserves joins of κ-sets, since

$$\kappa_L(\bigvee I_\lambda) = \bigvee(\bigcup_\lambda X_\lambda) = \bigvee_\lambda(\bigvee X_\lambda) = \bigvee_\lambda(\kappa_L(I_\lambda)).$$

Thus κ_L is a κ-frame homomorphism.

Also for any κ-frame map $h : L \to M$, $h \circ \kappa_L = \kappa_M \circ \mathcal{K}(h)$. Since, if I is generated by the κ-set X of L then

$$h \circ \kappa_L(I) = h(\bigvee X) = \bigvee(h[X]) = \kappa_M(<h[X]>) = \kappa_M \circ \mathcal{K}(h)(I).$$

Thus ε is a natural transformation. $\square$

To show that κ_L is couniversal we need the following lemma.

Lemma 2.7. *Let $h : M \to L$ be a κ-frame homomorphism with compact regular domain M. Then, there exists a κ-frame homomorphism $\overline{h} : M \to \mathcal{K}L$ such that $\kappa_L \circ \overline{h} = h$.*

Proof. Let $a \in M$. By regularity of M there exists a κ-set $\{a_\lambda : a_\lambda \prec a, \lambda \in \Lambda\}$ such that $a = \bigvee a_\lambda$. It is easy to show that, similar to compact regular frames, the rather below relation interpolates in any compact regular κ-frame, and so $\prec\prec = \prec$ in M. Hence for each λ there exists a set $\{a_{\lambda i} : i \in \mathbb{N}\}$ such that $a_\lambda = a_{\lambda_0} \prec\prec a_{\lambda_1} \prec\prec ... \prec\prec a$.

Let J_a be the ideal generated by $\{h(a_{\lambda i}) : \lambda \in \Lambda, i \in \mathbb{N}\}$. Then $J_a \in \mathcal{K}L$. We define $\overline{h} : M \to \mathcal{K}L$ by $\overline{h}(a) = J_a$ for each $a \in M$. We show that $\overline{h}$ is well-defined. Let

$$a = \bigvee\{x_{\lambda i} : x_{\lambda_0} \prec\prec ... \prec\prec a, \lambda \in \Lambda\}$$
$$= \bigvee\{y_{\beta j} : y_{\beta_0} \prec\prec ... \prec\prec a,\ \beta \in \Lambda\}.$$

Take $J_1 = <\{h(x_{\lambda i})\}>$ and $J_2 = <\{h(y_{\beta j})\}>$. Let $s_{\lambda i}$ be the separating element of $x_{\lambda i} \prec x_{\lambda i+1}$. Then $s_{\lambda i} \wedge x_{\lambda i} = 0$ and $s_{\lambda i} \vee x_{\lambda i+1} = e$. Thus $e = s_{\lambda i} \vee a = \bigvee\{s_{\lambda i} \vee y_{\beta j} : \beta \in \Lambda, j \in \mathbb{N}\}$. Compactness of M implies that $s_{\lambda i} \vee y_{\beta_1 j_1} \vee ... \vee y_{\beta_n j_n} = e$. This shows that $x_{\lambda i} \prec y_{\beta_1 j_1} \vee ... \vee y_{\beta_n j_n}$ and so $h(x_{\lambda i}) \prec h(y_{\beta_1 j_1}) \vee ... \vee h(y_{\beta_n j_n})$. Hence $J_1 \subseteq J_2$. Similarly $J_2 \subseteq J_1$. This gives that $\overline{h}$ is well-defined. Clearly

$$\kappa_L \circ \overline{h}(a) = \bigvee\{h(a_{\lambda i}) : \lambda \in \Lambda, i \in \mathbb{N}\}$$
$$= h(\bigvee\{a_{\lambda i} : \lambda \in \Lambda, i \in \mathbb{N}\}) = h(a).$$

