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SOME FIXED POINT THEOREMS FOR WEAKLY COMPATIBLE MULTIVALUED MAPPINGS SATISFYING SOME GENERAL CONTRACTIVE CONDITIONS OF INTEGRAL TYPE

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Abstract. We prove some common fixed point theorems for multivalued mappings satisfying some general contractive conditions of integral type under the condition of weak compatibility.

1. **Introduction and preliminaries**

In 1922, the Polish mathematician Stefan Banach proved a theorem which ensures, under appropriate conditions, the existence and uniqueness of a fixed point. His result is called Banach's fixed point theorem or the Banach contraction principle. This theorem provides a technique for solving a variety of applied problems in mathematical sciences and engineering. Many authors have extended, generalized and improved Banach's fixed point theorem in different ways. In [11], Jungck introduced more generalized commuting mappings, called compatible mappings, which are more general than commuting and weakly commuting mappings. This concept has been useful for obtaining more comprehensive fixed point theorems (see, [5], [10], [12], [13], [14], [16], [17], [18], [19]). Recently, Branciari [4] obtained a fixed point result for a

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single mapping satisfying an analogue of Banach's contraction principle for an integral type inequality. The authors in [1], [2], [3], [7], [15] and [20] proved some fixed point theorems involving more general contractive conditions. Here, we establish some fixed point theorems for weakly compatible multivalued maps satisfying some general contractive inequalities of integral type.

Throughout this paper, let (X, d) denote a metric space and $\mathcal{B}(X)$ stand for the set of all bounded subsets of X. The function δ and D of $\mathcal{B}(X) \times \mathcal{B}(X)$ into $[0, \infty)$ are defined to be:

$$\begin{aligned} \delta(A, B) &= \sup\{d(a, b) : a \in A, b \in B\}, \\ D(A, B) &= \inf\{d(a, b) : a \in A, b \in B\}, \end{aligned}$$

for all A, B in $\mathcal{B}(X)$. If $A = \{a\}$ is singleton, then we write $\delta(A, B) = \delta(a, B)$ and if $B = \{b\}$, then we put $\delta(A, B) = \delta(a, b) = d(a, b)$. It is easily seen that

$$\begin{aligned} \delta(A, B) &= \delta(B, A) \geq 0, \\ \delta(A, B) &\leq \delta(A, C) + \delta(C, B), \\ \delta(A, A) &= \operatorname{diam} A, \\ \delta(A, B) &= 0 \text{ implies } A = B = \{a\}, \end{aligned}$$

for all A, B, C in $\mathcal{B}(X)$. We recall some definitions and basic lemmas of Fisher [8] and Imdad et al. [9]. Let $\{A_n : n = 1, 2, ...\}$ be a sequence of subsets of X. We say that the sequence $\{A_n\}$ converges to a subset A of X if each point a in A is the limit of a convergent sequence $\{a_n\}$ with a_n in A_n, for $n = 1, 2, \ldots$, and if for any $\varepsilon > 0$, there exists an integer N such that $A_n \subseteq A_\varepsilon$, for $n > N$, A_ε being the union of all open spheres with centers in A and radius ε. The following lemmas hold.

Lemma 1.1 ([8]). *If $\{A_n\}$ and $\{B_n\}$ are sequences of bounded subsets of (X, d) which converge to the bounded subsets A and B, respectively, then the sequence $\{\delta(A_n, B_n)\}$ converges to $\delta(A, B)$.*

Lemma 1.2 ([9]). *If $\{A_n\}$ is a sequence of bounded sets in the complete metric space (X, d) and if $\lim_{n\to\infty} \delta(A_n, \{y\}) = 0$, for some $y \in X$, then $\{A_n\} \to \{y\}$.*

A set-valued mapping F of X into $\mathcal{B}(X)$ is continuous at the point x in X if whenever $\{x_n\}$ is a sequence of points of X converging to x, the sequence $\{Fx_n\}$ in $\mathcal{B}(X)$ converges to Fx. F is said to be continuous in X if it is continuous at each point x in X. We say that z is a fixed point of F if z is in Fz. Furthermore, if U is any nonempty subset of X, then we define the set $F(U)$ by

$$F(U) = \underset{x\in U}{\cup} Fx.$$

Also, if B is a self mapping of X, then by $F(X) \subseteq B(X)$, we mean

$$F(X) = \underset{x\in X}{\cup} Fx \subseteq B(X),$$

that is, for all $x \in X$, we have $Fx \subseteq B(X)$.

