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ITERATIVE METHODS FOR EQUILIBRIUM PROBLEMS, VARIATIONAL INEQUALITIES AND FIXED POINTS

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Abstract. We introduce iterative methods for finding a common element of the set of solutions of a system of equilibrium problems, the set of fixed points for an infinite family of nonexpansive mappings and a family of strictly pseudocontractive mappings, and the set of solutions of the variational inequalities for a family of α-inverse-strongly monotone mappings in a Hilbert space. We establish some weak and strong convergence theorems of the sequences generated by our proposed schemes. The strong convergence results are obtained via the CQ method.

1. Introduction

Let C be a nonempty closed convex subset of a Hilbert space H. Let $F : C \times C \to \mathbb{R}$ be a bifunction. The equilibrium problem for F is to determine its equilibrium points; i.e., the set

$$EP(F) := \{x \in C : F(x, y) \geq 0 \ \forall y \in C\}.$$

Let $\mathcal{G} = \{F_i\}_{i\in I}$ be a family of bifunctions from $C \times C$ to $\mathbb{R}$. The system of equilibrium problems for $\mathcal{G} = \{F_i\}_{i\in I}$ is to determine common

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equilibrium points for $\mathcal{G} = \{F_i\}_{i\in I}$; i.e., the set

$$EP(\mathcal{G}) := \{x \in C : F_i(x,y) \geq 0 \ \forall y \in C \ \forall i \in I\}. \tag{1.1}$$

Many problems in applied sciences, such as monotone inclusion problems, saddle point problems, variational inequality problems, minimization problems, Nash equilibria in noncooperative games, vector equilibrium problems, as well as certain fixed point problems reduce to finding some element of $EP(F)$; see [3, 10, 11]. The formulation (1.1) extends this formalism to systems of such problems, covering in particular various forms of feasibility problems [2, 9].

Recall that a mapping S of C into H is called nonexpansive if

$$\|Sx - Sy\| \leq \|x - y\|, \quad \forall x, y \in C.$$

We denote by $Fix(S)$, the set of fixed points of S.

S is strictly pseudocontractive if there exists κ with $0 \leq \kappa < 1$ such that

$$\|Sx - Sy\|^2 \leq \|x - y\|^2 + \kappa\|(I - S)x - (I - S)y\|^2, \text{ for all } x, y \in C.$$

If $k = 0$, then S is nonexpansive.

Finding an optimal point in the intersection of the fixed point sets of a family of nonexpansive mappings is a task arising frequently in various areas of mathematical sciences and engineering. For example, the well-known convex feasibility problem reduces to finding a point in the intersection of the fixed point sets of a family of nonexpansive mappings; see, e.g., [2, 8].

Recall that a mapping $A : C \to H$ is called α-inverse-strongly monotone [4], if there exists a positive real number α such that

$$\langle Ax - Ay, x - y\rangle \geq \alpha\|Ax - Ay\|^2, \ \ \forall x, y \in C.$$

It is easy to see that if $A : C \to H$ is α-inverse-strongly monotone, then it is a $\frac{1}{\alpha}$-Lipschitzian mapping.

Let $A : C \to H$ be a mapping. The classical variational inequality problem is to find $u \in C$ such that

$$\langle Au, v - u\rangle \geq 0, \ \ \forall v \in C. \tag{1.2}$$

The set of solutions of variational inequality (1.2) is denoted by $VI(C, A)$. Put $A = I - T$, where $T : C \to H$ is a strictly pseudocontractive mapping with κ. It is known that A is $\frac{1-\kappa}{2}$-inverse-strongly monotone and $A^{-1}(0) = Fix(T) = \{x \in C : Tx = x\}$.

Recently, under certain appropriate conditions, Tada and Takahashi [21] obtained weak and strong convergence theorems for finding a common element of $EP(F)$ and $Fix(S)$, where F is a bifunction and S a nonexpansive mapping. Related work can also be found in [6, 7, 14, 17, 18, 19, 21, 26].

Here, motivated by [18, 21], we introduce iterative algorithms for finding a common element of the set of solutions of a system of equilibrium problems, the set of common fixed points for an infinite family of nonexpansive mappings and the set of solutions of variational inequalities for a family of α-inverse-strongly monotone mappings from C into H. Moreover, we apply our results to the problem of finding a common fixed point of a family of strictly pseudocontractive mappings. Our results present extentions of several existing results.

2. **Preliminaries**

Let C be a nonempty closed and convex subset of H. Let $F : C \times C \to \mathbb{R}$ be a bifunction. The equilibrium problem for F is to determine its equilibrium points; i.e., the set

$$EP(F) := \{x \in C : F(x, y) \geq 0 \ \forall y \in C\}.$$

Given any $r > 0$, the operator $J_r^F : H \to C$ defined by

$$J_r^F(x) := \{z \in C : F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0 \ \forall y \in C\}$$

is called the resolvent of F; see [10].

Lemma 2.1. *(See [10]) Let C be a nonempty closed convex subset of H and $F : C \times C \to \mathbb{R}$ satisfy*

(A1) $F(x, x) = 0$ *for all* $x \in C$*;*
(A2) F *is monotone; i.e.,* $F(x, y) + F(y, x) \leq 0$ *for all* $x, y \in C$.
(A3) *for all* $x, y, z \in C$,

$$\liminf_{t\to 0} F(tz + (1 - t)x, y) \leq F(x, y);$$

(A4) *for all* $x \in C$, $y \longmapsto F(x, y)$ *is convex and lower semicontinuous.*

Then,

(1) J_r^F *is single-valued;*
(2) J_r^F *is firmly nonexpansive; i.e.,*

$$\|J_r^F x - J_r^F y\|^2 \le \langle J_r^F x - J_r^F y, x - y\rangle, \text{ for all } x, y \in H;$$

(3) $fix(J_r^F) = EP(F)$;
(4) $EP(F)$ *is closed and convex.*

Recall that the metric (nearest point) projection P_C from a Hilbert space H to a closed convex subset C of H is defined as follows: given $x \in H$, $P_C x$ is the only point in C with the property,

$$\|x - P_C x\| = \inf\{\|x - y\| : y \in C\}.$$

It is known that P_C is a nonexpansive mapping and satisfies:

$$\|P_C x - P_C y\|^2 \le \langle P_C x - P_C y, x - y\rangle, \ \ \forall x, y \in H. \tag{2.1}$$

P_C is characterized as follows:

$$y = P_C x \iff \langle x - y, y - z\rangle \ge 0, \ \ \forall z \in C.$$

In the context of the variational inequality problem, this implies that

$$u \in VI(C, A) \iff u = P_C(u - \lambda A u), \ \ \forall \lambda > 0. \tag{2.2}$$

A set-valued mapping $T : H \to 2^H$ is said to be monotone, if for all $x, y \in H$, $f \in Tx$, and $g \in Ty$ we have $\langle f - g, x - y\rangle \ge 0$. A monotone mapping $T : H \to 2^H$ is said to be maximal, if the graph $G(T)$ of T is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping is maximal, if and only if for $(x, f) \in H \times H$, $\langle f - g, x - y\rangle \ge 0$, $\forall (y, g) \in G(T)$ we have $f \in Tx$. Let $A : C \to H$ be an inverse-strongly monotone mapping and let $N_C v$ be the normal cone to C at $v \in C$; i.e.,

$$N_C v = \{w \in H : \langle v - u, w\rangle \ge 0, \forall u \in C\},$$

and define,

$$Tv = \begin{cases} Av + N_C v, & v \in C; \\ \emptyset, & v \notin C. \end{cases}$$

Then, T is maximal monotone and $0 \in Tv$ if and only if $v \in VI(C, A)$ (see [12, 16]). It is easy to show that for given $\lambda \in [0, 2\alpha]$, the mapping $(I - \lambda A) : C \to H$ is nonexpansive.

