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SUBORDINATION AND SUPERORDINATION RESULTS INVOLVING CERTAIN CONVOLUTION OPERATORS

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Abstract. We introduce a new convolution operator $\mathcal{L}_a^\lambda(b,c;\beta)$. Several subordination and superordination results involving this operator are proved.

1. **Introduction and Preliminaries**

Let $\mathcal{H}(\mathbb{U})$ be the linear space of all analytic functions in the open unit disk,

$$\mathbb{U} := \{z : \ z \in \mathbb{C} \quad \text{and} \quad |z| < 1\},$$

and

$$\mathcal{A} := \{f \in \mathcal{H}(\mathbb{U}) : \ f(0) = f'(0) - 1 = 0\}.$$

For a positive integer number n and $a \in \mathbb{C}$, let

$$\mathcal{H}[a,n] := \{\mathfrak{f} \in \mathcal{H}(\mathbb{U}) : \ \mathfrak{f}(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots\}.$$

Furthermore, denote Q to be the set of all analytic and univalent functions on the set $\overline{\mathbb{U}} \setminus E(f)$, where,

$$E(f) := \{\zeta \in \partial\mathbb{U} : \ \lim_{z\to\zeta} f(z) = \infty\},$$

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and such that $f'(\zeta) \neq 0$ for $\zeta \in \partial\mathbb{U} \setminus E(f)$. The subclass of Q for which $f(0) = a$ is denoted by $Q(a)$.

For two functions f and g, analytic in $\mathbb{U}$, we say that the function f is subordinate to g in $\mathbb{U}$, and write

$$f(z) \prec g(z) \qquad (z \in \mathbb{U}),$$

if there exists a Schwarz function ω, which is analytic in $\mathbb{U}$ with

$$\omega(0) = 0 \quad \text{and} \quad |\omega(z)| < 1 \quad (z \in \mathbb{U})$$

such that

$$f(z) = g\big(\omega(z)\big) \qquad (z \in \mathbb{U}).$$

Indeed, it is known that

$$f(z) \prec g(z) \quad (z \in \mathbb{U}) \Longrightarrow f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

Furthermore, if the function g is univalent in $\mathbb{U}$, then we have the following equivalence,

$$f(z) \prec g(z) \quad (z \in \mathbb{U}) \Longleftrightarrow f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

For two functions $f_j(z)$ $(j = 1, 2)$, given by

$$f_j(z) = z + \sum_{k=2}^{\infty} a_{k,j} z^k \qquad (j = 1, 2),$$

we define the Hadamard product (or convolution) of $f_1(z)$ and $f_2(z)$ by

$$(f_1 * f_2)(z) := z + \sum_{k=2}^{\infty} a_{k,1} a_{k,2} z^k =: (f_2 * f_1)(z) \qquad (z \in \mathbb{U}).$$

In terms of the Pochhammer symbol (or the shifted factorial), define $(\kappa)_n$ by

$$(\kappa)_0 = 1,$$

and

$$(\kappa)_n = \kappa(\kappa+1)(\kappa+2)\cdots(\kappa+n-1) \qquad (n \in \mathbb{N} := \{1, 2, 3, \ldots\}),$$

and then define a function $\phi_a^\lambda(b, c; z)$ by

$$\phi_a^\lambda(b, c; z) := 1 + \sum_{n=1}^{\infty} \left(\frac{a}{a+n}\right)^\lambda \frac{(b)_n}{(c)_n} z^n \qquad (z \in \mathbb{U}), \tag{1.1}$$

where

$$b \in \mathbb{R};\ c \in \mathbb{R} \setminus \mathbb{Z}_0^-;\ a \in \mathbb{C} \setminus \mathbb{Z}_0^-\ (\mathbb{Z}_0^- := \{0, -1, -2, ...\});\ \lambda \geq 0.$$

Corresponding to the function $\phi_a^{\lambda}(b,c;z)$, given by (1.1), we introduce the following convolution operator,
(1.2)

$$\mathcal{L}_a^{\lambda}(b,c;\beta)f(z) := \phi_a^{\lambda}(b,c;z) * \left(\frac{f(z)}{z}\right)^{\beta} \qquad (f \in \mathcal{A};\ \beta \in \mathbb{C}\setminus\{0\};\ z \in \mathbb{U}).$$