The following theorem was established by Chang [6].

Theorem 1.3. *Let (X,d) be a complete metric space, I and J be self-maps of X, and $S,T : X \to \mathcal{B}(X)$ be such that $S(X) \subset J(X)$ and $T(X) \subset I(X)$. Moreover, assume that for all $x, y \in X$,*

$$\delta(Sx,Ty) \le \psi\left(\begin{array}{r}\max\{d(Ix,Jy), \delta(Ix,Sx), \delta(Jy,Ty),\\ \frac{1}{2}[D(Ix,Ty)+D(Jy,Sx)]\end{array}\right),$$

where $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is nondecreasing and $\psi(0) = 0, \psi(t) < t$, for $t > 0$, is upper semicontinuous, both (I,S) and (T,J) are compatible, and at least one of I or J is continuous. Then I, J, S and T have a unique common fixed point z in X. Furthermore, $Sz = Tz = \{Iz\} = \{Jz\} = \{z\}$.

The following definition was given by Jungck and Rhoades [12].

Definition 1.4. *Let $A : X \to X$ and $F : X \to \mathcal{B}(X)$ be two mappings. The pair (A,F) is weakly compatible if A and F commute at coincidence points; i.e., for each point u in X such that $Fu = \{Au\}$, we have $FAu = AFu$. (Note that the equation $Fu = \{Au\}$ implies that Fu is a singleton).*

2. **Common fixed point theorems**

Theorem 2.1. *Let A and B be mappings of a metric space (X,d) into itself, and F, G be mappings from X into $\mathcal{B}(X)$ such that*

$$F(X) \subseteq B(X) \text{ and } G(X) \subseteq A(X). \tag{2.1}$$

Also, the mappings A, B, F and G satisfy the following inequality,

$$\begin{aligned} \int_0^{\delta(Fx,Gy)} \varphi(t)dt \;&\leq\; \alpha \int_0^{\max\{d(Ax,By),\delta(Ax,Fx),\delta(By,Gy)\}} \varphi(t)dt \\ &+\; (1-\alpha)\left[a \int_0^{\frac{D(Ax,Gy)}{2}} \varphi(t)dt + b \int_0^{\frac{D(By,Fx)}{2}} \varphi(t)dt\right], \end{aligned} \tag{2.2}$$

for all $x, y \in X$, where $0 \leq \alpha < 1, a \geq 0, b \geq 0, a + b < 1$ and $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a Lebesgue-integral mapping which is summable, non-negative and such that

$$\int_0^{\varepsilon} \varphi(t)dt > 0 \text{ for all } \varepsilon > 0. \tag{2.3}$$

Suppose that any one of $A(X)$ or $B(X)$ is complete. If both pairs (A, F) and (B, G) are weakly compatible, then there exists a unique $z \in X$ such that $\{z\} = \{Az\} = \{Bz\} = Fz = Gz$.