Below, lemmas 2.2 and 2.3 were proved in [24].

Lemma 2.2. *Let* $\{\alpha_n\}$ *be a sequence of real numbers such that* $0 < a \leq \alpha_n \leq b < 0$*, for all* $n \in \mathbb{N}$*. Let* $\{v_n\}$ *and* $\{w_n\}$ *be sequences of* H *such that*

$$\limsup_{n\to\infty} \|v_n\| \leq c, \quad \limsup_{n\to\infty} \|w_n\| \leq c$$

and

$$\limsup_{n\to\infty} \|\alpha_n v_n + (1-\alpha_n)w_n\| = c, \text{ for some } c > 0.$$

Then, $\lim_{n\to\infty} \|v_n - w_n\| = 0$.

Lemma 2.3. *Let* C *be a nonempty closed convex subset of* H*. Let* $\{x_n\}$ *be a sequence in* H*. Suppose that, for all* $y \in C$*,*

$$\|x_{n+1} - y\| \leq \|x_n - y\|,$$

for every $n \in \mathbb{N}$*. Then,* $\{P_C(x_n)\}$ *converges strongly to some* $z \in C$*.*

Definition 2.4. Let $\{S_i : C \to C\}$ be an infinite family of nonexpansive mappings and $\{\mu_i\}$ be a nonnegative real sequence with $0 \leq \mu_i < 1$, $\forall i \geq 1$. For any $n \geq 1$, define a mapping $W_n : C \to C$ as follows:

$$\begin{aligned}
&U_{n,n+1} := I,\\
&U_{n,n} := \mu_n S_n U_{n,n+1} + (1-\mu_n)I,\\
&U_{n,n-1} := \mu_{n-1} S_{n-1} U_{n,n} + (1-\mu_{n-1})I,\\
&\quad\vdots\\
&U_{n,k} := \mu_k S_k U_{n,k+1} + (1-\mu_k)I,\\
&U_{n,k-1} := \mu_{k-1} S_{k-1} U_{n,k} + (1-\mu_{k-1})I,\\
&\quad\vdots\\
&U_{n,2} := \mu_2 S_2 U_{n,3} + (1-\mu_2)I,\\
&W_n := U_{n,1} = \mu_1 S_1 U_{n,2} + (1-\mu_1)I. \qquad (2.3)
\end{aligned}$$

Such a mapping W is nonexpansive from C to C and is called the W-mapping generated by $S_n, S_{n-1}, \ldots, S_1$ and $\mu_n, \mu_{n-1}, \ldots, \mu_1$.

The concept of W-mappings was introduced in [22, 23]. It is now one of the main tools in studying convergence of iterative methods for approaching a common fixed of nonlinear mappings; more recent progresses can be found in [1, 5, 7, 13, 25] and the references cited therein.

Lemma 2.5. *(Shimoji et al., [20]) Let C be a nonempty closed convex subset of a Hilbert space H, $\{S_i : C \to C\}$ be an infinite family of nonexpansive mappings with $\cap_{i=1}^{\infty} Fix(S_i) \neq \emptyset$, and $\{\mu_i\}$ be a real sequence such that $0 < \mu_i \leq b < 1$, $\forall i \geq 1$. Then,*

(1) W_n is nonexpansive and $Fix(W_n) = \cap_{i=1}^{n} Fix(S_i)$ for each $n \geq 1$;
(2) for each $x \in C$ and for each positive integer k, $\lim_{n\to\infty} U_{n,k}x$ exists;
(3) the mapping $W : C \to C$ defined by

$$Wx := \lim_{n\to\infty} W_n x = \lim_{n\to\infty} U_{n,1}x,$$

is a nonexpansive mapping satisfying $Fix(W) = \cap_{i=1}^{\infty} Fix(S_i)$ and it is called the W-mapping generated by $S_1, S_2, \ldots$ and $\mu_1, \mu_2, \ldots$.

3. Strong convergence of a general iterative method

The following is our main result.

Theorem 3.1. *Let C be a nonempty closed convex subset of a Hilbert space H, $\varphi = \{S_i : C \to C\}$ an infinite family of nonexpansive mappings, $\mathcal{G} = \{F_j : j = 1, \ldots, M\}$ a finite family of bifunctions from $C \times C$ into $\mathbb{R}$ which satisfy (A1)-(A4), $\mathcal{A} = \{A_k : k = 1 \ldots N\}$ a finite family of α-inverse-strongly monotone mappings from C into H, and $\mathcal{F} := \cap_{k=1}^{N} VI(C, A_k) \cap Fix(\varphi) \cap EP(\mathcal{G}) \neq \emptyset$.*

Let $\{\alpha_n\}$ be a sequence in $[a, 1]$ for some $a \in (0, 1)$, $\{\lambda_{k,n}\}_{k=1}^{N}$ sequences in $[c, d] \subset (0, 2\alpha)$ and $\{r_{j,n}\}_{j=1}^{M}$ sequences in $(0, \infty)$ such that $\liminf_n r_{j,n} > 0$ for every $j \in \{1, \ldots, M\}$. For every $n \in \mathbb{N}$, let W_n be the W-mapping defined by (2.3).

If $\{x_n\}$ is the sequence generated by $x_1 = x \in H$ and $\forall n \geq 1$,

$$\begin{cases} u_n = J_{r_{M,n}}^{F_M} \ldots J_{r_{2,n}}^{F_2} J_{r_{1,n}}^{F_1} x_n, \\ v_n = P_C(I - \lambda_{N,n} A_N) \ldots P_C(I - \lambda_{2,n} A_2) P_C(I - \lambda_{1,n} A_1) u_n, \\ y_n = (1 - \alpha_n) x_n + \alpha_n W_n v_n, \\ C_n = \{z \in H : \|y_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in H : \langle x_n - z, x - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n}(x), \end{cases}$$

then the sequences $\{x_n\}$ and $\{y_n\}$ converge strongly to $P_{\mathcal{F}}(x)$.

