Bulletin of the Iranian Mathematical Society Vol. 29 *No. 1 (2003), pp 65-76.*

COMPLETE GROWTH SERIES OF COXETER GROUPS WITH MORE THAN THREE GENERATORS

M. J. MAMAGHANI

ABSTRACT. In this paper invoking several methods we compute the complete growth series of Coxeter groups with more than three generators. In the most general method we apply the algebraic version of Moebius inversion formula of number theory and obtain a recursive analog of the usual growth series for complete growth series. This formula will show that the complete growth series is a rational function.

1. Introduction

In [3] Grigorchuk and Nagnibeda introduced the notions of complete growth series and operator growth series for discrete groups and proved several facts about surface groups and hyperbolic groups using these notions. In [6] we computed complete growth series of Coxeter groups which are generated with a set of cardinality 3 using a complete set of rewriting rules and showed that these series are all rational. In this paper we continue our work in [6] for Coxeter groups with more than three generators and noting that in this

MSC(2000): Primary 20F55; Secondary 20F05, 68Q42
Keywords: Coxeter system, Cayley graph, Complete rewriting system, Spanning tree, Moebius inversion formula
Received: 7 December 2002 , Accepted: 14 August 2003

case the technique of rewriting rules cannot be invoked as in [6] efficiently we apply other methods.

2. **Prelminaries**

In this section we recall the various notions of growth and growth series in groups that we will use in the course of our work. For more information we refer to [3] and [6].

Definition 2.1. To each element g of the finitely presented group $G = \langle S, \phi \rangle$ we assign the length $|g|$ of g as follows:

$$|g| = \inf\{l \mid g = t_1 t_2 \cdots t_l,\ t_i \in S \cup S^{-1},\ 1 \leq i \leq l\}$$

With this definition of length we define the notion of growth function of G as follows

Definition 2.2. The function $\gamma : \mathbb{N} \cup \{0\} \to \mathbb{N}$ defined by

$$\gamma(0) = a_0 = 1$$

$$\gamma(n) = a_n = \#\{g \in G \mid |g| = n\} \qquad (n \neq 0)$$

is called the growth function of G with respect to S, and the series

$$F(z) = \sum_{n=0}^{\infty} a_n z^n \tag{2.1}$$

is called the growth series of G.

We observe that $a_n \in \mathbb{Z}$ for each $n \in \mathbb{N} \cup \{0\}$ and $F(z)$ belongs to the ring of formal power series $\mathbb{Z}[[z]]$.
Now let R be any ring with identity and let $R[G]$ be the group ring with coefficients in R. Recall that each element of $R[G]$ is a finite sum of the form $\sum_1^k a_i g_i$ where $a_i \in R$ and $g_i \in G$.

Definition 2.3. The function

$$\gamma^c : \mathbb{N} \cup \{0\} \to R[G]$$

$$\gamma^c(0) = 1$$
$$\gamma^c(n) = \sum_{g\in G, |g|=n>0} g \qquad (2.2)$$

is called the complete growth function and the series

$$F^c(z) = \sum_{g\in G} g z^{|g|} = \sum_{n=0}^{\infty} \left(\sum_{|g|=n} g \right) z^n \qquad (2.3)$$

is called the complete growth series of G with respect to S and R.

Observe that the coefficients of this series belong to $R[G]$ and so $F^c(z) \in R[G][[z]]$. Comparing (2.1) and (2.3) we observe that taking $R = \mathbb{Z}$ the image of γ^c under the augmentation function $\epsilon : \mathbb{Z}[G] \to \mathbb{Z}$, $\epsilon(g) = 1, g \in G$ is γ and therefore we have $\epsilon(F^c(z)) = F(z)$. We deduce that the notion of complete growth series generalizes the notion of growth series. For more information on complete growth series refer to [3].

Definition 2.4. Let $S = \{s_1, s_2, \dots, s_m\}$ be a finite set. The pair (W, S) is called a Coxeter system if

(1) W is a group generated by S ;
(2) for each $1 \leq i \leq m$ and $1 \leq j \leq m$, there are $m_{ij} \in \{1, 2, 3, \dots, \infty\}$ with $m_{ii} = 1$ and $m_{ij} \in \{2, 3, \dots, \infty\}$ when $i \neq j$ such that the relation $(s_i s_j)^{m_{ij}} = 1$ holds in W.