Proof. Let x_0 be an arbitrary point in X. By (2.1), we choose a point x_1 in X such that $Bx_1 \in Fx_0 = Z_0$. For this point x_1, there exists a point x_2 in X such that $Ax_2 \in Gx_1 = Z_1$, and so on. Continuing in this manner, we can define a sequence $\{x_n\}$ as follows:

$$Bx_{2n+1} \in Fx_{2n} = Z_{2n},\;\; Ax_{2n+2} \in Gx_{2n+1} = Z_{2n+1}, \tag{2.4}$$

for $n = 0, 1, ...$ For simplicity, we put $V_n = \delta(Z_n, Z_{n+1})$, for $n = 0, 1, ...$

By (2.2) and (2.4), we have

$$
\begin{aligned}
(2.5)\qquad & \int_0^{V_{2n}} \varphi(t)dt \\
&= \int_0^{\delta(Z_{2n},Z_{2n+1})} \varphi(t)dt \\
&= \int_0^{\delta(Fx_{2n},Gx_{2n+1})} \varphi(t)dt \\
&\leq \alpha \int_0^{\max\{d(Ax_{2n},Bx_{2n+1}),\delta(Ax_{2n},Fx_{2n}),\delta(Bx_{2n+1},Gx_{2n+1})\}} \varphi(t)dt \\
&\quad +(1-\alpha)\left[a \int_0^{\frac{D(Ax_{2n},Gx_{2n+1})}{2}} \varphi(t)dt \right. \\
&\quad \left. + b \int_0^{\frac{D(Bx_{2n+1},Fx_{2n})}{2}} \varphi(t)dt \right] \\
&\leq \alpha \int_0^{\max\{\delta(Gx_{2n-1},Fx_{2n}),\delta(Fx_{2n},Gx_{2n+1})\}} \varphi(t)dt \\
&\quad +(1-\alpha)a \int_0^{\frac{\delta(Gx_{2n-1},Gx_{2n+1})}{2}} \varphi(t)dt \\
(2.6)\qquad &\leq \alpha \int_0^{\max\{V_{2n-1},V_{2n}\}} \varphi(t)dt + (1-\alpha)a \int_0^{\frac{V_{2n-1}+V_{2n}}{2}} \varphi(t)dt
\end{aligned}
$$

for $n = 1, 2, ...$

Now if $V_{2n} \geq V_{2n-1}$, then from (2.6) we have,

$$
\begin{aligned}
\int_0^{V_{2n}} \varphi(t)dt &\leq \alpha \int_0^{V_{2n}} \varphi(t)dt + (1-\alpha)a \int_0^{V_{2n}} \varphi(t)dt \\
&= (\alpha + (1-\alpha)a) \int_0^{V_{2n}} \varphi(t)dt \\
&< \int_0^{V_{2n}} \varphi(t)dt,
\end{aligned}
$$

which is a contradiction. Thus, $V_{2n} < V_{2n-1}$ and so from (2.6) we have,

$$
(2.7)\qquad \int_0^{V_{2n}} \varphi(t)dt \leq (\alpha + (1-\alpha)a) \int_0^{V_{2n-1}} \varphi(t)dt.
$$

Similarly, we have

$$\int_0^{V_{2n+1}} \varphi(t)dt \leq (\alpha + (1-\alpha)b) \int_0^{V_{2n}} \varphi(t)dt. \tag{2.8}$$

From (2.7) and (2.8), we have

$$\int_0^{V_n} \varphi(t)dt \leq c \int_0^{V_{n-1}} \varphi(t)dt,$$

and so,

$$\int_0^{V_n} \varphi(t)dt \leq c^n \int_0^{V_0} \varphi(t)dt,$$

for $n = 1, 2, ...,$ where $c = \max\{\alpha + (1-\alpha)a, \alpha + (1-\alpha)b\}$. Thus, we have

$$\lim_{n\to\infty} \int_0^{V_n} \varphi(t)dt = 0,$$

which from (2.3) implies that

$$\lim_{n\to\infty} V_n = \lim_{n\to\infty} \delta(Z_n, Z_{n+1}) = 0.$$

Now, we want to show that

$$\lim_{n,m\to\infty} \delta(Z_n, Z_m) = 0.$$

For this, it is sufficient to show that

$$\lim_{n,m\to\infty} \delta(Z_{2n}, Z_{2m}) = 0.$$

Suppose that this not true. Then, there is an $\varepsilon > 0$ such that for an even integer $2k$ there exist even integers $2m(k) > 2n(k) > 2k$ such that

