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APPROXIMATE CONVEXITY AND SUBMONOTONICITY IN LOCALLY CONVEX SPACES

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ABSTRACT. We introduce some new concepts of locally Lipschitz mappings, Clarke subdifferential, approximate convexity and submonotonocity in locally convex spaces. We show that, if f is approximately convex and bounded above, then f is locally Lipschitz. We also prove that a Lipschitz function is approximately convex if and only if its Clarke subdifferential is a submonotone operator. Several properties of approximate convexity are discussed. Our results can be viewed as extensions and refinements of the previously known results from Banach spaces to locally convex spaces.

1. Introduction

It is well known that the convexity plays a fundamental and crucial role in several aspects of optimization and other related fields. This serves as a motivation for relaxing convexity assumptions imposed on the functions arising in optimality and duality. Here, we introduce the concept of locally Lipschitz mapping, approximate convexity and other related results in locally convex spaces. Under suitable conditions, we show that one can extend and generalize the results from Banach spaces

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to locally convex spaces. Our results are more general, flexible and unifying.

A locally Lipschitz function $f : U \to \mathbb{R}$, where U is an open subset of $\mathbb{R}^n$, is called *lower-*C^1, if for every $x_0 \in U$, there exists a neighborhood V of x_0, a compact set S and a jointly continuous function $g : V \times S \to \mathbb{R}$, such that for all $x \in V$ we have $f(x) = \max_{s \in S} g(x, s)$ and derivative $D_x g$ exists and is jointly continuous.
The above class of functions has been introduced by Spingarn [8]. It has been shown [8] that a locally lipschitz function $f : U \to \mathbb{R}$ is lower-C^1 if and only if its Clarke subdifferential ∂f is submonotone at every $x \in U$. A multivalued operator $T : \mathbb{R}^n \to 2^{\mathbb{R}^n}$ is called submonotone at $x_0 \in X$, if for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$\langle x_1^* - x_2^*, x_1 - x_2 \rangle \geq -\epsilon \|x_1 - x_2\|,$$

for all $x_i \in B(x_0, \delta)$ and all $x_i^* \in T(x_i),\ i = 1, 2$.

Ngai et al. [6] introduced and studied the class of approximately convex functions defined in a Banach space X.

Definition 1.1. function $f : X \to \mathbb{R} \cup \{\infty\}$ is called approximately convex at $x_0 \in X$, if for every $\epsilon > 0$ there exists $\delta > 0$ such that the following implication for all $x, y \in B(x_0, \delta)$ and $t \in (0, 1)$ holds,

$$f(tx + (1 - t)y) \leq tf(x) + (1 - t)f(y) + \epsilon t(1 - t)\|x - y\|.$$

Our aim here is to generalize some basic notions of convex analysis as locally Lipschitz mappings, Clarke subdifferential, approximate convexity and submonotonocity from Banach spaces to locally convex spaces. We first define a locally Lipschitz functions f defined on a locally convex space and introduce its generalized Clarke subdifferential. Then, we deduce a Lebourge's type mean value theorem in the setting of locally convex spaces. In section 3, we extend the concept of approximate convexity to locally convex spaces and prove that each proper approximate convex function f, which is bounded above, is locally Lipschitz. Using the ideas and technique of Daniilidis and Georgiev [4], we introduce submonotone operators in locally convex spaces and give a characterization of approximate convexity via submonotonocity of ∂f. These results can be viewed as an extension and refinement of the results of Daniilidis and Georgiev [4].

Throughout the paper, let X denote a Hausdorff locally convex space whose topology is generated by a family of seminorms $\{p_i, i \in I\}$ and

$$x_0 + B(\delta, p_1, ..., p_n) = x_0 + \{x \in X| \max_{1\le i\le n} p_i(x) < \delta\}.$$

Let X^* be dual space of X and $\langle ., . \rangle$ denotes duality pair of X and X^*.

2. **Locally Lipschitz function in locally convex spaces**

Let $f : X \to \mathbb{R} \cup \{+\infty\}$. We denote by dom $f = \{x \in X : f(x) \neq +\infty\}$, the effective domain of f. The function f is proper if it has a nonempty domain.