It is easy to see that

$$z\left(\phi_a^{\lambda+1}(b,c;z)\right)' = a\phi_a^{\lambda}(b,c;z) - a\phi_a^{\lambda+1}(b,c;z), \tag{1.3}$$

and

$$z\left(\phi_a^{\lambda}(b,c;z)\right)' = b\phi_a^{\lambda}(b+1,c;z) - b\phi_a^{\lambda}(b,c;z). \tag{1.4}$$

Hence, from (1.3) and (1.4), we easily obtain:

$$z\left(\mathcal{L}_a^{\lambda+1}(b,c;\beta)f\right)'(z) = a\mathcal{L}_a^{\lambda}(b,c,\beta)f(z) - a\mathcal{L}_a^{\lambda+1}(b,c,\beta)f(z), \tag{1.5}$$

and

$$z\left(\mathcal{L}_a^{\lambda}(b,c;\beta)f\right)'(z) = b\mathcal{L}_a^{\lambda}(b+1,c;\beta)f(z) - b\mathcal{L}_a^{\lambda}(b,c;\beta)f(z). \tag{1.6}$$

The operator $\mathcal{L}_a^{\lambda}(b,c,\beta)$ includes, as its special cases, Komatu integral operator (see [3, 4, 10]), some fractional calculus operators (see [2, 11, 12]) and Carlson-Shaffer operator (see [1]).

The following definition and lemmas play key roles in the proofs of our main results.

Definition 1.1. A function $P(z,t)$ $(z \in \mathbb{U};\ t \geq 0)$ is said to be a subordination chain if $P(.,t)$ is analytic and univalent in $\mathbb{U}$, for all $t \geq 0$, $P(z,0)$ is continuously differentiable on $[0,\infty)$, for all $z \in \mathbb{U}$, and $P(z,t_1) \prec P(z,t_2)$, for all $0 \leq t_1 \leq t_2$.

Lemma 1.2. (See [9]) *The function,*

$$P(z,t):\ \mathbb{U}\times[0,\infty) \to \mathbb{C}$$

of the form

$$P(z,t) = a_1(t)z + a_2(t)z^2 + \cdots \qquad (a_1(t) \neq 0;\ t \geq 0)\,,$$

and $\lim_{t\to\infty} |a_1(t)| = \infty$, *is a subordination chain if and only if*

$$\Re\left(\frac{z\,\partial P/\partial z}{\partial P/\partial t}\right) > 0 \qquad (z \in \mathbb{U};\ t \geq 0).$$

Lemma 1.3. (See [5]) *Suppose that the function* $H : \mathbb{C}^2 \to \mathbb{C}$ *for all real* s *and for all*

$$t \leq -\frac{n(1+s^2)}{2} \qquad (n \in \mathbb{N})$$

satisfies the condition $\Re(H(is,t)) \leq 0$. *If the function*

$$p(z) = 1 + p_n z^n + p_{n+1} z^{n+1} + \cdots$$

is analytic in $\mathbb{U}$ *and*

$$\Re\left(H(p(z), zp'(z))\right) > 0 \qquad (z \in \mathbb{U}),$$

then

$$\Re(p(z)) > 0 \qquad (z \in \mathbb{U}).$$

Lemma 1.4. (See [6]) *Let* $\kappa, \gamma \in \mathbb{C}$ *with* $\kappa \neq 0$ *and let* $h \in \mathcal{H}(\mathbb{U})$ *with* $h(0) = c$. *If*

$$\Re(\kappa h(z) + \gamma) > 0 \qquad (z \in \mathbb{U}),$$

then the solution of the following differential equation,

$$q(z) + \frac{zq'(z)}{\kappa q(z) + \gamma} = h(z) \quad (z \in \mathbb{U};\ q(0) = c)$$

is analytic in $\mathbb{U}$ *and satisfies the inequality given by*

$$\Re(\kappa q(z) + \gamma) > 0 \qquad (z \in \mathbb{U}).$$

Lemma 1.5. (See [7]) *Let* $p \in Q(a)$ *and*

$$q(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots \qquad (q \neq a;\ n \in \mathbb{N})$$

be analytic in $\mathbb{U}$. *If* q *is not subordinate to* p, *then there exist two points*

$$z_0 = r_0 e^{i\theta} \in \mathbb{U} \quad \text{and} \quad \xi_0 \in \partial\mathbb{U} \backslash E(f)$$

such that

$$q(\mathbb{U}_{r_0}) \subset p(\mathbb{U}), \quad q(z_0) = p(\xi_0) \quad \text{and} \quad z_0 q'(z_0) = m\xi_0 p'(\xi_0) \quad (m \geq n).$$