$$\delta(Z_{2n(k)}, Z_{2m(k)}) \geq \varepsilon. \tag{2.9}$$

For every even integer $2k$, let $2m(k)$ be the least positive integer exceeding $2n(k)$ satisfying (2.9) and such that

$$\delta(Z_{2n(k)}, Z_{2m(k)-2}) < \varepsilon. \tag{2.10}$$

Now,

$$\begin{aligned} \varepsilon &\leq \delta(Z_{2n(k)}, Z_{2m(k)}) \\ &\leq \delta(Z_{2n(k)}, Z_{2m(k)-2}) + \delta(Z_{2m(k)-2}, Z_{2m(k)-1}) + \delta(Z_{2m(k)-1}, Z_{2m(k)}). \end{aligned}$$

Then, by (2.9) and (2.10), it follows that

$$\lim_{k\to\infty} \delta(Z_{2n(k)}, Z_{2m(k)}) = \varepsilon. \tag{2.11}$$

Also, by triangular inequality, we have

$$\left|\delta(Z_{2n(k)}, Z_{2m(k)-1}) - \delta(Z_{2n(k)}, Z_{2m(k)})\right| \leq \delta(Z_{2m(k)-1}, Z_{2m(k)}).$$

By using (2.11), we get

$$\lim_{k\to\infty} \delta(Z_{2n(k)}, Z_{2m(k)-1}) = \varepsilon. \tag{2.12}$$

Now,

$$\begin{aligned}
&0 < \delta := \int_0^{\varepsilon} \varphi(t)dt = \lim_{k\to\infty} \int_0^{\delta(Z_{2n(k)}, Z_{2m(k)})} \varphi(t)dt \\
&\leq \lim_{k\to\infty} \int_0^{\delta(Z_{2n(k)}, Z_{2n(k)+1}) + \delta(Z_{2n(k)+1}, Z_{2m(k)})} \varphi(t)dt \\
&= \lim_{k\to\infty} \int_0^{\delta(Z_{2n(k)+1}, Z_{2m(k)})} \varphi(t)dt \\
&= \lim_{k\to\infty} \int_0^{\delta(Fx_{2m(k)}, Gx_{2n(k)+1})} \varphi(t)dt \\
&\leq \lim_{k\to\infty} \left(\alpha \int_0^{\max\{d(Ax_{2m(k)}, Bx_{2n(k)+1}), \delta(Ax_{2m(k)}, Fx_{2m(k)}) \delta(Bx_{2n(k)+1}, Gx_{2n(k)+1})\}} \varphi(t)dt \right.\\
&\left. +(1-\alpha)\left[a\int_0^{\frac{D(Ax_{2n(k)}, Gx_{2m(k)+1})}{2}} \varphi(t)dt + b\int_0^{\frac{D(Bx_{2m(k)+1}, Fx_{2n(k)})}{2}} \varphi(t)dt\right]\right) \\
&\leq \lim_{k\to\infty} \left(\alpha \int_0^{\max\{\delta(Gx_{2n(k)-1}, Fx_{2m(k)}), \delta(Gx_{2n(k)-1}, Fx_{2n(k)}), \delta(Fx_{2m(k)}, Gx_{2m(k)+1})\}} \varphi(t)dt \right.\\
&\left. +(1-\alpha)\left[a\int_0^{\frac{\delta(Gx_{2m(k)-1}, Gx_{2n(k)+1})}{2}} \varphi(t)dt + b\int_0^{\frac{\delta(Fx_{2n(k)}, Fx_{2m(k)})}{2}} \varphi(t)dt\right]\right) \\
&\leq \lim_{k\to\infty} \left(\alpha \int_0^{\max\{\delta(Z_{2m(k)-1}, Z_{2n(k)}), \delta(Z_{2m(k)-1}, Z_{2m(k)}) \delta(Z_{2n(k)}, Z_{2n(k)+1})\}} \varphi(t)dt \right.\\
&\left. +(1-\alpha)\left[a\int_0^{\frac{\delta(Z_{2m(k)-1}, Z_{2n(k)+1})}{2}} \varphi(t)dt + b\int_0^{\frac{\delta(Z_{2n(k)}, Z_{2m(k)})}{2}} \varphi(t)dt\right]\right)
\end{aligned}$$