Definition 2.1. A function $f : X \to \mathbb{R}$ is said to be locally Lipschitz near $x_0 \in X$ of rank K, if there exists a neighborhood $V = B(\delta, p_1, p_2, ..., p_n)$ of 0 such that

$$| f(x) - f(y) | \le K \max_{1\le i\le n} p_i(x - y), \quad \forall\, x, y \in x_0 + V.$$

For $Y \subseteq X$, we say that f is Lipschitz on Y, if f is Lipschitz near each $x \in Y$.

Remark 2.2. Obviously, if f is Lipschitz near x_0, then it is continuous at x_0. In Section 3, as a consequence of Theorem 3.5, we show that each convex function which is locally bounded at x_0, is Lipschitz near x_0.

Definition 2.3. If $f : X \to \mathbb{R} \cup \{+\infty\}$ is a function, then the generalized Clarke-Rockafellar directional derivative of f at $x_0 \in \mathrm{dom} f$ in the direction v denoted by $f^{\uparrow}(x; v)$, is defined to be

$$f^{\uparrow}(x; v) = \sup_U \limsup_{y\to_f x,\, t\to 0^+} \inf_{u\in U} \frac{f(y + tu) - f(y)}{t},$$

where U is an arbitrary neighborhood of v, $y \to_f x$ means that both $y \to x$ and $f(y) \to f(x)$. Let $f^{\circ}(x; v)$ denote the generalized Clarke derivative of f at x in direction v, defined as

$$f^{\circ}(x; v) = \limsup_{y\to x,\, t\to 0^+} \frac{f(y + tv) - f(y)}{t}.$$

Remark 2.4. If f is locally Lipschitz near x, then by the definition of the generalized Clarke-Rockafellar directional derivative and Remark 2.2, we obtain $f^{\uparrow}(x;v) \leq f^{\circ}(x;v)$. For the converse of the inequality, since f is locally Lipschitz near x, we deduce

$$\frac{f(y+tv)-f(y)}{t} \leq \frac{f(y+tu)-f(y)}{t} + K \max_{1\leq i\leq n} p_i(u-v),$$

for $t > 0$ small enough, and y, u sufficiently near x, v, respectively. This inequality, and continuity of semi-norms p_i give

$$f^{\circ}(x;v) \leq f^{\uparrow}(x;v).$$

In the following theorem we give some properties of $f^{\circ}(x;v)$, where f is a locally Lipschitz mapping, the proof of which is similar to the Banach space case.

Theorem 2.5. *Let f be Lipschitz of rank K near x. Then*

(i) The function $v \to f^{\circ}(x;v)$ is finite, positively homogeneous, subadditive on X and there are semi-norms $p_1, p_2, ..., p_n$ such that $\mid f^{\circ}(x;v) \mid \leq K \max p_i(v)$, for all $i = 1, 2, ..., n$.

(ii) $f^{\circ}(x;-v) = (-f)^{\circ}(x;v)$.

(iii) $f^{\circ}(x;v)$ as a function of v alone is Lipschitz of rank K on X.

The following theorem shows that a function such as $v \to f^{\circ}(x;v)$ which is positively homogeneous and subadditive on X is the support function of a uniquely determined closed convex set in X^*, extending proposition 1.3 in [2].

Definition 2.6. Given a nonempty subset Σ of X^*, its support function is the function $H_\Sigma : X \to \mathbb{R} \cup \{+\infty\}$, defined as

$$H_\Sigma(v) = \sup\{\langle \zeta, v\rangle : \zeta \in \Sigma\}.$$

Theorem 2.7. *(i) Let Σ be a nonempty subset of X^*. Then, H_Σ is positively homogeneous, subadditive, and lower semicontinuous.*

(ii) If Σ is convex and w^*-closed, then a point $\zeta \in X^*$ belongs to Σ if and only if we have $H_\Sigma(v) \geq \langle \zeta, v\rangle$, for all v in X.

