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A NOTE ON FINSLER MODULES

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Abstract. Let E be a full Finsler B-module, $\phi : A \longrightarrow B$ a $*$-isomorphism of C^*-algebras. Define the module action by $ax = \phi(a)x$ and the map $x \mapsto \rho_A(x)$ by $\rho_A(x) = \phi^{-1}(\rho_B(x))$. Then it is straightforward to show that E is a full Finsler A-module. In this paper we shall establish a converse statement to the above.

1. Introduction

Hilbert C^*-modules are significant keys in theory of operator algebras, operator K-theory, group representation theory (via strong Morita equivqlence), theory of operator spaces and so on (see [1] and [2]).

Recall that a pre-Hilbert module over a C^*-algebra A is a complex linear space E which is a left A-module (and $\lambda(ax) = (\lambda a)x = a(\lambda x)$ where $\lambda \in \mathcal{C}, a \in A$ and $x \in E$) equipped with an A-valued inner product $< .,. >: E \times E \longrightarrow A$ satisfying :
(i) $< x,x >\geq 0,$ and $< x,x >= 0$ iff $x = 0$;
(ii) $< \lambda x + y, z >= \lambda < x,z > + < y,z >$;
(iii) $< ax,y >= a < x,y >$;
(iv) $< y,x >=< x,y >^*$.
A pre-Hilbert A-module is caled a Hilbert A-module or Hilbert C^*-module over A, if it is complete with respect to the norm $\|x\| =$

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$\| < x, x > \|^{\frac{1}{2}}$.
If the linear span of the set $\{< x, y >;\ x, y \in E\}$ is dense in A then E is called full. For example every C^*-algebra A is a full Hilbert A-module whenever we define $< x, y >= xy^*$.

Finsler modules over C^*-algebras are generalization of Hilbert C^*-modules. Let A_+ be the positive cone of a C^*-algebra A. Suppose that E is a complex linear space which is a left A-module (and $\lambda(ax) = (\lambda a)x = a(\lambda x)$ where $\lambda \in \mathcal{C}, a \in A$ and $x \in E$) equipped with a map $\rho_A : E \to A_+$ such that
(i) The map $\|.\|_E : x \mapsto \|\rho_A(x)\|^{\frac{1}{2}}$ is a norm on E, and
(ii) $\rho_A(ax) = a\rho_A(x)a^*$ for each $a \in A$ and $x \in E$.
Then E is called a pre-Finsler A-module. If $(E, \|.\|_E)$ is complete then E is called a Finsler A-module. This definition is a modification of one introduced by N.C. Phillips and N. Weaver [3]. Indeed it is routine by using an interesting theorem of C. Akemann [3, Theorem 4] to show that the norm completion of a pre-Finsler A-module is a Finsler A-module.

A Finsler A-module is said to be full if the linear span of $\{\rho_A(x); x \in E\}$, denoted by $< \rho_A(E) >$, is dense in A.
For example, if E is a (full) Hilbert C^*-module over A then E together with $\rho_A(x) =< x, x >$ is a (full) Finsler module because of $\rho_A(ax) =< ax, ax >= a < x, x > a^* = a\rho_A(x)a^*$. If A has a nontrivial commutative ideal, then there exists a Finsler A-module which can not be regard as a Hilbert A-module, It follows from [3, corollary 18] that if A is a C^*-algebra,then the class of Finsler A-modules equals the class of Hilbert A-modules if and only if A has no nonzero commutative ideals. In particular, this holds if A is simple with $\dim(A) > 1$, approximately divisible, or a von Neumann algebra with no abelian summand.

The following lemmas which are interesting in its own right, will be used in our main result.

Lemma 1.1. *Every Finsler A-module is a Banach A-module.*

Proof. Suppose E is a Finsler A-module. E is a Banach space by the definition of Finsler module. It remains to show that $\|ax\|_E \leq$

$\|a\|\|x\|_E$ for all $a \in A, x \in E$. For this, note that

$$\|ax\|_E^2 = \|\rho_A(ax)\| = \|a\rho_A(x)a^*\| \leq \|a\|^2\|\rho_A(x)\| = \|a\|^2\|x\|_E^2.\Box$$

Lemma 1.2. *Let E be a full Finsler module over a C^*-algebra A and $a \in A$. Then $ax = 0$ for all $x \in E$ iff $a = 0$.*