Lemma 1.6. (See [8]) *Let* $q \in \mathcal{H}[a,1]$ *and* $\phi : \mathbb{C}^2 \to \mathbb{C}$. *Also, set*

$$\phi\left(q(z), zq'(z)\right) \equiv h(z) \qquad (z \in \mathbb{U}).$$

If $P(z,t) := \phi(q(z), tzq'(z))$ *is a subordination chain and* $p \in \mathcal{H}[a,1] \cap Q(a)$, *then*

$$h(z) \prec \phi\left(p(z), zp'(z)\right) \qquad (z \in \mathbb{U})$$

implies that

$$q(z) \prec p(z) \qquad (z \in \mathbb{U}).$$

Furthermore, if $\phi(q(z), zq'(z)) = h(z)$ *has a univalent solution* $q \in Q(a)$, *then* q *is the best subordination.*

2. Main results

We begin by presenting our first subordination property given by Theorem 2.1 below. For convenience, let

$$\mathcal{A}_0 := \{f \in \mathcal{A} : \ \mathcal{L}_a^{\lambda}(b,c;\beta)f(z) \neq 0 \quad (z \in \mathbb{U})\}.$$

Theorem 2.1. *Let $f,\ g \in \mathcal{A}_0,\ \lambda \geq 0,\ a \in \mathbb{C},\ \Re(a) > 0$ and $\beta \in \mathbb{C}\setminus\{0\}$. If*

$$\Re\left(1 + \frac{z\psi''(z)}{\psi'(z)}\right) > -\delta \qquad \left(z \in \mathbb{U};\ \psi(z) := \mathcal{L}_a^{\lambda}(b,c;\beta)g(z)\right), \tag{2.1}$$

where,

$$\delta := \frac{1+|a|^2 - |1-a^2|}{4\Re(a)}, \tag{2.2}$$

then the following subordination relationship,

$$\mathcal{L}_a^{\lambda}(b,c;\beta)f(z) \prec \mathcal{L}_a^{\lambda}(b,c;\beta)g(z) \qquad (z \in \mathbb{U}) \tag{2.3}$$

implies that

$$\mathcal{L}_a^{\lambda+1}(b,c;\beta)f(z) \prec \mathcal{L}_a^{\lambda+1}(b,c;\beta)g(z) \qquad (z \in \mathbb{U}).$$

Moreover, the function $\mathcal{L}_a^{\lambda+1}(b,c;\beta)g$ is the best dominant.

Proof. Let $F,\ G$ and q be defined by

$$F := \mathcal{L}_a^{\lambda+1}(b,c;\beta)f,\ G := \mathcal{L}_a^{\lambda+1}(b,c;\beta)g \ \text{ and } \ q := 1 + \frac{zG''(z)}{G'}. \tag{2.4}$$

We can assume, without loss of generality, that G is analytic and univalent on $\overline{\mathbb{U}}$ and that $G'(\zeta) \neq 0$ $(|\zeta| = 1)$. If not, then we can replace F and G by $F(\rho z)$ and $G(\rho z)$ with $0 < \rho < 1$. These new functions have the desired properties on $\overline{\mathbb{U}}$, and we can use them in the proof of our result. Therefore, our result would follow by letting $\rho \to 1$.

We first show that

$$\Re(q(z)) > 0 \qquad (z \in \mathbb{U}).$$

In view of (1.5) and the definitions of G and ψ, we know that

$$\psi(z) = G(z) + \frac{1}{a}zG'(z). \tag{2.5}$$

Differentiating both sides of (2.5), we get

$$\psi'(z) = \left(1 + \frac{1}{a}\right)G'(z) + \frac{1}{a}zG''(z). \tag{2.6}$$

After a simple manipulation, we obtain the following relation,

$$1+\frac{z\psi''(z)}{\psi'(z)}=q(z)+\frac{zq'(z)}{a+q(z)}:=\mathfrak{h}(z)\qquad(z\in\mathbb{U}).\tag{2.7}$$