(iii) More generally, if Σ and Λ are two nonempty, convex and w^*- closed subsets of X^*, then $\Sigma \supseteq \Lambda$ if and only if $H_\Sigma(v) \geq H_\Lambda(v)$, for all v in X.

(iv) If $P : X \to \mathbb{R}$ is positively homogeneous and subadditive and bounded on a neighborhood W of 0, then there is a uniquely defined nonempty, convex and w^*-compact subset Σ of X^* such that $P = H_\Sigma$.

Proof. The proofs of (*i*),(*ii*) and (*iii*) are similar to the Banach space case (Proposition 3.1 in [1]). There remains (*iv*). By our assumption, P is bounded on neighborhood W of 0, which we may assume it is absorbing and balanced. Then, we can assume that

$$|P(v)| \le M, \quad \text{for each } v \in W. \tag{2.1}$$

Let

$$\Sigma = \{\zeta \in X^* : P(v) \ge \langle \zeta, v\rangle \ \forall v \in X\}.$$

Since W is an absorbing and balanced neighborhood of 0 and P is positively homogeneous, we have

$$\Sigma = \{\zeta \in X^* : P(v) \ge \langle \zeta, v\rangle \ \forall v \in W\}.$$

Now, by Banach- Alaoglu-Bourbaki's theorem, Σ is w^*-compact. Clearly, we have $P \ge H_\Sigma$. For the equality, let $v \in X$ and suppose that $M = \{tv : t \in \mathbb{R}\}$, and define linear functional f on M as

$$f(tv) = tP(v), \quad \text{for every } t \in \mathbb{R}. \tag{2.2}$$

By Hahn-Banach theorem, we can extend f to X with property

$$f(x) \le P(x), \qquad \text{for all } x \in X. \tag{2.3}$$

From (2.1) and (2.3), we get

$$M \ge P(x) \ge f(x) \ge -P(-x) \ge -M, \ \forall v \in W.$$

Thus,

$$|f(x)| \le M, \quad \forall v \in W. \tag{2.4}$$

If V be a neighborhood of 0, in $\mathbb{R}$, then by (2.4) there exists positive integer n such that $f(W) \subset nV$. The last inclusion implies that f is continuous in 0. Hence, $f \in X^*$. Consequently, by (2.2), $P(x) = f(x)$ and hence $P \le H_\Sigma$. □

We define the generalized Clarke subdifferential of f at $x \in X$, denoted by $\partial f(x)$, to be the w^*-compact subset of X^* whose support function is $f^\circ(x;.)$. In other words,

$$\partial f(x) = \{x^* \in X^* : \langle x^*, u\rangle \leq f^\circ(x;u), \forall u \in X\}.$$

In the following, we establish a basic calculus about subdifferential of two functions. The proofs similar to the ones for Propositions 2.1 and 2.2 in [2].

Remark 2.8. If f is locally Lipschitz near x and also it attains a local minimum or maximum at x, then $0 \in \partial f(x)$. Indeed, let $v \in X$. Then, by definition of the generalized directional derivative of f at x, we have

$$f^\circ(x;v) \geq \limsup_{t\to 0^+} \frac{f(x+tv) - f(x)}{t} \geq 0 = \langle 0, v\rangle,\ \forall v \in X.$$

Hence, $0 \in \partial f(x)$.

Proposition 2.9.

(i) For any scalar λ, $\partial \lambda f(x) = \lambda \partial f(x)$

(ii) Let f_i $(i = 1, 2, ..., n)$ be Lipschitz near x, and let λ_i, $(i = 1, 2, ..., n)$ be scalars. Then,

$$\partial(\sum_{i=1}^{n} \lambda_i f_i)(x) \subset \sum_{i=1}^{n} \lambda_i \partial f_i(x).$$

In the following, we obtain a Lebourg's type mean value theorem in locally convex spaces, the proof of which follows along the lines of the Banach space case and thus is omitted (see Theorem 2.4 in [2]).