Proof. Let $b \in A$ be arbitrary. Since E is full, there exists $\{u_n\}$ in $< \rho_A(E) >$ such that $b = \lim\limits_n u_n$. Each u_n is of the form $u_n = \sum\limits_{i=1}^{k_n}\lambda_{i,n}\rho_A(x_{i,n})$ in which $x_{i,n} \in E$ and $\lambda_{i,n} \in \mathcal{C}$. Hence

$$aba^* = \lim_n au_na^* = \lim_n (a\sum_{i=1}^{k_n}\lambda_{i,n}\rho_A(x_{i,n})a^*) = \lim_n \sum_{i=1}^{k_n}\lambda_{i,n}\rho_A(ax_{i,n}) = 0.$$

Hence for $b = a^*a$, we have $\|a\|^4 = \|aa^*\|^2 = \|aa^*(aa^*)^*\| = 0$. We conclude that $a = 0.\Box$

2. Main Theorem

Let E be a (full) Finsler B-module, $\phi : A \longrightarrow B$ is a $*$-isomorphism of C^*-algebras. Define the module action by $ax = \phi(a)x$ and the map $x \mapsto \rho_A(x)$ by $\rho_A(x) = \phi^{-1}(\rho_B(x))$. Then it is straightforward to show that E is a (full) Finsler A-module. We shall establish a converse statement to the above.

Theorem. *Let E be both a full Finsler A-module and a full Finsler B-module such that $\|\rho_A(x)\| = \|\rho_B(x)\|$ for each $x \in E$, and let $\phi : A \rightarrow B$ be a map such that $ax = \phi(a)x$ and $\phi(\rho_A(x)) = \rho_B(x)$,where $x \in E, a \in A$. Then ϕ is an $*$-isomorphism of C^*-algebras.*

Proof. Assume $\{a_n\}$is a sequence in A such that $a_n \rightarrow 0$ and $\phi(a_n) \rightarrow b$. By lemma 1.1 $a_nx \rightarrow 0$ and $\phi(a_n)x \rightarrow bx$. By the definition of module action, $\phi(a_n)x \rightarrow 0$. Hence $bx = 0$. Applying the lemma 1.2, $b = 0$. It follows from closed graph theorem that ϕ is continuous.
Since $(\phi(a + b) - \phi(a) - \phi(b))x = ((a + b)x - ax - bx) = 0$, by

lemma 1.2 $\phi(a+b) = \phi(a) + \phi(b)$. Similarly, $\phi(\lambda a) = \lambda\phi(a)$ and $\phi(ab) = \phi(a)\phi(b)$. Therefore ϕ is a homomorphism.
If $a \in A$, then we may assume that $a = \lim_n u_n$, Each u_n is of the form $u_n = \sum_{i=1}^{k_n} \lambda_{i,n}\rho_A(x_{i,n})$ where $x_{i,n} \in E$ and $\lambda_{i,n} \in \mathcal{C}$. Hence

$\phi(a^*) = \lim_n \phi(u_n^*) = \lim_n \sum_{i=1}^{k_n} \overline{\lambda_{i,n}}\phi(\rho_A(x_{i,n})) = \lim_n \sum_{i=1}^{k_n} \overline{\lambda_{i,n}}\rho_B(x_{i,n}) =$
$(\lim_n \sum_{i=1}^{k_n} \lambda_{i,n}\rho_B(x_{i,n}))^* = (\phi(a))^*$. Then ϕ is a $*$-homomorphism.
If $\phi(a) = 0$, then $ax = \phi(a)x = 0$ for all $x \in E$. Hence $a = 0$. ϕ is therefore one to one.
Given $b \in B$ and $\epsilon > 0$. Since E is a full Finsler B-module, there is $\{x_i\}_{1 \leq i \leq n}$ in E such that $\|b - \sum_{i=1}^{n} \lambda_i \rho_B(x_i)\| < \epsilon$, hence $\|b - \phi(\sum_{i=1}^{n} \lambda_i \rho_A(x_i))\| < \epsilon$.
Therefore ϕ has a dense range. But ϕ is a $*$-homomorphism from A into B, so that its range is closed. ϕ is therefore surjective.
Thus ϕ is a $*$-isomorphism. $\square$

Remark. We could not drop the condition of fullness. For instance, let $B = C([0,1])$ and $A = E = \{f \in B; f(0) = 0\}$. Then E is a full Finsler A-module with respect to $\rho_A(f) =\mid f \mid^2$, and E is a Finsler B-module with respect to $\rho_B(f) =\mid f \mid^2$. It is clear that E is not a full Finsler B-module. In addition the inclusion map $\phi : A \longrightarrow B$ satisfies $a.f = \phi(a).f$ and $\phi(\rho_A(f)) = \rho_B(f)$, whereas ϕ is not surjective.(thus is not isomorphism).

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