The Coxeter group W associated with the Coxeter system (W,S) is the group with presentation

$$W = \langle s_1, s_2, \dots, s_m | (s_i s_j)^{m_{ij}} = 1; 1 \leq i \leq m; 1 \leq j \leq m \rangle.$$

Thus the order of each generator in S is 2, $m_{ij} = m_{ji}$ and when there is no relation between s_i and s_j we have $m_{ij} = \infty$. S is called a Coxeter generating set of the Coxeter group W.

Now let (W, S) be a Coxeter system and $X \subset S$. In [4 section 5.5] it is shown that W_X, the subgroup of W generated by X, is itself a Coxeter group and (W_X, X) is a Coxeter system. This subgroup is called a parabolic subgroup of W.
Coxeter groups enjoy a rich literature. We refer the interested

reader to [1], [4] and [7] for more information on the subject.

3. **Complete growth series of a free product of groups**

In this section we compute the complete growth series of a group which is the free product of two Coxeter groups amalgamating a Coxeter subgroup.

Theorem 3.1. *Let W_1 and W_2 be two finitely presented Coxeter groups such that*

(1) $H = W_1 \cap W_2$ *is a parabolic subgroup of both* W_1 *and* W_2*;*
(2) W_1 *and* W_2 *admit the finite complete rewriting systems* R_1 *and* R_2 *respectively in such a way that* $R_3 = R_1 \cap R_2$ *is a complete rewriting system for* H*.*
Then we have

$$\frac{1}{F^{c}_{W_1 *_H W_2}(z)} = \frac{1}{F^{c}_{W_1}(z)} + \frac{1}{F^{c}_{W_2}(z)} - \frac{1}{F^{c}_{H}(z)}$$

where $W_1 *_H W_2$ is the amalgamated free product of W_1 and W_2 amalgamating H and $F^{c}_{X}(z)$is the complete growth series of X.

Proof. The rewriting system $R_1 \cup R_2$ is a finite complete rewriting system for $W_1 *_H W_2$. Therefore by theorem 2 of [3] the proof is complete.□
As a special case of the theorem we have the following corollary.

Corollary 3.2. *Let (W_i, A_i), $i = 1, 2, \ldots n$ be Coxeter systems such that $A_i \cap A_j = \emptyset$ for $i, j \in \{1, 2, \ldots n\}$ and $i \neq j$. Let $W = W_1 * W_2 * \ldots * W_n$ be the free product of $W_i, i = 1, 2, \ldots n$. If each of the groups W_i admits a finite complete rewriting system R_i with respect to A_i, then we have*

$$\frac{1}{F^{c}_{W}(z)} = \frac{1}{F^{c}_{W_1}(z)} + \frac{1}{F^{c}_{W_2}(z)} + \cdots + \frac{1}{F^{c}_{W_n}(z)} - (n-1)$$

Proof. In fact the set of rules $R = \cup_{i=1}^{n} R_i$ is a finite complete rewriting system for W with respect to $A = \cup_{i=1}^{n} A_i$. Therefore using induction and noting that $A_i \cap A_j = \emptyset$ for $1 \leq i < j \leq n$ the proof is complete.□

Example. We would like to find the complete growth series of the group

$W = \langle a, b, c, d \mid a^2 = b^2 = c^2 = d^2 = (ab)^p = (ac)^q = (bc)^r = (cd)^s = 1 \rangle$

where p, q, r and s are positive odd integers.
We observe that $W = W_1 *_H W_2$, where

$$W_1 = \langle a, b, c \mid a^2 = b^2 = c^2 = (ab)^p = (ac)^q = (bc)^r = 1 \rangle$$