Theorem 2.10. *Let x, y belong to X, and suppose that f is Lipschitz on an open set containing the line segment $[x, y]$. Then, there exists a point $u \in (x, y)$ such that*

$$f(y) - f(x) \in \langle \partial f(u), y - x\rangle.$$

3. Characterization of approximate convexity

The notion of an approximate convex function in a Banach space was first introduced by Ngai et al. [6].

Definition 3.1. Let $f : X \to \mathbb{R}\cup\{+\infty\}$. We say that f is approximately convex at $x_0 \in X$ if for every $\epsilon > 0$ there exists an open neighborhood $V = B(\delta, p_1, p_2, ..., p_n)$ of 0 such that

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) + \epsilon t(1-t) \max_{1\leq i\leq n} p_i(x-y),$$

for all $x, y \in x_0 + V,\ t \in (0,1)$. The function f is called approximately convex on $Y \subseteq X$ if f is approximately convex at every point of Y.

Definition 3.2. A function $f : X \to \mathbb{R}$ is strictly differentiable at $x_0 \in X$ if there exists $Df(x_0) \in X^*$ such that for each $\epsilon > 0$, there exists $V = B(\delta, p_1, p_2, ..., p_n)$ together with the following implication for every $x, y \in x_0 + V$,

$$|f(x) - f(y) - Df(x_0)(x-y)| \leq \epsilon \max p_i(x-y).$$

The following proposition, which is easy to prove, gives some sufficient conditions for f to be approximate convex and is similar to Proposition 3.1 in [6].

Proposition 3.3. *Let $f : X \to \mathbb{R}$. Each of the following conditions is sufficient for f to be approximate convex at $x_0 \in X$:*

(i) f is strictly differentiable at x_0;

(ii) $f = f_1 + f_2$ or $f = \max(f_1, f_2)$ where f_1 and f_2 are approximately convex at x_0;

(iii) $f = g \circ A$ where A is a continuous affine from X to a locally convex space Y and g is a function from Y to $\mathbb{R}\cup\{\infty\}$ which is approximate convex at $Ax_0 \in Y$.

In the next theorem, we establish a Lipschitz property of approximate convex functions which generalizes a result in [6]. The following definition is necessary for obtaining our result.

Definition 3.4. The function f is called bounded above near x, if there exist an open neighborhood U of x and real number r such that

$$f(y) \leq r,\ \forall y \in U.$$

Theorem 3.5. *Suppose that $f : X \to \mathbb{R} \cup \{\infty\}$ is a proper function. If f is approximate convex at $x_0 \in Int(dom\ f)$ and bounded above near x_0, then f is locally Lipschitz at x_0.*

Proof. By our assumption, there exists a neighborhood

$$V = B(\delta, p_1, p_2, p_3, ..., p_n)$$

of 0 and positive number r such that

$$f(y) \leq r, \quad \forall y \in x_0 + V,$$

$$f(tx+(1-t)y) \leq \tag{3.1}$$
$$tf(x) + (1-t)f(y) + t(1-t)\max\ p_i(x-y),\ \forall x, y \in x_0 + V.$$

If $x \in x_0 + V$, then there exists $y \in x_0 + V$ such that $\frac{x+y}{2} = x_0$. Then, by (2.1), we deduce that

$$f(x_0) \leq \frac{1}{2}f(x) + \frac{1}{2}f(y) + \max\ p_i(x-y) \leq$$
$$\frac{1}{2}(f(x) + f(y)) + 2\delta \leq \frac{1}{2}f(x) + \frac{r}{2} + 2\delta.$$