It is enough to show that $\overline{h}$ is a κ-frame homomorphism. Trivially $\overline{h}(0) = \{0\}$, and $\overline{h}(e) = <\{h(e)\}> = <\{e\}> = L$. Let $a = \bigvee\{x_{\lambda i} : x_{\lambda_0} \prec\prec ... \prec\prec a,\ \lambda \in \Lambda\}$ and $b = \bigvee\{y_{\beta j} : y_{\beta_0} \prec\prec ... \prec\prec b,\ \beta \in \Lambda\}$. Then $a \wedge b = \bigvee\{x_{\lambda i} \wedge y_{\beta j} : \lambda \in \Lambda, \beta \in \Lambda, i, j \in \mathbb{N}\}$. Hence $J_{a \wedge b} = <\{h(x_{\lambda i} \wedge y_{\beta j})\}> = J_a \wedge J_b$, since h preserves meets.

Now let $\{a_t : t \in K\}$ be a κ-subset of M and

$$a_t = \bigvee\{a^t{}_{\lambda i} : a^t{}_{\lambda 0} \prec\prec ... \prec\prec a_t, \lambda \in \Lambda\}.$$

Then $J_{a_t} = <\{h(a^t{}_{\lambda i})\}>$. Thus $\bigvee(J_{a_t}) = <\{h(a^t{}_{\lambda i}), t \in K\}> = J_{\bigvee a_t}$, since $\bigvee a_t = \bigvee\{a^t{}_{\lambda i} : \lambda \in \Lambda, t \in K\}$. Therefore $\overline{h}$ is a κ-frame homomorphism. $\square$

Proposition 2.8. *The map κ_L is an $\mathcal{I}$-couniversal arrow for the κ-frame L.*

Proof. We have that $\mathcal{K}L$ is a compact regular κ-frame and κ_L is a κ-frame homomorphism. Let $h : M \to L$ be a κ-frame homomorphism with compact regular domain M. Then $\overline{h} : M \to \mathcal{K}L$, given by $\overline{h}(a) = J_a$, is a κ-frame homomorphism such that $\kappa_L \circ \overline{h} = h$. It is enough to show that $\overline{h}$ is unique. Let $g : M \to \mathcal{K}L$ be a κ-frame homomorphism such that $\kappa_L \circ g = h$. Let

$$a = \bigvee\{a_{\lambda i} : a_{\lambda 0} \prec\prec \dots \prec\prec a, \lambda \in \Lambda\} \in M$$

since in compact regular κ-frames $\prec = \prec\prec$. For each $\lambda \in \Lambda, i \in \mathbb{N}$

$$\kappa_L \circ g(a_{\lambda i}) = \bigvee g(a_{\lambda i}) = h(a_{\lambda i}) \in J_a = \overline{h}(a).$$

Thus $g(a_{\lambda i}) \subseteq \overline{h}(a)$ for each $\lambda \in \Lambda, i \in \mathbb{N}$, and so $g(a) \subseteq \overline{h}(a)$. Now let $s_{\lambda i}$ be the separating element of $a_{\lambda i} \prec a_{\lambda i+1}$. Then $g(a_{\lambda i}) \wedge g(s_{\lambda i}) = 0$ and $g(a_{\lambda i+1}) \vee g(s_{\lambda i}) = L$. Take $z \in g(a_{\lambda i+1}),\ t \in g(s_{\lambda i})$ such that $z \vee t = e$. Then for each $y \in g(a_{\lambda i}),\ y = (y \wedge z) \vee (y \wedge t) = y \wedge z \leq z$, and so $g(a_{\lambda i}) \subseteq\downarrow z$. Thus $h(a_{\lambda i}) = \bigvee g(a_{\lambda i}) \leq z$. Also we have that $z \in g(a_{\lambda i+1}) \subseteq g(a)$. Therefore $h(a_{\lambda i}) \in g(a)$ for each $\lambda \in \Lambda, i \in \mathbb{N}$, and so $\overline{h}(a) \subseteq g(a)$. This shows that $\overline{h}$ is unique. $\square$

Corollary 2.9. (a) *The category* **KRκFrm** *is a coreflective subcategory of the category* **κFrm**.