$$W_2 = \langle d, b, c \mid d^2 = b^2 = c^2 = (bc)^r = (cd)^s = 1 \rangle$$

and

$$H = \langle b, c \mid b^2 = c^2 = (bc)^r = 1 \rangle$$

Here the set of rewriting rules for W are

$$\overbrace{aba \dots a}^{p} \to \overbrace{bab \dots b}^{p} \tag{3.1}$$

$$\overbrace{cac \dots c}^{q} \to \overbrace{aca \dots a}^{q} \tag{3.2}$$

$$\overbrace{bcb \dots b}^{r} \to \overbrace{cbc \dots c}^{r} \tag{3.3}$$

$$a^2 \to 1 \tag{3.4}$$

$$b^2 \to 1 \tag{3.5}$$

$$c^2 \to 1 \tag{3.6}$$

$$\overbrace{dcd \dots d}^{s} \to \overbrace{cdc \dots c}^{s} \tag{3.7}$$

$$d^2 \to 1 \tag{3.8}$$

From these the rules $1, 2, 3, 4, 5, 6$ are the rules for W_1, $3, 5, 6, 7, 8$ are the rules for W_2 and $3, 5, 6$ are the rules for H. Since H is a finite dihedral group with Coxeter element(the element with greatest length) $w = cb \dots c$ of length r we have

$$F_H^c(z) = 1+(b+c)z+(bc+cb)z^2+\cdots+((bc\dots bc)+(cb\dots cb))z^{r-1}+wz^r.$$

If $r = \infty$ then we have

$$F_H^c(z) = \frac{1}{1-bcz^2}(1+bz) + \frac{1}{1-cbz^2}cz(1+bz).$$

Now using Theorem 3.1 we observe that the complete growth series of W will be computed if we obtain the complete growth series of W_1 and W_2. To this end one can invoke the methods obtained in [6] for triangle Coxeter groups.
For more information on rewriting systems we refer to [2], [5] and [6].

4. **Smallest spanning tree**

In this section using a spanning tree for the Cayley graph of a Coxeter group we obtain the complete growth series of the group.

Theorem 4.1. *The complete growth series of the Coxeter group*

$$W_m = \langle s_1, s_2, \dots, s_m \mid s_1^2 = s_2^2 = \dots = s_m^2 = 1 \rangle$$

with respect to the Coxeter generating set $S = \{s_1, s_2, \cdots s_m\}$ *is*

$$W_m^c(z) = \frac{1-z^2}{1-Az+(m-1)\cdot z^2} \tag{4.1}$$

where $A = s_1 + s_2 + \cdots + s_m$.

Figure 1

Proof. Consider the rooted tree T whose vertices are the elements of W_m (the root corresponds to the neutral element) and the vertex g is connected to the vertex h if there exists $s \in \{s_1, s_2 \ldots s_m\}$ such that $h = gs$. Figure 1 shows a small part of the spanning tree for W_3. Using this tree we obtain

$$W_m^c(z) = 1 + F_1(z) + F_2(z) + \ldots + F_m(z),$$

where $F_i(z)$ is that part of the complete growth series of the group whose spanning tree has s_i as its root. It is clear that for $i = 1, 2, \ldots m$ we have

$$F_i(z) = s_i z + s_i(\sum_{j=1, j\neq i}^{m} F_j(z))z \tag{4.2}$$

From these and the relations $s_i^2 = 1$ for $i = 1, 2, \ldots, m$ in the group we have

$$s_i F_i(z) z = z^2 + (\sum_{j=1, j\neq i}^{m} F_j(z)) z^2. \tag{4.3}$$

Summing these up we obtain

$$\left(\sum_{i=1}^{m} s_i F_i(z)\right) z = mz^2 + (m-1)\left(\sum_{i=1}^{m} F_i(z)\right) z^2 \qquad (4.4)$$

Therefore we have

$$(\sum_{i=1}^{m} s_i F_i(z))z = mz^2 + (m-1)(W_m^c(z) - 1)z^2 \qquad (4.5)$$

On the other hand letting $A = s_1 + s_2 + \ldots + s_m$ we have

$$\sum_{i=1}^{m} F_i(z) = Az + \sum_{i=1}^{m}(A - s_i)F_i(z)z = Az + A\sum_{i=1}^{m} F_i(z)z - \sum_{i=1}^{m} s_i F_i(z)z \qquad (4.6)$$

Therefore we conclude that

$$(1 - Az + (m-1)z^2)W_m^c(z) = 1 - z^2 \qquad (4.7)$$

The proof is complete. □

We note that we can use the method of the last section and obtain the same result. In fact

$$s_i^2 \to 1, i = 1, 2, \ldots m \qquad (4.8)$$

is a complete rewriting system for W_m.