Thus,

$$2f(x_0) - r - 4\delta \leq f(x)\ \forall x \in x_0 + V.$$

So, f is bounded below. Consequently, f is locally bounded, say by M; i.e.,

$$|f(x)| \leq M, \quad \forall x \in x_0 + V.$$

If $x, y \in x_0 + B(\frac{\delta}{2}, p_1, p_2, ..., p_n)$, then $z = x + \frac{\delta}{2\eta+\theta}(x-y) \in x_0 + V$ with $\eta = \max p_i(x-y)$. Hence, for each positive number θ,

$$f(x) = f(\frac{2\eta+\theta}{\delta+2\eta+\theta}z + \frac{\delta}{\delta+2\eta+\theta}y) \leq$$
$$\frac{2\eta+\theta}{\delta+2\eta+\theta}f(z) + \frac{\delta}{\delta+2\eta+\theta}f(y) + \frac{\epsilon(2\eta+\theta)\delta}{(\delta+2\eta+\theta)^2}\max p_i(z-y).$$

It follows that

$$f(x)-f(y)\le \frac{2\eta+\theta}{\delta+2\eta+\theta}(f(z)-f(y))+\frac{\epsilon\delta}{\delta+2\eta+\theta}\max p_i(x-y)\le$$
$$(\frac{4M}{\delta}+\epsilon)\max p_i(x-y)+\frac{2M\theta}{\delta}.$$

By interchanging the roles of x and y, we have

$$|f(x)-f(y)|\le(\frac{4M}{\delta}+\epsilon)\max p_i(x-y)+\frac{2M\theta}{\delta}.$$

Since θ is an arbitrary positive number, we obtain the required result. $\square$

Remark 3.6. Let X be complete metrizable l.c.s. and $f:X\to\mathbb{R}\cup\{\infty\}$ be proper and lower semicontinuous. Using Baire category theorem, one can show that f is bounded above near $x_0\in \mathrm{Int}(\mathrm{dom\ f})$ if f is approximately convex at x_0 (see proof of Proposition 3.2 in [6]).

Definition 3.7. The set-valued map $T:X\to 2^{X^*}$ is called submonotone at $x_0\in X$, if for every $\epsilon>0$ there exists an open neighborhood $V=B(\delta,p_1,p_2,...,p_n)$ of 0 such that

$$\langle x_1^*-x_2^*,x_1-x_2\rangle\ge-\epsilon\max p_i(x_1-x_2),$$

for all $x_1,x_2\in x_0+V$ and all $x_i^*\in T(x_i),\ i=1,2$.

The following theorem extends Theorem 2 in [4] which is a characterization of approximate convexity. In order to obtain our theorem, we need the following definition and lemma.

Definition 3.8. The function f is said to be regular at x provided that f is locally Lipschitz near x and admits directional derivatives $f^{'}(x;v)$ at x for all v, with $f^{'}(x;v)=f^{\circ}(x;v)$, where $f^{'}(x;v)$ is defined to be

$$f^{'}(x;v)=\lim_{t\searrow 0^+}\frac{f(x+tv)-f(x)}{t},$$

when the limit exists.

The following lemma states that under suitable conditions the function f is regular, extending Proposition 4.1 in [2] to l.c.s..

Lemma 3.9. *If f is approximately convex and locally Lipschitz near x, then f is regular at x.*

Proof. Let $\epsilon > 0$ be sufficiently small. By our assumptions, there exists an open neighborhood $V = B(\delta, p_1, p_2, ..., p_n)$ of 0 such that f admits definitions 2.1 and 3.1 on $x+V$. Suppose $0 < h_1 \leq h_2$, sufficiently small, and $u \in X$. From approximate convexity, we have,

$$f(\frac{h_1}{h_2}(x+h_2u)+(1-\frac{h_1}{h_2})x) \leq$$
$$\frac{h_1}{h_2}f(x+h_2u)+1-\frac{h_1}{h_2}f(x)+\epsilon\frac{h_1}{h_2}(1-\frac{h_1}{h_2})\max_{1\leq i\leq n} p_i(u).$$

Then,

$$\frac{f(x+h_1u)-f(x)}{h_1} \leq \frac{f(x+h_2u)-f(x)}{h_2}+\epsilon(1-\frac{h_1}{h_2})\max_{1\leq i\leq n} p_i(u)$$

Now, for fixed h_2 and h_1 tending to 0, we have

$$\limsup_{h_1\to 0}\frac{f(x+h_1u)-f(x)}{h_1} \leq \frac{f(x+h_2u)-f(x)}{h_2}+\epsilon\max_{1\leq i\leq n} p_i(u).$$