Proof. Take

$$\mathcal{J}_n^k := J_{r_{k,n}}^{F_k} \dots J_{r_{2,n}}^{F_2} J_{r_{1,n}}^{F_1},\ \forall k \in \{1, \dots, M\},$$

$$\mathcal{J}_n^0 := I,$$

and

$$\mathcal{P}_n^k := P_C(I - \lambda_{k,n} A_k) \dots P_C(I - \lambda_{2,n} A_2) P_C(I - \lambda_{1,n} A_1),\ \forall k \in \{1, \dots, N\},$$

$$\mathcal{P}_n^0 := I.$$

So, we can write

$$y_n = (1 - \alpha_n) x_n + \alpha_n W_n \mathcal{P}_n^N \mathcal{J}_n^M x_n.$$

We shall divide the proof into several steps.

Step 1. The sequence $\{x_n\}$ is well defined.

Proof of Step 1. The sets C_n and Q_n are closed and convex subsets of H for every $n \in \mathbb{N}$; see [15, 21]. So, $C_n \cap Q_n$ is a closed convex subset of H for any $n \in \mathbb{N}$. Let $v \in \mathcal{F}$. Since, for each $k \in \{1, \dots, M\}$, $J_{r_{k,n}}^{F_k}$ is nonexpansive, and from Lemma 2.1, we have

$$\|u_n - p\| = \|\mathcal{J}_n^M x_n - v\| = \|\mathcal{J}_n^M x_n - \mathcal{J}_n^M v\| \le \|x_n - v\|. \tag{3.1}$$

On the other hand, since $A_k : C \to H$ is α-inverse-strongly monotone and $\lambda_{k,n} \in [c, d] \subset [0, 2\alpha]$, then $P_C(I - \lambda_{k,n} A_k)$ is nonexpansive. Thus $\mathcal{P}_n^N$ is nonexpansive. From Lemma 2.5 and (2.2), we have $\mathcal{P}_n^N v = v = W_n v$. It follows that

$$\begin{aligned} \|y_n - v\| &\le (1 - \alpha_n)\|x_n - v\| + \alpha_n \|W_n \mathcal{P}_n^N \mathcal{J}_n^M x_n - v\| \\ &\le (1 - \alpha_n)\|x_n - v\| + \alpha_n \|x_n - v\| = \|x_n - v\|. \end{aligned} \tag{3.2}$$

So, we have $v \in C_n$; thus, $\mathcal{F} \subset C_n$, for every $n \in \mathbb{N}$. Next, we show by induction that

$$\mathcal{F} \subset C_n \cap Q_n$$

for each $n \in \mathbb{N}$. Since $\mathcal{F} \subset C_1$ and $Q_1 = H$, we get $\mathcal{F} \subset C_1 \cap Q_1$. Suppose that $\mathcal{F} \subset C_k \cap Q_k$ for $k \in \mathbb{N}$. Then, there exists $x_{k+1} \in C_k \cap Q_k$ such that $x_{k+1} = P_{C_k \cap Q_k}(x)$. Therefore, for each $z \in C_k \cap Q_k$, we have

$$\langle x_{k+1} - z, x - x_{k+1} \rangle \ge 0.$$

So, we get

$$\mathcal{F} \subset C_k \cap Q_k \subset Q_{k+1}.$$

From this and $\mathcal{F} \subset C_n$ $(\forall n)$, we have $\mathcal{F} \subset C_{k+1} \cap Q_{k+1}$. This means that the sequence $\{x_n\}$ is well defined.

Step 2. The sequences $\{x_n\}$, $\{y_n\}$, $\{\mathcal{J}_n^k x_n\}_{k=1}^M$ and $\{\mathcal{P}_n^k u_n\}_{k=1}^N$ are bounded and

$$\lim_{n\to\infty} \|x_n - x\| = c, \quad \text{for some } c \in \mathbb{R}. \tag{3.3}$$

Proof of Step 2. From $x_{n+1} = P_{C_n \cap Q_n}(x)$, we have

$$\|x_{n+1} - x\| \leq \|z - x\|, \ \forall z \in C_n \cap Q_n.$$

Since $P_{\mathcal{F}}(x) \in \mathcal{F} \subset C_n \cap Q_n$, we have

$$\|x_{n+1} - x\| \leq \|P_{\mathcal{F}}(x) - x\|, \tag{3.4}$$

for every $n \in \mathbb{N}$. Therefore, $\{x_n\}$ is bounded. From (3.1) and (3.2), the sequences $\{\mathcal{J}_n^k x_n\}_{k=1}^M$, $\{\mathcal{P}_n^k u_n\}_{k=1}^N$ and $\{y_n\}$ are also bounded.

It is easy to show $x_n = P_{Q_n}(x)$. From this and $x_{n+1} \in Q_n$, we have

$$\|x - x_n\| \leq \|x - x_{n+1}\|,$$

for every $n \in \mathbb{N}$. Since $\{x_n\}$ is bounded, there exists $c \in \mathbb{R}$ such that (3.3) holds.

Step 3. $\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0$.

Proof of Step 3. Since $x_n = P_{Q_n}(x)$, $x_{n+1} \in Q_n$ and $(x_n + x_{n+1})/2 \in Q_n$, we have

$$\begin{aligned} \|x - x_n\|^2 &\leq \|x - \frac{x_n + x_{n+1}}{2}\|^2 \\ &= \|\frac{1}{2}(x - x_n) + \frac{1}{2}(x - x_{n+1}\|^2 \\ &= \frac{1}{2}\|x - x_n\|^2 + \frac{1}{2}\|x - x_{n+1}\|^2 - \frac{1}{4}\|x_n - x_{n+1}\|^2. \end{aligned}$$

So, we get

$$\frac{1}{4}\|x_n - x_{n+1}\|^2 \leq \frac{1}{2}\|x - x_{n+1}\|^2 - \frac{1}{2}\|x - x_n\|^2.$$

From (3.3), we obtain $\lim_{n\to\infty} \|x_n - x_{n+1}\|^2 = 0$.

Step 4. $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

Proof of Step 4. From $x_{n+1} \in C_n$, we have

$$\|x_n - y_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\| \leq 2\|x_n - x_{n+1}\|.$$

Now, apply Step 3.

Step 5. $\lim_{n\to\infty} \|\mathcal{J}_n^k x_n - \mathcal{J}_n^{k+1} x_n\| = 0$, $\forall k \in \{0, 1, \ldots, M-1\}$.