From (2.1), we deduce that

$$\Re\left(\mathfrak{h}(z)+a\right)>0\qquad(z\in\mathbb{U}).\tag{2.8}$$

Furthermore, by Lemma 1.4, we conclude that the differential equation (2.7) has a solution $q\in\mathcal{H}(\mathbb{U})$ with $\mathfrak{h}(0)=q(0)=1$. Let

$$H(u,v)=u+\frac{v}{u+a}+\delta,$$

where δ is given by (2.2). From (2.7) and (2.8), we obtain

$$\Re\left(H(q(z),zq'(z))\right)>0\qquad(z\in\mathbb{U}).$$

To verify the condition $\Re(H(is,t))\leq 0$, we proceed as follows:

$$\Re(H(is,t))=\Re\left(is+\frac{t}{is+a}+\delta\right)=\frac{t\Re(a)}{|a+is|^2}+\delta\leq-\frac{k}{2|a+is|^2},$$

where,

$$k=|\Re(a)-2\delta|s^2-4\delta\Im(a)s-2\delta|a|^2+\Re(a).$$

But in view of the value of δ given by (2.2), we know that k is a perfect square, which implies that

$$\Re(H(is,t))\leq 0\qquad\left(s\in\mathbb{R};\ t\leq-\frac{1+s^2}{2}\right).$$

Now, by Lemma 1.3, we conclude that

$$\Re(q(z))>0\qquad(z\in\mathbb{U}).$$

By the definition of q, we know that G is convex. To prove $F\prec G$, we let the function $P(z,t)$ be defined by

$$P(z,t):=G(z)+\left(\frac{1+t}{a}\right)zG'(z)\qquad(z\in\mathbb{U};\ 0\leq t<\infty),$$

Since G is convex and $\Re(a)>0$, we have

$$\frac{\partial P(z,t)}{\partial z}|_{z=0}=G'(0)\left(1+\frac{1+t}{a}\right)\neq 0\qquad(z\in\mathbb{U};\ 0\leq t<\infty)$$

and

$$\Re\left(\frac{z\,\partial P(z,t)/\partial z}{\partial P(z,t)/\partial t}\right)=\Re(a+(1+t)q(z))>0\qquad(z\in\mathbb{U}).$$

Therefore, by Lemma 1.2, we obtain that $P(z,t)$ is a subordination chain. Now, from the definition of subordination chain, it follows that

$$\psi(z) = G(z) + \frac{1}{a} zG'(z) = P(z,0),$$

and

$$P(z,0) \prec P(z,t) \qquad (z \in \mathbb{U};\ 0 \leq t < \infty),$$

which implies:

$$P(\zeta,t) \notin P(\mathbb{U},0) = \psi(\mathbb{U}) \qquad (\zeta \in \partial\mathbb{U};\ 0 \leq t < \infty). \tag{2.9}$$

If F is not subordinate to G, then by Lemma 1.5, there exist two points $z_0 \in \mathbb{U}$ and $\zeta_0 \in \partial\mathbb{U}$ such that

$$F(z_0) = G(\zeta_0) \quad \text{and} \quad z_0 F(z_0) = (1+t)\zeta_0 G'(\zeta_0) \quad (0 \leq t < \infty). \tag{2.10}$$

Hence, by virtue of (1.5) and (2.10), we have

$$\begin{aligned} P(\zeta_0,t) &= G(\zeta_0) + \frac{1+t}{a}\zeta_0 G'(\zeta_0) \\ &= F(z_0) + \frac{1}{a} z_0 F'(z_0) \\ &= \mathcal{L}_a^{\lambda}(b,c;\beta)f(z_0) \in \psi(\mathbb{U}). \end{aligned}$$

But, this contradicts (2.9), and thus we deduce that $F \prec G$. Considering $F = G$, we see that the function G is the best dominant. The proof of Theorem 2.1 is thus complete.

Suppose that $\gamma \in \mathbb{C}$. By setting $a = \gamma + \beta,\ \lambda = 0,\ b = c = 1$ in Theorem 2.1, we get the following result.

Corollary 2.2. *Let $f,\ g \in \mathcal{A}_0$, and $\beta \in \mathbb{C} \setminus \{0\}$. If*

$$\Re\left(1 + \frac{z\psi''(z)}{\psi'(z)}\right) > -\frac{1 + |\gamma+\beta|^2 - |1 - (\gamma+\beta)^2|}{4\Re(\gamma+\beta)}$$

$$\left(z \in \mathbb{U};\ \psi(z) := \left(\frac{f(z)}{z}\right)^{\beta}\right),$$

then

$$\left(\frac{f(z)}{z}\right)^{\beta} \prec \left(\frac{g(z)}{z}\right)^{\beta}$$

implies:

$$\frac{\gamma+\beta}{z^{\gamma+\beta}} \int_0^z u^{\gamma-1}(f(u))^{\beta} du \prec \frac{\gamma+\beta}{z^{\gamma+\beta}} \int_0^z u^{\gamma-1}(g(u))^{\beta} du.$$

By using the relationship (1.6) and applying the similar method of the Theorem 2.1, we easily get the following result.