Hence, by the last inequality, we have

$$\limsup_{h_1\to 0}\frac{f(x+h_1u)-f(x)}{h_1} \leq \inf_{h_2>0}\frac{f(x+h_2u)-f(x)}{h_2}+\epsilon\max_{1\leq i\leq n} p_i(u)$$
$$\leq \liminf_{h_2\to 0}\frac{f(x+h_2u)-f(x)}{h_2}+\epsilon\max_{1\leq i\leq n} p_i(u).$$

Since ϵ is arbitrary, the above inequality implies that

$$f^{'}(x;u) = \lim_{t\to 0^+}\frac{f(x+tu)-f(x)}{t} = \inf_{t>0}\frac{f(x+tu)-f(x)}{t}.$$

Using the locally Lipschitz property, for t, ϵ (sufficiently small) and $x' \in x+B(\epsilon\delta)$, we have

$$\mid \frac{f(x'+\epsilon u)-f(x)}{\epsilon} - \frac{f(x+\epsilon u)-f(x)}{\epsilon} \mid\leq 2K\max_{1\leq i\leq n} p_i(x'-x).$$

Thus,

$$f^{\circ}(x;u)=\limsup_{x'\to x,\epsilon\to 0}\frac{f(x'+\epsilon u)-f(x)}{\epsilon}\leq$$

$$\limsup_{x'\to x,\epsilon\to 0}\frac{f(x+\epsilon u)-f(x)}{\epsilon}+2K\max_{1\leq i\leq n}p_i(x'-x)$$

$$\leq\limsup_{\epsilon\to 0}\frac{f(x+\epsilon u)-f(x)}{\epsilon}+\limsup_{x'\to x}2K\max_{1\leq i\leq n}p_i(x'-x).$$

The last inequality completes the proof. □

Theorem 3.10. *Assume that f is locally Lipschitz at $x_0\in X$. The followings are equivalent:*

(i) f is approximately convex at x_0.

(ii) For every $\epsilon>0$, there exists an open neighborhood $V=B(\delta,p_1,p_2,...,p_n)$ of 0 such that for each $x\in x_0+V$ and $x^*\in\partial f(x)$,

$$f(x+u)-f(x)\geq\langle x^*,u\rangle-\epsilon\max_{1\leq i\leq n}p_i(u),$$

whenever $u\in V$ is such that $x+u\in x_0+V$.

(iii) ∂f is submonotone at x_0.

Proof. $(i)\Rightarrow(ii)$. Let $\epsilon>0$ be given. By *(i)* there exists $V=B(\delta,p_1,p_2,...,p_n)$ of 0 such that for each $x,x+tu\in x_0+V$ and $t\in(0,1)$, we have

$$f(x+tu)-f(x)=f(t(x+u)+(1-t)x)-f(x)\leq$$

$$tf(x+u)+(1-t)f(x)-f(x)+\varepsilon t(1-t)\max_{1\leq i\leq n}p_i(u).$$

Then,

$$\frac{f(x+tu)-f(x)}{t}\leq f(x+u)-f(x)+\epsilon(1-t)\max_{1\leq i\leq n}p_i(u),$$

Since f is locally Lipshcitz and approximately convex at x_0, then f is regular at x_0, and so

$$\langle x^*,u\rangle\leq f^0(x;u)=\limsup_{t\to 0^+}\frac{f(x+tu)-f(x)}{t}$$

$$\leq f(x+u)-f(x)+\epsilon\max_{1\leq i\leq n}p_i(u),\quad\forall x^*\in\partial f(x),u\in X.$$

(ii) $\Rightarrow$ *(iii)*. Let $\epsilon > 0$ and take $V = B(\delta, p_1, p_2, ..., p_n)$ as given in *(ii)*. Now, for all $x, y \in x_0 + B(\frac{\delta}{2}, p_1, p_2, ..., p_n)$ and $x^* \in \partial f(x)$, $y^* \in \partial f(y)$, by *(ii)*, we have

$$f(y) - f(x) \geq \langle x^*, y - x\rangle - \frac{\epsilon}{2} \max_{1\leq i\leq n} p_i(y - x),$$

and

$$f(x) - f(y) \geq \langle y^*, x - y\rangle - \frac{\epsilon}{2} \max_{1\leq i\leq n} P_i(x - y).$$

By adding the above inequalities, the result follows.