Proof of Step 5. Let $v \in \mathcal{F}$ and $k \in \{0, 1, \ldots, M-1\}$. Since $J_{r_{k+1,n}}^{F_{k+1}}$ is firmly nonexpansive, we obtain

$$\begin{aligned}
\|v - \mathcal{J}_n^{k+1} x_n\|^2 &= \|J_{r_{k+1,n}}^{F_{k+1}} v - J_{r_{k+1,n}}^{F_{k+1}} \mathcal{J}_n^k x_n\|^2 \\
&\leq \langle J_{r_{k+1,n}}^{F_{k+1}} \mathcal{J}_n^k x_n - v, \mathcal{J}_n^k x_n - v \rangle \\
&= \frac{1}{2}(\|J_{r_{k+1,n}}^{F_{k+1}} \mathcal{J}_n^k x_n - v\|^2 + \|\mathcal{J}_n^k x_n - v\|^2 - \|\mathcal{J}_n^k x_n - J_{r_{k+1,n}}^{F_{k+1}} \mathcal{J}_n^k x_n\|^2).
\end{aligned}$$

It follows that

$$\|\mathcal{J}_n^{k+1} x_n - v\|^2 \leq \|x_n - v\|^2 - \|\mathcal{J}_n^k x_n - \mathcal{J}_n^{k+1} x_n\|^2.$$

Therefore, by the convexity of $\|.\|^2$, we have

$$\begin{aligned}
\|y_n - v\|^2 &\leq (1-\alpha_n)\|x_n - v\|^2 + \alpha_n \|W_n v_n - v\|^2 \\
&\leq (1-\alpha_n)\|x_n - v\|^2 + \alpha_n \|\mathcal{J}_n^{k+1} x_n - v\|^2 \\
&\leq (1-\alpha_n)\|x_n - v\|^2 + \alpha_n(\|x_n - v\|^2 - \|\mathcal{J}_n^k x_n - \mathcal{J}_n^{k+1} x_n\|^2) \\
&= \|x_n - v\|^2 - \alpha_n \|\mathcal{J}_n^k x_n - \mathcal{J}_n^{k+1} x_n\|^2.
\end{aligned}$$

Since $\{\alpha_n\} \subset [a, 1]$, we get

$$\begin{aligned}
a\|\mathcal{J}_n^k x_n - \mathcal{J}_n^{k+1} x_n\|^2 &\leq \alpha_n \|\mathcal{J}_n^k x_n - \mathcal{J}_n^{k+1} x_n\|^2 \\
&\leq \|x_n - v\|^2 - \|y_n - v\|^2 \leq \|x_n - y_n\|(\|x_n - v\| + \|y_n - v\|).
\end{aligned}$$

From this and Step 4, we get the desired result.

Step 6. $\lim_{n\to\infty} \|x_n - W_n v_n\| = 0$.

Proof of Step 6. Observe that

$$\alpha_n W_n v_n = y_n - (1-\alpha_n) x_n.$$

So, we have

$$\begin{aligned}
a\|x_n - W_n v_n\| &\leq \alpha_n \|x_n - W_n v_n\| \\
&= \|y_n - (1-\alpha_n)x_n - \alpha_n x_n\| \leq \|y_n - x_n\|.
\end{aligned}$$

From this and Step 4, we obtain $\lim_{n\to\infty} \|x_n - W_n v_n\| = 0$.

Step 7. $\lim_{n\to\infty} \|\mathcal{P}_n^k u_n - \mathcal{P}_n^{k+1} u_n\| = 0$, $\forall k \in \{0, 1, \ldots, N-1\}$.

Proof of Step 7. Since $\{A_k : k = 1 \ldots N\}$ is α-inverse-strongly monotone, by the the choice of $\{\lambda_{k,n}\}$ for given $v \in \mathcal{F}$ and $k \in \{0, 1, \ldots, N-1\}$, we have

$$\begin{aligned}
&\|\mathcal{P}_n^{k+1} u_n - v\|^2 \\
&= \|P_C(I - \lambda_{k+1,n} A_{k+1})\mathcal{P}_n^k u_n - P_C(I - \lambda_{k+1,n} A_{k+1})v\|^2 \\
&\leq \|(I - \lambda_{k+1,n} A_{k+1})\mathcal{P}_n^k u_n - (I - \lambda_{k+1,n} A_{k+1})v\|^2 \\
&\leq \|\mathcal{P}_n^k u_n - v\|^2 + \lambda_{k+1,n}(\lambda_{k+1,n} - 2\alpha)\|A_{k+1}\mathcal{P}_n^k u_n - A_{k+1}v\|^2 \\
&\leq \|x_n - v\|^2 + c(d - 2\alpha)\|A_{k+1}\mathcal{P}_n^k u_n - A_{k+1}v\|^2.
\end{aligned}$$

From this, we have

$$\begin{aligned}
\|y_n - v\|^2 &\leq (1 - \alpha_n)\|x_n - v\|^2 + \alpha_n \|W_n \mathcal{P}_n^N u_n - v\|^2 \\
&\leq (1 - \alpha_n)\|x_n - v\|^2 + \alpha_n \|\mathcal{P}_n^{k+1} u_n - v\|^2 \\
&\leq (1 - \alpha_n)\|x_n - v\|^2 + \alpha_n(\|x_n - v\|^2 + c(d - 2\alpha)\|A_{k+1}\mathcal{P}_n^k u_n - A_{k+1}v\|^2) \\
&= \|x_n - v\|^2 + c(d - 2\alpha)\alpha_n \|A_{k+1}\mathcal{P}_n^k u_n - A_{k+1}v\|^2.
\end{aligned}$$

So,

$$\begin{aligned}
c(2\alpha - d)\alpha_n \|A_{k+1}\mathcal{P}_n^k u_n - A_{k+1}v\|^2 &\leq \|x_n - v\|^2 - \|y_n - v\|^2 \\
&\leq \|x_n - y_n\|(\|x_n - v\| + \|y_n - v\|).
\end{aligned}$$

Since $\alpha_n \subset [a, 1]$, we obtain

$$\|A_{k+1}\mathcal{P}_n^k u_n - A_{k+1}v\| \to 0 \;\; (n \to \infty). \tag{3.5}$$

From (2.1) and the fact that $I - \lambda_{k+1,n} A_{k+1}$ is nonexpansive, we have