$$\leq \lim_{k\to\infty}\left(\alpha\int_0^{\delta(Z_{2m(k)-1},Z_{2n(k)})}\varphi(t)dt\right.$$
$$+(1-\alpha)\left[a\int_0^{\frac{\delta(Z_{2m(k)-1},Z_{2m(k)})+\delta(Z_{2m(k)},Z_{2n(k)})+\delta(Z_{2n(k)},Z_{2n(k)+1})}{2}}\varphi(t)dt\right.$$
$$\left.\left.+b\int_0^{\frac{\delta(Z_{2n(k)},Z_{2m(k)})}{2}}\varphi(t)dt\right]\right)$$
$$\leq \alpha\int_0^{\varepsilon}\varphi(t)dt+(1-\alpha)(a+b)\int_0^{\varepsilon}\varphi(t)dt$$
$$=(\alpha+(1-\alpha)(a+b))\delta,$$

which is a contradiction. Therefore, we have

$$\lim_{n,m\to\infty}\delta(Z_n,Z_m)=0.$$

Thus, if z_n is an arbitrary point in the set Z_n, for $n=0,1,\cdots$, it follows that

$$\lim_{n,m\to\infty}d(z_n,z_m)\leq\lim_{n,m\to\infty}\delta(Z_n,Z_m)=0.$$

Therefore, the sequence $\{z_n\}$ and hence any subsequence thereof is a Cauchy sequence in X.

Now, suppose $B(X)$ is complete. Let $\{x_n\}$ be the sequence defined by (2.4). Since $Bx_{2n+1}\in Fx_{2n}=Z_{2n}$, for $n=0,1,...$, we have

$$d(Bx_{2m+1},Bx_{2n+1})\leq\delta(Z_{2m},Z_{2n})<\varepsilon,$$

for $m,n\geq n_0, n_0=1,2,....$ Therefore, by the above, the sequence $\{Bx_{2n+1}\}$ is Cauchy, and hence $Bx_{2n+1}\to p=Bq\in B(X)$, for some $q\in X$. But, $Ax_{2n}\in Gx_{2n-1}=Z_{2n-1}$, by (2.4), so that we have

$$d(Ax_{2n},Bx_{2n+1})\leq\delta(Z_{2n-1},Z_{2n})=V_{2n-1}\to 0,$$

as $n\to\infty$. Consequently, $Ax_{2n}\to p$. Moreover, we have for $n=1,2,3,...,$

$$\delta(Fx_{2n},p)\leq\delta(Fx_{2n},Ax_{2n})+d(Ax_{2n},p)=V_{2n}+d(Ax_{2n},p).$$

Therefore, $\delta(Fx_{2n}, p) \to 0$. In a similar manner, it follows that $\delta(Gx_{2n-1}, p) \to 0$. Now, using the inequality (2.2), we have

$$\begin{aligned}\int_0^{\delta(Fx_{2n},Gq)} \varphi(t)dt \ &\leq \alpha \int_0^{\max\{d(Ax_{2n},Bq),\delta(Ax_{2n},Fx_{2n}),\delta(Bq,Gq)\}} \varphi(t)dt \\ &+(1-\alpha)\left[a\int_0^{\frac{D(Ax_{2n},Gq)}{2}} \varphi(t)dt + b\int_0^{\frac{D(Bq,Fx_{2n})}{2}} \varphi(t)dt\right]\end{aligned}$$

and so we have

$$\begin{aligned}\int_0^{\delta(Fx_{2n},Gq)} \varphi(t)dt \ &\leq \alpha \int_0^{\max\{d(Ax_{2n},Bq),\delta(Ax_{2n},Fx_{2n}),\delta(Bq,Gq)\}} \varphi(t)dt \\ &+(1-\alpha)\left[a\int_0^{\frac{\delta(Ax_{2n},Gq)}{2}} \varphi(t)dt + b\int_0^{\frac{\delta(Bq,Fx_{2n})}{2}} \varphi(t)dt\right].\end{aligned}$$