5. **Complete growth series using the Moebius inversion formula**

In this section, after proving the Moebius inversion formula, we follow [6] and compute the complete growth series of the Coxeter group W with Coxeter generating set $S = \{s_1, s_2, \cdots, s_m\}$.
First we prove a combinatorial lemma.

Lemma 5.1. *If Z is a finite set and $X \subset Z$ is fixed then*

$$\sum_{Y}(-1)^{|Z-Y|+1} = (-1)^{|Z-X|},$$

where the sum is taken over all subsets Y satisfying $Z \supseteq Y \supset X$

Proof. Let $|Z-X| = n$. By assumption we have $0 \leq |Z-Y| \leq n-1$. We note that $|Z-Y| = 0$ corresponds to the choice $Y = Z$. Therefore the number of choices of Y with $|Z-Y| = i, i = 0, \dots n-1$ is $\left(\begin{array}{c} n \\ i \end{array}\right)$. Now we have

$\sum_{Z \supseteq Y \supset X} (-1)^{|Z-Y|+1} = -\sum_0^{n-1} \left(\begin{array}{c} n \\ i \end{array}\right) (-1)^i$

$= -\left[\sum_0^n \left(\begin{array}{c} n \\ i \end{array}\right) (-1)^i - (-1)^n\right] = (-1)^n.$

The proof is complete.□

Lemma 5.2. Mobius inversion formula. *Let $P(S)$ be the set of all subsets of S and let G be an abelian group. If the functions $f : P(S) \to G$ and $g : P(S) \to G$ satisfy*

$$f(X) = \sum_{Y \supseteq X} g(Y), \qquad \forall X \in P(S)$$

then they satisfy

$$g(X) = \sum_{Y \supseteq X} (-1)^{|Y-X|} f(Y), \qquad \forall X \in P(S)$$

where $|Y-X|$ is the number of elements in the set $Y-X$.

Proof. We use induction on $n = |S-X|$. If $|S-X| = 0$ then $S = X$ and $Y = S$ and the conclusion follows immediately. Let the lemma be true for $n < k$ and let $X \subset S$ be such that $|S-X| = k$. Then we have $f(X) = g(X) + \sum_{Y \supset X} g(Y)$. Since $Y \supset X$ we have $|S-Y| < k$ and so by induction

$$g(Y) = \sum_{Z \supseteq Y} (-1)^{|Z-Y|} f(Z)$$

Thus

$$f(X) = g(X) + \sum_{Y \supset X} \sum_{Z \supseteq Y} (-1)^{|Z-Y|} f(Z).$$

Therefore by substitution we have

$$g(X) = f(X) + \sum_{Z \supseteq Y \supset X} (-1)^{|Z-Y|+1} f(Z).$$

Thus using the result of lemma 5.1 and changing the symbol Z to Y we obtain

$$g(X) = f(X) + \sum_{Y \supset X} (-1)^{|Y-X|} f(Y) = \sum_{Y \supseteq X} (-1)^{|Y-X|} f(Y).$$

The proof is complete.□

Definition 5.3. Let (W, S) be a Coxeter system and $X \subset S$. We say that $v \in W$ is a left $X-$ minimal element if $|v| = \min_{w \in vW_X} |w|$.

Let A_X be the set of all left $X-$ minimal elements of W and let B_X be the set of all left $X-$ minimal elements that are not left $Y-$ minimal for every $Y \supset X$. Consider the functions $A^c(z)$ and $B^c(z)$ from $P(S)$ to $\mathbb{Z}[\mathbb{W}][[z]]$, the formal power series ring with coefficients in the group ring $\mathbb{Z}[\mathbb{W}]$, defined as follows

$$B_X^c(z) = \sum_{v \in B_X} v z^{|v|}, \quad A_X^c(z) = \sum_{v \in A_X} v z^{|v|}.$$

It is easily seen that $A_X = \bigsqcup_{Y \supseteq X} B_Y$. Therefore we have $A_X^c(z) = \sum_{Y \supseteq X} B_Y(z)$ and by the Mobius inversion formula we compute

$$B_X^c(z) = \sum_{Y \supset X} (-1)^{|Y-X|} A_Y^c(z). \tag{5.1}$$

Theorem 5.4. *Let (W, S) be a Coxeter system with finite Coxeter generating set S. If W is infinite and $W^c(z)$ is the complete growth series of W with respect to S then*

$$W^c(z) = A_X^c(z) W_X^c(z),$$

where $W_X^c(z)$ is the complete growth series of W_X with respect to $X \subset S$.