$$\begin{aligned}
&\|\mathcal{P}_n^{k+1} u_n - v\|^2 = \|P_C(I - \lambda_{k+1,n} A_{k+1})\mathcal{P}_n^k u_n - P_C(I - \lambda_{k+1,n} A_{k+1})v\|^2 \\
&\leq \langle (\mathcal{P}_n^k u_n - \lambda_{k+1,n} A_{k+1}\mathcal{P}_n^k u_n) - (v - \lambda_{k+1,n} A_{k+1}v), \mathcal{P}_n^{k+1} u_n - v\rangle \\
&= \frac{1}{2}\{\|(\mathcal{P}_n^k u_n - \lambda_{k+1,n} A_{k+1}\mathcal{P}_n^k u_n) - (v - \lambda_{k+1,n} A_{k+1}v)\|^2 + \|\mathcal{P}_n^{k+1} u_n - v\|^2 \\
&\quad - \|(\mathcal{P}_n^k u_n - \lambda_{k+1,n} A_{k+1}\mathcal{P}_n^k u_n) - (v - \lambda_{k+1,n} A_{k+1}v) - (\mathcal{P}_n^{k+1} u_n - v)\|^2\} \\
&\leq \frac{1}{2}\{\|\mathcal{P}_n^k u_n - v\|^2 + \|\mathcal{P}_n^{k+1} u_n - v\|^2 \\
&\quad - \|\mathcal{P}_n^k u_n - \mathcal{P}_n^{k+1} u_n - \lambda_{k+1,n}(A_{k+1}\mathcal{P}_n^k u_n - A_{k+1}v)\|^2\} \\
&= \frac{1}{2}\{\|\mathcal{P}_n^k u_n - v\|^2 + \|\mathcal{P}_n^{k+1} u_n - v\|^2 - \|\mathcal{P}_n^k u_n - \mathcal{P}_n^{k+1} u_n\|^2 \\
&\quad + 2\lambda_{k+1,n}\langle \mathcal{P}_n^k u_n - \mathcal{P}_n^{k+1} u_n, A_{k+1}\mathcal{P}_n^k u_n - A_{k+1}v\rangle \\
&\quad - \lambda_{k+1,n}^2 \|A_{k+1}\mathcal{P}_n^k u_n - A_{k+1}v\|^2\}.
\end{aligned}$$

This implies that

$$\begin{aligned}\|\mathcal{P}_n^{k+1}u_n - v\|^2 &\le \|\mathcal{P}_n^k u_n - v\|^2 - \|\mathcal{P}_n^k u_n - \mathcal{P}_n^{k+1}u_n\|^2 \\ &+2\lambda_{k+1,n}\langle \mathcal{P}_n^k u_n - \mathcal{P}_n^{k+1}u_n, A_{k+1}\mathcal{P}_n^k u_n - A_{k+1}v\rangle \\ &-\lambda_{k+1,n}^2\|A_{k+1}\mathcal{P}_n^k u_n - A_{k+1}v\|^2 \\ &\le \|x_n - v\|^2 - \|\mathcal{P}_n^k u_n - \mathcal{P}_n^{k+1}u_n\|^2 \\ &+2\lambda_{k+1,n}\langle \mathcal{P}_n^k u_n - \mathcal{P}_n^{k+1}u_n, A_{k+1}\mathcal{P}_n^k u_n - A_{k+1}v\rangle.\end{aligned}$$

From this, we have

$$\begin{aligned}\|y_n - v\|^2 &\le (1-\alpha_n)\|x_n - v\|^2 + \alpha_n\|\mathcal{P}_n^{k+1}u_n - v\|^2 \\ &\le (1-\alpha_n)\|x_n - v\|^2 + \alpha_n\{\|x_n - v\|^2 - \|\mathcal{P}_n^k u_n - \mathcal{P}_n^{k+1}u_n\|^2 \\ &+2\lambda_{k+1,n}\langle \mathcal{P}_n^k u_n - \mathcal{P}_n^{k+1}u_n, A_{k+1}\mathcal{P}_n^k u_n - A_{k+1}v\rangle\} \\ &\le \|x_n - v\|^2 - \alpha_n\|\mathcal{P}_n^k u_n - \mathcal{P}_n^{k+1}u_n\|^2 \\ &+2\lambda_{k+1,n}\|\mathcal{P}_n^k u_n - \mathcal{P}_n^{k+1}u_n\|\|A_{k+1}\mathcal{P}_n^k u_n - A_{k+1}v\|,\end{aligned}$$

which implies,

$$\begin{aligned}a\|\mathcal{P}_n^k u_n - \mathcal{P}_n^{k+1}u_n\|^2 &\le \alpha_n\|\mathcal{P}_n^k u_n - \mathcal{P}_n^{k+1}u_n\|^2 \le \|x_n - v\|^2 - \|y_n - v\|^2 \\ &+2\lambda_{k+1,n}\|\mathcal{P}_n^k u_n - \mathcal{P}_n^{k+1}u_n\|\|A_{k+1}\mathcal{P}_n^k u_n - A_{k+1}v\|.\end{aligned}$$

Hence it follows from Step 4 and (3.5) that $\|\mathcal{P}_n^k u_n - \mathcal{P}_n^{k+1}u_n\| \to 0$.

Step 8. The weak ω-limit set of $\{x_n\}$, $\omega_w(x_n)$, is a subset of $\mathcal{F}$.

Proof of Step 8. Let $z_0 \in \omega_w(x_n)$ and let $\{x_{n_m}\}$ be a subsequence of $\{x_n\}$ weakly converging to z_0. From steps 5 and 7, we also obtain that

$$\mathcal{J}_{n_m}^k x_{n_m} \rightharpoonup z_0,$$

for all $k \in \{1, \dots, M\}$, and

$$\mathcal{P}_{n_m}^k u_{n_m} \rightharpoonup z_0,$$

for all $k \in \{1, \dots, N\}$. In particular, $u_{n_m} \rightharpoonup z_0$ and $v_{n_m} \rightharpoonup z_0$. We need to show that $z_0 \in \mathcal{F}$. First, we prove $z_0 \in \cap_{i=1}^N VI(C, A_i)$. For this purpose, let $k \in \{1, \dots, N\}$ and T_k be the maximal monotone mapping defined by

$$T_k x = \begin{cases} A_k z + N_C z, & z \in C; \\ \emptyset, & z \notin C. \end{cases}$$

Hence, for any given $(z, u) \in G(T_k)$, we have $u - A_k z \in N_C z$. Since $\mathcal{P}_n^k u_n \in C$, by the definition of N_C, we have

$$\langle z - \mathcal{P}_n^k u_n, u - A_k z\rangle \ge 0. \tag{3.6}$$

On the other hand, since $\mathcal{P}_n^k u_n = P_C(\mathcal{P}_n^{k-1} u_n - \lambda_{k,n} A_k \mathcal{P}_n^{k-1} u_n)$, we have

$$\langle z - \mathcal{P}_n^k u_n, \mathcal{P}_n^k u_n - (\mathcal{P}_n^{k-1} u_n - \lambda_{k,n} A_k \mathcal{P}_n^{k-1} u_n)\rangle \geq 0.$$

So,

$$\langle z - \mathcal{P}_n^k u_n, \frac{\mathcal{P}_n^k u_n - \mathcal{P}_n^{k-1} u_n}{\lambda_{k,n}} + A_k \mathcal{P}_n^{k-1} u_n\rangle \geq 0.$$