We get as $n \to \infty$

$$\begin{aligned}\int_0^{\delta(p,Gq)} \varphi(t)dt \ &\leq \ \alpha\int_0^{\delta(p,Gq)} \varphi(t)dt + (1-\alpha)a\int_0^{\frac{\delta(p,Gq)}{2}} \varphi(t)dt \\ &\leq \ (\alpha + (1-\alpha)a)\int_0^{\delta(p,Gq)} \varphi(t)dt,\end{aligned}$$

which is a contradiction if $\delta(p, Gq) > 0$. Thus, we have $\delta(p, Gq) = 0$ and so we have $\{p\} = Gq = \{Bq\}$.

But $G(X) \subseteq A(X)$, and so $r \in X$ exists such that $\{Ar\} = Gq = \{Bq\}$. Now, if $\delta(Fr, Gq) > 0$ so that we have

$$\begin{aligned}\int_0^{\delta(Fr,Gq)} \varphi(t)dt \ &\leq \alpha \int_0^{\max\{d(Ar,Bq),\delta(Ar,Fr),\delta(Bq,Gq)\}} \varphi(t)dt \\ &+(1-\alpha)\left[a\int_0^{\frac{D(Ar,Gq)}{2}} \varphi(t)dt + b\int_0^{\frac{D(Bq,Fr)}{2}} \varphi(t)dt\right],\end{aligned}$$

then we have

$$\begin{aligned}\int_0^{\delta(Fr,p)} \varphi(t)dt \ &\leq alpha\int_0^{\delta(p,Fr)} \varphi(t)dt + (1-\alpha)b\int_0^{\frac{\delta(p,Fr)}{2}} \varphi(t)dt \\ &\leq \ (\alpha + (1-\alpha)b)\int_0^{\delta(p,Fr)} \varphi(t)dt,\end{aligned}$$

which is a contradiction. Thus, we have $\delta(Fr, p) = 0$. It follows that $Fr = \{p\} = Gq = \{Ar\} = \{Bq\}$.

Since $Fr = \{Ar\}$ and the pair (A, F) is weakly compatible, then we obtain $Fp = FAr = AFr = Ap$. Now, using (2.2) we have, if $\delta(Fp, Gq) > 0$,

$$\begin{aligned}\int_0^{\delta(Fp,Gq)} \varphi(t)dt \quad &\leq \alpha \int_0^{\max\{d(Ap,Bq),\delta(Ap,Fp),\delta(Bq,Gq)\}} \varphi(t)dt \\ &+(1-\alpha)\left[a\int_0^{\frac{D(Ap,Gq)}{2}} \varphi(t)dt + b\int_0^{\frac{D(Bq,Fp)}{2}} \varphi(t)dt\right],\end{aligned}$$

and so

$$\begin{aligned}\int_0^{\delta(Fp,p)} \varphi(t)dt \quad &\leq \alpha \int_0^{d(Fp,p)} \varphi(t)dt \\ &+(1-\alpha)\left[a\int_0^{\frac{\delta(Fp,p)}{2}} \varphi(t)dt + b\int_0^{\frac{\delta(Fp,p)}{2}} \varphi(t)dt\right] \\ &\leq \ (\alpha + (1-\alpha)(a+b))\int_0^{d(Fp,p)} \varphi(t)dt,\end{aligned}$$

which is a contradiction. Thus, $\delta(Fp, p) = 0$ and so $Fp = \{p\} = \{Ap\}$. Similarly, $\{p\} = Gp = \{Bp\}$ if the pair (B, G) is weakly compatible. Therefore, we obtain $\{p\} = \{Ap\} = \{Bp\} = Fp = Gp$.