Proof. Using the definition of the complete growth series and the identity $W = \bigsqcup_{v \in A_X} vW_X$ we have

$$W^c(z) = \sum_{w \in W} w z^{|w|} = \sum_{v \in A_X} \sum_{u \in W_X} vu z^{|v|+|u|} = A_X^c(z) W_X^c(z). \tag{5.2}$$

The proof is complete.□
We note that in general one cannot change the order of multiplication factors. Observe that from equation 17 we have

$$B^c_X(z) = \sum_{Y \supseteq X} (-1)^{|Y-X|} \frac{W^c(z)}{W^c_Y(z)}$$

Since W is infinite we have $B_\emptyset = \emptyset$ and therefore $B^c_\emptyset(z) = 0$. This implies that

$$0 = B^c_\emptyset(z) = \sum_{Y \in P(S)} (-1)^{|Y|} \frac{W^c(z)}{W^c_Y(z)}$$

In this summation we separate the term corresponding to $Y = S$ and obtain

$$\frac{(-1)^{|S|}}{W^c(z)} = \sum_{Y \in P(S) \setminus S} (-1)^{|Y|+1} \frac{1}{W^c_Y(z)}.$$

The proof is complete.□

Corollary 5.5. *The complete growth series of W is rational.*

Proof. We prove this assertion using induction on $|S|$. For $S = \{r, s\}$, W is the infinite dihedral group. Using corollary 3.2 we have

$$\frac{1}{W^c(z)} = \frac{1}{1+rz} + \frac{1}{1+sz} - 1,$$

which is rational. Let the assertion be true for all S with $|S| < n$. Then for any $Y \in P(S) \setminus S$ the function $\frac{1}{W^c_Y(z)}$ and so the sum on the right hand side of equation 19 and therefore $W^c(z)$ is rational. The proof is complete.□

One can observe that (as in page 4) the complete growth series of any finite Coxeter group is a polynomial. Therefore our final conclusion is

Corollary 5.6. *The complete growth series of any Coxeter group with respect to its Coxeter generators is rational.*

Acknowledgment

The author wishes to express his sincere thanks to professor R.I. Grigorchuk of Texas A & M and Steklov Institute of Mathematics for his advice and encouragement. Also, he thanks the referees for their invaluable comments. This research supported by the National Research Council of I. R. Iran(NRCI), National Research Project No. 836.

References

[1] N. Bourbaki, *Groups et Algebres de Lie,* Chapitres 4, 5, et 6, Hermann, Paris, 1968.

[2] D. Epstein et. al., *Word Processing in Groups*, Jones and Bartlett, Boston, 1992.

[3] R. I. Grigorchuk and T. Nagnibeda, Complete growth functions of hyperbolic groups, *Invent. Math.,* **130** (1997), 159-188.

[4] J. E. Humphreys, *Reflection Groups and Coxeter Groups,* Cambridge University Press, Cambridge, 1992.

[5] R. I. Grigorchuk, Growth functions, Rewriting systems and The Euler characteristic, *Mat. Zam,* **58**, no. 5(1995), 653-668.

[6] M. J. Mamaghani, Rewriting systems and complete growth series for triangle Coxeter groups, *Mat. Zam*, vol **71**, no. 3(2002), 431-439.

[7] L. Paris, Growth series of Coxeter groups, in *Group Theory from Geometrical Viewpoint*, E. Ghys, A. Haelfliger and A. Veriovsky (eds.), World Scientific, Singapore, 1991.

[8] J. Tits, Le problém des mots dans les groupes de Coxeter, *Symp. Math.* **1**, Acad. Press, (1968), 175-185.

M.J. Mamaghani
Dept. of Mathematics and Statistics
Univ. of Allameh Tabatabaii
Tehran
Iran
e-mail:Imamaghan@yahoo.com