By (3.6) and the α-inverse monotonicity, we have

$$\begin{aligned}
\langle z - \mathcal{P}_{n_m}^k u_{n_m}, u\rangle &\geq \langle z - \mathcal{P}_{n_m}^k u_{n_m}, A_k z\rangle \\
&\geq \langle z - \mathcal{P}_{n_m}^k u_{n_m}, A_k z\rangle \\
&-\langle z - \mathcal{P}_{n_m}^k u_{n_m}, \frac{\mathcal{P}_{n_m}^k u_{n_m} - \mathcal{P}_{n_m}^{k-1} u_{n_m}}{\lambda_{k,n_m}} + A_k \mathcal{P}_{n_m}^{k-1} u_{n_m}\rangle \\
&= \langle z - \mathcal{P}_{n_m}^k u_{n_m}, A_k z - A_k \mathcal{P}_{n_m}^k u_{n_m}\rangle \\
&+\langle z - \mathcal{P}_{n_m}^k u_{n_m}, A_k \mathcal{P}_{n_m}^k u_{n_m} - A_k \mathcal{P}_{n_m}^{k-1} u_{n_m}\rangle \\
&-\langle z - \mathcal{P}_{n_m}^k u_{n_m}, \frac{\mathcal{P}_{n_m}^k u_{n_m} - \mathcal{P}_{n_m}^{k-1} u_{n_m}}{\lambda_{k,n_m}}\rangle \\
&\geq \langle z - \mathcal{P}_{n_m}^k u_{n_m}, A_k \mathcal{P}_{n_m}^k u_{n_m} - A_k \mathcal{P}_{n_m}^{k-1} u_{n_m}\rangle \\
&-\langle z - \mathcal{P}_{n_m}^k u_{n_m}, \frac{\mathcal{P}_{n_m}^k u_{n_m} - \mathcal{P}_{n_m}^{k-1} u_{n_m}}{\lambda_{k,n_m}}\rangle.
\end{aligned}$$

Since $\|\mathcal{P}_n^k \mathcal{J}_n^M x_n - \mathcal{P}_n^{k-1} \mathcal{J}_n^M x_n\| \to 0$, $\mathcal{P}_{n_m}^k u_{n_m} \rightharpoonup z_0$ and $\{A_k : k = 1, \ldots, N\}$ are Lipschitz continuous, we have

$$\lim_{m\to\infty} \langle z - \mathcal{P}_{n_m}^k u_{n_m}, u\rangle = \langle z - z_0, u\rangle \geq 0.$$

Again, since T_k is maximal monotone, then $0 \in T_k z_0$. This shows that $z_0 \in VI(C, A_k)$. From this, it follows that

$$z_0 \in \cap_{i=1}^N VI(C, A_i). \tag{3.7}$$

Now, we note that by (A2) for given $y \in C$ and $k \in \{0, 1, \ldots, M-1\}$, we have

$$\frac{1}{r_{k+1,n}} \langle y - \mathcal{J}_n^{k+1} x_n, \mathcal{J}_n^{k+1} x_n - \mathcal{J}_n^k x_n\rangle \geq F_{k+1}(y, \mathcal{J}_n^{k+1} x_n).$$

Thus,

$$\langle y - \mathcal{J}_{n_m}^{k+1} x_{n_m}, \frac{\mathcal{J}_{n_m}^{k+1} x_{n_m} - \mathcal{J}_{n_m}^k x_{n_m}}{r_{k+1,n_m}}\rangle \geq F_{k+1}(y, \mathcal{J}_{n_m}^{k+1} x_{n_m}). \tag{3.8}$$

By condition (A4), $F_i(y,.)$, $\forall i$, is lower semicontinuous and convex, and thus weakly semicontinuous. Step 5 and the condition $\liminf_n r_{j,n} > 0$ imply that

$$\frac{\mathcal{J}_{n_m}^{k+1}x_{n_m} - \mathcal{J}_{n_m}^{k}x_{n_m}}{r_{k+1,n_m}} \to 0,$$

in norm. Therefore, letting $m \to \infty$ in (3.8) yields

$$F_{k+1}(y, z_0) \le \lim_m F_{k+1}(y, \mathcal{J}_{n_m}^{k+1}x_{n_m}) \le 0,$$

for all $y \in C$ and $k \in \{0, 1, \ldots, M-1\}$. Replacing y with $y_t := ty + (1-t)z_0$, $t \in (0,1)$, and using (A1) and (A4), we obtain

$$0 = F_{k+1}(y_t, y_t) \le tF_{k+1}(y_t, y) + (1-t)F_{k+1}(y_t, z_0) \le tF_{k+1}(y_t, y).$$

Hence, $F_{k+1}(ty + (1-t)z_0, y) \ge 0$, for all $t \in (0,1)$ and $y \in C$. Letting $t \to 0^+$ and using (A3), we conclude $F_{k+1}(z_0, y) \ge 0$, for all $y \in C$ and $k \in \{0, \ldots, M-1\}$. Therefore,

$$z_0 \in \bigcap_{k=1}^{M} EP(F_k) = EP(\mathcal{G}). \tag{3.9}$$

We next show $z_0 \in \cap_{i=1}^{\infty} Fix(S_i)$. By Lemma 2.5, we have, for every $z \in C$,

$$W_{n_m} z \to Wz, \tag{3.10}$$

and $Fix(W) = \cap_{i=1}^{\infty} Fix(T_i)$. Assume that $z_0 \notin Fix(W)$. Then, $z_0 \ne Wz_0$. From the Opial' s property of Hilbert space, (3.7), (3.9), (3.10) and Step 6, we have

$$\begin{aligned}
\liminf_m \|x_{n_m} - z_0\| &< \liminf_m \|x_{n_m} - Wz_0\| \\
&\le \liminf_m (\|x_{n_m} - W_{n_m} v_{n_m}\| \\
&\quad + \|W_{n_m}\mathcal{P}_{n_m}^N \mathcal{J}_{n_m}^M x_{n_m} - W_{n_m}\mathcal{P}_{n_m}^N \mathcal{J}_{n_m}^M z_0\| + \|W_{n_m} z_0 - Wz_0\|) \\
&\le \liminf_m \|x_{n_m} - z_0\|.
\end{aligned}$$

This is a contradiction. Therefore, z_0 must belong to $Fix(W) = \cap_{i=1}^{\infty} Fix(S_i)$.

Step 9. The sequences $\{x_n\}$ and $\{y_n\}$ converge strongly to $P_{\mathcal{F}}(x)$.