To see that p is unique, suppose that $\{p'\} = \{Ap'\} = \{Bp'\} = Fp' = Gp'$, for some $p' \in X$. Then, from (2.2), we have

$$\begin{aligned}\int_0^{\delta(Fp,Gp')} \varphi(t)dt \quad &\leq \alpha \int_0^{\max\{d(Ap,Bp'),\delta(Ap,Fp),\delta(Bp',Gp')\}} \varphi(t)dt \\ &+(1-\alpha)\left[a\int_0^{\frac{D(Ap,Gp')}{2}} \varphi(t)dt + b\int_0^{\frac{D(Bp',Fp)}{2}} \varphi(t)dt\right]\end{aligned}$$

and so,

$$\int_0^{d(p,p')} \varphi(t)dt \leq (\alpha + (1-\alpha)(a+b))\int_0^{d(p,p')} \varphi(t)dt,$$

which is a contradiction. Thus, we have $p = p'$.

The other case (that is assuming the completeness of $A(X)$) can be proved by a similar argument as above.

We can prove the following theorems as in proof of Theorem 2.1.

Theorem 2.2. *Let A and B be mappings of a metric space (X, d) into itself and F and G be mappings from X into $\mathcal{B}(X)$ such that*

$$F(X) \subseteq B(X) \text{ and } G(X) \subseteq A(X).$$

Also, the mappings A, B, F and G satisfy the following inequality,

$$\int_0^{\delta(Fx,Gy)} \varphi(t)dt$$

$$\leq \alpha \int_0^{\max\{d(Ax,By),\delta(Ax,Fx),\delta(By,Gy),\frac{D(Ax,Gy)}{2},\frac{D(By,Fx)}{2}\}} \varphi(t)dt,$$

for all $x, y \in X$, where $0 \leq \alpha < 1$ and $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a Lebesgue-integral mapping which is summable, non-negative and such that

$$\int_0^{\varepsilon} \varphi(t)dt > 0 \text{ for all } \varepsilon > 0.$$

Suppose that any one of $A(X)$ or $B(X)$ is complete. If both pairs (A, F) and (B, G) are weakly compatible, then there exists a unique $z \in X$ such that $\{z\} = \{Az\} = \{Bz\} = Fz = Gz$.

Theorem 2.3. *Let A and B be mappings of a metric space (X, d) into itself and F and G be mappings from X into $\mathcal{B}(X)$ such that*

$$F(X) \subseteq B(X) \text{ and } G(X) \subseteq A(X).$$

Also, the mappings A, B, F and G satisfy the following inequality

$$\int_0^{\delta(Fx,Gy)} \varphi(t)dt$$

$$\leq \psi\left(\int_0^{\max\{d(Ax,By),\delta(Ax,Fx),\delta(By,Gy),\frac{D(Ax,Gy)}{2},\frac{D(By,Fx)}{2}\}} \varphi(t)dt\right),$$

for all $x, y \in X$, where $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is nondecreasing and $\psi(0) = 0, \psi(t) < t$, for $t > 0$, and $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a Lebesgue-integral mapping which is summable, non-negative and such that

$$\int_0^{\varepsilon} \varphi(t)dt > 0 \text{ for all } \varepsilon > 0.$$

Suppose that any one of $A(X)$ or $B(X)$ is complete. If both pairs (A, F) and (B, G) are weakly compatible, then there exists a unique $z \in X$ such that $\{z\} = \{Az\} = \{Bz\} = Fz = Gz$.

Remark 2.4. If $\varphi(t) = 1$ in Theorem 2.3, then we obtain a generalized version of Theorem 1.3.

Remark 2.5. By Theorem 2.1 (or Theorem 2.2 or Theorem 2.3), we have a generalized version of and Theorem 2.1 of [4], Theorem 2.1 of [15] for multivalued mappings.

Remark 2.6. Similarly, we can have several fixed point theorems in the literature as special cases of Theorems 2.1, 2.2 and 2.3.

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