(iii) $\Rightarrow$ *(i)*. Let $\varepsilon > 0$. By *(iii)*, there exists an open neighborhood $V = x_0 + B(\delta, p_1, p_2, ..., p_n)$ such that the relation in *(iii)* holds. Let $x, y \in x_0 + B(\delta, p_1, p_2, ..., p_n)$ and $t \in (0, 1)$ and set $x_t = tx + (1 - t)y$. By Theorem 2.10, there exist a point $z_1 \in [x, x_t[$ and $z_1^* \in \partial f(x_1)$ such that

$$\langle z_1^*, x_t - x\rangle = f(x_t) - f(x). \tag{3.1}$$

Similarly, there exists a point $z_2 \in [x_t, y[$ and $z_2^* \in \partial f(z_2)$ such that

$$\langle z_2^*, x_t - y\rangle = f(x_t) - f(y). \tag{3.2}$$

By multiplying the relations (3.1) and (3.2), by t and $(1-t)$, respectively, and adding the resulting equations, we obtain

$$tf(x) + (1 - t)f(y) - f(x_t) = t(1 - t)\langle z_1^* - z_2^*, x - y\rangle. \tag{3.3}$$

Now, from the fact that z_1 and z_2 are distinct points of the line segment $[x, y]$, there exists a positive number c such that $z_1 - z_2 = c(x - y)$. Then, by (3.3) and submonotonicity of ∂f, we obtain

$$tf(x) + (1 - t)f(y) - f(x_t) = t(1 - t)\langle z_1^* - z_2^*, \frac{z_1 - z_2}{c}\rangle \geq$$

$$\frac{-\varepsilon}{c} t(1 - t) \max_{1\leq i\leq n} p_i(z_1 - z_2) = -\varepsilon t(1 - t) \max_{1\leq i\leq n} p_i(x - y),$$

Thus,

$$f(tx + (1 - t)y) \leq tf(x) + (1 - t)f(y) + \epsilon t(1 - t) \max_{1\leq i\leq n} p_i(x - y).$$

□

The following theorem is obtained by using the fact that a Hausdorff vector topology on a finite dimensional vector space is unique.

Theorem 3.11. *Let X be a finite dimensional Hausdorff locally convex space. Then, a real-valued, locally Lipschitz function f on X is approximately convex if and only if it is lower-C^1.*

Proof. Since X is finite dimensional, then X is normable. Let $\|.\|$ denote the norm of X. We first show that f is locally Lipschitz with respect to the norm of X. To see this, let $x_0 \in X$. Then, there exists a neighborhood $V = B(\delta, p_1, p_2, ..., p_n)$ of 0 such that

$$| f(x) - f(y) | \leq K \max_{1 \leq i \leq n} p_i(x - y), \quad \forall\, x, y \in x_0 + V.$$

Let $P(x) = \max p_i(x)$ and S be the unit ball of X. Since $P(x)$ is continuous and S is compact, then P is bounded on S, say by M. Then, for each nonzero element $x \in X$ we have $P(\frac{x}{\|x\|}) \leq M$. Consequently,

$$\max_{1 \leq i \leq n} p_i(x) \leq M \parallel x \parallel \;\; \forall x \in X,$$

which shows that

$$| f(x) - f(y) | \leq K \max p_i(x - y) \leq KM \parallel x - y \parallel, \; \forall\, x, y \in x_0 + V.$$

Therefore f is locally Lipschitz at x_0 and so locally Lipschitz on X (note that x_0 is an arbitrary element of X). Now, the result follows by Corollary 3 in [4]. □

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