Proof of Step 9. Let $z_0 \in \omega_w(x_n)$ and let $\{x_{n_m}\}$ be a subsequence of $\{x_n\}$ weakly converging to z_0. From Step 8 and (3.4), we have

$$\|x - P_{\mathcal{F}}(x)\| \leq \|x - z_0\| \leq \liminf_{m\to\infty} \|x - x_{n_m}\|$$
$$\leq \limsup_{m\to\infty} \|x - x_{n_m}\| \leq \|x - P_{\mathcal{F}}(x)\|.$$

Hence,

$$\lim_{m\to\infty} \|x - x_{n_m}\| = \|x - z_0\| = \|x - P_{\mathcal{F}}(x)\|.$$

Since $z_0 \in \mathcal{F}$ and H is a Hilbert space, we obtain

$$x_{n_m} \longrightarrow z_0 = P_{\mathcal{F}}(x).$$

Since $z_0 \in \omega_w(x_n)$ was arbitrary, we get $x_n \longrightarrow P_{\mathcal{F}}(x)$. $\square$

Corollary 3.2. *Let C, φ, $\mathcal{G}$, $\{\alpha_n\}$ and $\{r_{n,j}\}_{j=1}^M$ be as in Theorem 3.1. Let $\psi = \{T_j : j = 1 \ldots N\}$ be a finite family of strictly pseudocontractive mappings with $0 \leq \kappa < 1$ from C into C such that $\mathcal{F} := Fix(\varphi) \cap Fix(\psi) \cap EP(\mathcal{G}) \neq \emptyset$ and $\{\lambda_{k,n}\}_{k=1}^N$ be sequences in $[c,d] \subset (0, 1-\kappa)$.*

If $\{x_n\}$ is the sequence generated by $x_1 = x \in H$ and $\forall n \geq 1$,

$$\begin{cases} u_n = J^{F_M}_{r_{M,n}} \ldots J^{F_2}_{r_{2,n}} J^{F_1}_{r_{1,n}} x_n, \\ v_n = ((1-\lambda_{N,n})I + \lambda_{N,n}T_N) \ldots ((1-\lambda_{2,n})I + \lambda_{2,n}T_2)((1-\lambda_{1,n})I \\ \qquad\qquad\qquad +\lambda_{1,n}T_1)u_n, \\ y_n = (1-\alpha_n)x_n + \alpha_n W_n v_n, \\ C_n = \{z \in H : \|y_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in H : \langle x_n - z, x - x_n\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n}(x), \end{cases}$$

then the sequences $\{x_n\}$ and $\{y_n\}$ converge strongly to $P_{\mathcal{F}}(x)$.

Proof. Put $A_j = I - T_j$ for every $j \in \{1, \ldots, N\}$. Then A_j is $\frac{1-k}{2}$-inverse-strongly monotone. We have that $Fix(T_j)$ is the solution set of $VI(C, A_j)$; i.e., $Fix(T_j) = VI(C, A_j)$. Therefore, $Fix(\psi) = \cap_{k=1}^N VI(C, A_k)$ and it suffices to apply Theorem 3.1. $\square$

The following is a weak convergence theorem.

Theorem 3.3. *Let C, φ, $\mathcal{G}$, $\mathcal{A}$, $\mathcal{F}$, $\{\lambda_{k,n}\}_{k=1}^N$ and $\{r_{n,j}\}_{j=1}^M$ be as in Theorem 3.1. Let $\{\alpha_n\}$ be a sequence in $[a,b]$ for some $a, b \in (0,1)$.*

If $\{x_n\}$ is the sequence generated by $x_1 = x \in H$ and $\forall n \geq 1$,

$$\begin{cases} u_n = J^{F_M}_{r_{M,n}} \dots J^{F_2}_{r_{2,n}} J^{F_1}_{r_{1,n}} x_n, \\ v_n = P_C(I - \lambda_{N,n} A_N) \dots P_C(I - \lambda_{2,n} A_2) P_C(I - \lambda_{1,n} A_1) u_n, \\ x_{n+1} = (1 - \alpha_n) x_n + \alpha_n W_n v_n, \end{cases}$$

then the sequence $\{x_n\}$ *converges weakly to* $z_0 \in \mathcal{F}$*, where* $z_0 = \lim_{n\to\infty} P_{\mathcal{F}}(x_n)$.

Proof. We apply the notations used in the proof of Theorem 3.1. Let $v \in \mathcal{F}$. Then,

$$\|x_{n+1} - v\| \leq \alpha_n \|x_n - v\| + (1-\alpha_n)\|W_n \mathcal{P}_n^N \mathcal{J}_n^M x_n - v\| \leq \|x_n - v\|. \tag{3.11}$$

So, there exists $c \in \mathbb{R}$ such that

$$c = \lim_{n\to\infty} \|x_n - v\|. \tag{3.12}$$

Hence, $\{x_n\}$ is bounded. Next, for $v \in \mathcal{F}$, as in Step 5 of Theorem 3.1, we get

$$\|\mathcal{J}_n^{k+1} x_n - v\|^2 \leq \|x_n - v\|^2 - \|\mathcal{J}_n^k x_n - \mathcal{J}_n^{k+1} x_n\|^2,$$

for all $k \in \{0, 1, \dots, M-1\}$. Therefore, we have

$$\begin{aligned} \|x_{n+1} - v\|^2 &\leq \alpha_n \|x_n - v\|^2 + (1-\alpha_n)\|W_n \mathcal{P}_n^N \mathcal{J}_n^M x_n - v\|^2 \\ &\leq \alpha_n \|x_n - v\|^2 + (1-\alpha_n)\|\mathcal{J}_n^{k+1} x_n - v\|^2 \\ &\leq \alpha_n \|x_n - v\|^2 + (1-\alpha_n)(\|x_n - v\|^2 - \|\mathcal{J}_n^k x_n - \mathcal{J}_n^{k+1} x_n\|^2) \\ &= \|x_n - v\|^2 - (1-\alpha_n)\|\mathcal{J}_n^k x_n - \mathcal{J}_n^{k+1} x_n\|^2. \end{aligned}$$

So, we obtain

$$(1-b)\|\mathcal{J}_n^k x_n - \mathcal{J}_n^{k+1} x_n\| \leq \|x_n - v\|^2 - \|x_{n+1} - v\|^2.$$

From (3.12), we get

$$\lim_{n\to\infty} \|\mathcal{J}_n^k x_n - \mathcal{J}_n^{k+1} x_n\| = 0, \tag{3.13}$$

for all $k \in \{0, 1, \dots, M-1\}$. Since

$$\|W_n \mathcal{P}_n^N \mathcal{J}_n^M x_n - v\| \leq \|\mathcal{J}_n^M x_n - v\| \leq \|x_n - v\|,$$

from (3.12), we have

$$\limsup_{n\to\infty} \|W_n \mathcal{P}_n^N \mathcal{J}_n^M x_n - v\| \leq c.$$

Moreover, we have

$$\lim_{n\to\infty} \|\alpha_n(x_n - v) + (1-\alpha_n)(W_n \mathcal{P}_n^N \mathcal{J}_n^M x_n - v)\| = \lim_{n\to\infty} \|x_{n+1} - v\| = c.$$

By Lemma 2.2, we obtain

$$\lim_{n\to\infty} \|W_n \mathcal{P}_n^N \mathcal{J}_n^M x_n - x_n\| = 0. \tag{3.14}$$

We can show that

$$\lim_{n\to\infty} \|\mathcal{P}_n^k u_n - \mathcal{P}_n^{k+1} u_n\| = 0, \ \forall k \in \{0, 1, \ldots, N-1\}. \tag{3.15}$$