(b) *The category* **KRκFrm** *has all products and all equalizers and hence it is a complete category.*

3. Compactification of κ-frames

In this section a compactification for regular and proximal κ-frames are given.

Proposition 3.1. *For any completely regular κ-frame L, the map $\kappa_L : \mathcal{K}L \to L$ is a compactification of L.*

Proof. By Proposition 2.4, $\mathcal{K}L$ is a compact regular κ-frame. Trivially κ_L is a dense map. It is enough to show that it is surjective. Let $a \in L$. By complete regularity of L there exists a κ-subset $\{a_\lambda : a_\lambda \prec\prec a, \lambda \in \Lambda\}$ such that $a = \bigvee\{a_\lambda : \lambda \in \Lambda\}$. For each λ there exists a sequence $\{a_{\lambda i} : i \in \mathbb{N}\}$ such that $a_\lambda = a_{\lambda_0} \prec\prec a_{\lambda_1} \prec\prec ... \prec\prec a$. Take I_a to be the ideal generated by $\{a_{\lambda i} : \lambda \in \Lambda, i \in \mathbb{N}\}$. Then $I_a \in \mathcal{K}L$ and $\kappa_L(I_a) = a$. Hence κ_L is surjective. □

Here we recall the definition of proximal κ-frames from [6] and show that any proximal κ-frame has a compactification.

Definition 3.2. A *strong inclusion* on a κ-frame L is a binary relation $\lhd$ on L such that

i) If $x \leq a \lhd b \leq y$ then $x \lhd y$.
ii) $\lhd \subseteq L \times L$ is a sublattice. That is, $0 \lhd 0, e \lhd e, x, y \lhd a, b$ imply $x \lhd a \wedge b, x \vee y \lhd a$.
iii) If $x \lhd a$ then $x \prec y$.
iv) If $x \lhd y$ then there exists z with $x \lhd z \lhd y$.
v) If $x \lhd y$ then there exist $a, b \in L$ with $b \lhd a, b \vee y = e$ and $a \wedge x = 0$.
vi) Each $a \in L$ is a join of a κ-set of elements strongly included in a.

The pair $(L, \lhd)$ is called a *proximal κ-frame.*

Definition 3.3. For a strong inclusion $\lhd$ on a κ-frame L, an ideal I of L is said to be *strongly regular* if for each element x in I there exists an element y in I with $x \lhd y$.

Proposition 3.4. *Any proximal κ-frame has a compactification.*

Proof. For a proximal κ-frame L consider $C_\kappa L$, the set of all κ-set generated strongly regular ideals of L. Taking strongly below relation instead of completely rather below relation one can show that $C_\kappa L$ is a compact regular κ-frame. Also the join map $\rho_L : C_\kappa L \to L$ given by $\rho_L(I) = \bigvee X$ whenever I is generated by the κ-set X of L, is a compactification of L. $\square$

Remark 3.5. In the case of zero-dimensoinal κ-frames, that is, κ-frames L which are generated by their Boolean part $\mathcal{B}L$ of the complemented elements of L, the relation $x \lhd y$ iff $x \leq b \leq y$ for some $b \in \mathcal{B}L$ is a strong inclusion. Thus κ-subset generated strongly regular ideals of L are exactly ideals of L which are generated by a κ-subset of $\mathcal{B}L$. Hence $\mathcal{K}L$ is the set of ideals of L generated by a κ-subset of $\mathcal{B}L$.

Acknowledgment

The authors would like to thank the referees for their suggestions and correcting the statement of Proposition 3.1 for completely regular κ-frames rather than regular ones. Also we would like to thank one of the referees for suggesting us to consider κ-frames as partial frames and then using the more general approach of [5]. But we preferred to leave it as it is which is more common to us for σ and κ-frames. We are also grateful for the financial support of Shahid Beheshti University.

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