Indeed, for $k \in \{0, 1, \ldots, N-1\}$, like Step 7 of Theorem 3.1, we get

$$\|\mathcal{P}_n^{k+1} u_n - v\|^2 \leq \|x_n - v\|^2 + c(d - 2\alpha)\|A_{k+1}\mathcal{P}_n^k u_n - A_{k+1}v\|^2.$$

From this, we have

$$\begin{aligned}
\|x_{n+1} - v\|^2 &\leq (1-\alpha_n)\|x_n - v\|^2 + \alpha_n\|v_n - v\|^2 \\
&\leq (1-\alpha_n)\|x_n - v\|^2 + \alpha_n\|\mathcal{P}_n^{k+1} u_n - v\|^2 \\
&\leq (1-\alpha_n)\|x_n - v\|^2 + \alpha_n\{\|x_n - v\|^2 + c(d-2\alpha)\|A_{k+1}\mathcal{P}_n^k u_n - A_{k+1}v\|^2\} \\
&= \|x_n - v\|^2 + c(d - 2\alpha)\alpha_n\|A_{k+1}\mathcal{P}_n^k u_n - A_{k+1}v\|^2.
\end{aligned}$$

So, from (3.12), we have

$$\begin{aligned}
&c(2\alpha - d)\alpha_n\|A_{k+1}\mathcal{P}_n^k u_n - A_{k+1}v\|^2 \\
&\leq \|x_n - v\|^2 - \|x_{n+1} - v\|^2 \to 0.
\end{aligned}$$

Since $0 < a \leq \alpha_n$, we obtain

$$\|A_{k+1}\mathcal{P}_n^k u_n - A_{k+1}v\| \to 0 \ \ (n \to \infty). \tag{3.16}$$

Again, like Step 7 of Theorem 3.1, we have

$$\begin{aligned}
\|\mathcal{P}_n^{k+1} u_n - v\|^2 &\leq \|x_n - v\|^2 - \|\mathcal{P}_n^k u_n - \mathcal{P}_n^{k+1} u_n\|^2 \\
&+2\lambda_{k+1,n}\langle \mathcal{P}_n^k u_n - \mathcal{P}_n^{k+1} u_n, A_{k+1}\mathcal{P}_n^k u_n - A_{k+1}v\rangle.
\end{aligned}$$

Then,

$$\begin{aligned}
\|x_{n+1} - v\|^2 &\leq (1-\alpha_n)\|x_n - v\|^2 + \alpha_n\|\mathcal{P}_n^{k+1} u_n - v\|^2 \\
&\leq (1-\alpha_n)\|x_n - v\|^2 + \alpha_n\{\|x_n - v\|^2 - \|\mathcal{P}_n^k u_n - \mathcal{P}_n^{k+1} u_n\|^2 \\
&+2\lambda_{k+1,n}\langle \mathcal{P}_n^k u_n - \mathcal{P}_n^{k+1} u_n, A_{k+1}\mathcal{P}_n^k u_n - A_{k+1}v\rangle\} \\
&\leq \|x_n - v\|^2 - a\|\mathcal{P}_n^k u_n - \mathcal{P}_n^{k+1} u_n\|^2 \\
&+2\lambda_{k+1,n}\|\mathcal{P}_n^k u_n - \mathcal{P}_n^{k+1} u_n\|\|A_{k+1}\mathcal{P}_n^k u_n - A_{k+1}v\|,
\end{aligned}$$

which implies

$$\begin{aligned}
a\|\mathcal{P}_n^k u_n - \mathcal{P}_n^{k+1} u_n\|^2 &\leq \|x_n - v\|^2 - \|x_{n+1} - v\|^2 \\
&+2\lambda_{k+1,n}\|\mathcal{P}_n^k u_n - \mathcal{P}_n^{k+1} u_n\|\|A_{k+1}\mathcal{P}_n^k u_n - A_{k+1}v\|.
\end{aligned}$$

Hence, it follows from (3.12) and (3.16) that $\|\mathcal{P}_n^k u_n - \mathcal{P}_n^{k+1} u_n\| \to 0$. This completes the proof of (3.15).

Now, applying (3.13), (3.14) and (3.15), as in Step 8 of Theorem 3.1, we can show that the weak ω-limit set of $\{x_n\}$, $\omega_w(x_n)$, is a subset of $\mathcal{F}$. Now, (3.12) and the Opial's property of Hilbert space imply that $\omega_w(x_n)$ is singleton. Therefore, $x_n \rightharpoonup z_0$ for some $z_0 \in \mathcal{F}$.

Let $z_n = P_{\mathcal{F}}(x_n)$. Since $z_0 \in \mathcal{F}$, we have

$$\langle x_n - z_n, z_n - z_0 \rangle \geq 0.$$

Using (3.11) and Lemma 2.3, we have that $\{z_n\}$ converges strongly to some $y_0 \in \mathcal{F}$. Since $x_n \rightharpoonup z_0$, we have

$$\langle z_0 - y_0, y_0 - z_0 \rangle \geq 0.$$

Therefore, we obtain $z_0 = y_0 = \lim_{n\to\infty} P_{\mathcal{F}}(x_n)$. □

Corollary 3.4. *Let C, φ, $\mathcal{G}$, $\{\alpha_n\}$ and $\{r_{n,j}\}_{j=1}^{M}$ be as in Theorem 3.3. Let $\psi = \{T_j : j = 1 \dots N\}$ be a finite family of strictly pseudocontractive mappings with $0 \leq \kappa < 1$ from C into C such that $\mathcal{F} := Fix(\varphi) \cap Fix(\psi) \cap EP(\mathcal{G}) \neq \emptyset$ and $\{\lambda_{k,n}\}_{k=1}^{N}$ be sequences in $[c,d] \subset (0, 1-\kappa)$.*

If $\{x_n\}$ is the sequence generated by $x_1 = x \in H$ and $\forall n \geq 1$,

$$\begin{cases} u_n = J_{r_{M,n}}^{F_M} \dots J_{r_{2,n}}^{F_2} J_{r_{1,n}}^{F_1} x_n, \\ v_n = ((1-\lambda_{N,n})I + \lambda_{N,n}T_N) \dots ((1-\lambda_{2,n})I + \lambda_{2,n}T_2)((1-\lambda_{1,n})I \\ \qquad\qquad +\lambda_{1,n}T_1)u_n, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n W_n v_n, \end{cases}$$

then the sequence $\{x_n\}$ converges weakly to $z_0 \in \mathcal{F}$, where $z_0 = \lim_{n\to\infty} P_{\mathcal{F}}(x_n)$.

Remark 3.5. We may put

$$v_n = P_C(I-\lambda_{N,n}(I-T_N)) \dots P_C(I-\lambda_{2,n}(I-T_2))P_C(I-\lambda_{1,n}(I-T_1))u_n,$$

in the schemes of corollaries 3.2 and 3.4, and obtain schemes for families of non-self strictly pseudocontractive mappings.

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