Corollary 2.3. *Let* $f,\ g \in \mathcal{A}_0,\ \lambda \geq 0,\ \beta \in \mathbb{C} \setminus \{0\}$ *and* $b > 0$. *If*

$$\Re\left(1 + \frac{z\varphi''(z)}{\varphi'(z)}\right) > -\eta \qquad (z \in \mathbb{U};\ \varphi(z) := \mathcal{L}_a^\lambda(b+1, c; \beta)g(z)), \tag{2.11}$$

where,

$$\eta := \frac{1 + |b|^2 - |1 - b^2|}{4b}, \tag{2.12}$$

then the following subordination relationship,

$$\mathcal{L}_a^\lambda(b+1, c; \beta)f(z) \prec \mathcal{L}_a^\lambda(b+1, c; \beta)g(z) \tag{2.13}$$

implies:

$$\mathcal{L}_a^\lambda(b, c; \beta)f(z) \prec \mathcal{L}_a^\lambda(b, c; \beta)g(z) \quad (z \in \mathbb{U}).$$

Moreover, the function $\mathcal{L}_a^\lambda(b, c; \beta)g$ *is the best dominant.*

We note that η given by (2.12) satisfies the condition $0 < \eta \leq \frac{1}{2}$. By setting $a = 1,\ \lambda = 1,\ b = 2,\ c = 1$ in Corollary 2.3, we get the following result.

Corollary 2.4. *Let* $f,\ g \in \mathcal{A}_0$ and $\beta \in \mathbb{C} \setminus \{0\}$. *If*

$$\begin{gathered}\Re\left(1 + \frac{z\varphi''(z)}{\varphi'(z)}\right) > -\frac{3}{8} \\ \left(z \in \mathbb{U};\ \varphi(z) := \frac{\beta}{2}\left(\left(\frac{f(z)}{z}\right)^\beta + \left(\frac{f(z)}{z}\right)^{\beta-1} f'(z)\right)\right),\end{gathered} \tag{2.14}$$

then the following subordination,

$$\left(\frac{f(z)}{z}\right)^\beta + \left(\frac{f(z)}{z}\right)^{\beta-1} f'(z) \prec \left(\frac{g(z)}{z}\right)^\beta + \left(\frac{g(z)}{z}\right)^{\beta-1} f'(z),$$

implies

$$\left(\frac{f(z)}{z}\right)^\beta \prec \left(\frac{g(z)}{z}\right)^\beta \qquad (z \in \mathbb{U}).$$

If f is subordinate to F, then F is superordinate to f. We now derive the following superordination result.

Theorem 2.5. *Let $f,\ g \in \mathcal{A}_0,\ \beta \in \mathbb{C} \setminus \{0\}$ and $\Re(a) > 0$. If*

$$\Re\left(1 + \frac{z\psi''(z)}{\psi'(z)}\right) > -\delta \qquad (z \in \mathbb{U};\ \psi(z) := \mathcal{L}_a^{\lambda}(b, c; \beta)g(z)),$$

where δ is given by (2.2), and if the function $\mathcal{L}_a^{\lambda}(b, c; \beta)f$ is univalent in $\mathbb{U}$ and $\mathcal{L}_a^{\lambda+1}(b, c; \beta)f \in Q$, then the following subordination relationship

$$\mathcal{L}_a^{\lambda}(b, c; \beta)g(z) \prec \mathcal{L}_a^{\lambda}(b, c; \beta)f(z) \qquad (z \in \mathbb{U})$$

implies:

$$\mathcal{L}_a^{\lambda+1}(b, c; \beta)g(z) \prec \mathcal{L}_a^{\lambda+1}(b, c; \beta)f(z) \qquad (z \in \mathbb{U}).$$

Moreover, the function $\mathcal{L}_a^{\lambda+1}(b, c; \beta)g$ is the best subordinant.

Proof. Let the functions F, G and q be defined by (2.4). By applying the similar method as in the proof of Theorem 2.1, we get

$$\Re(q(z)) > 0 \qquad (z \in \mathbb{U}).$$

Next, to arrive at our desired result, we show that $G \prec F$. For this, let the function $P(z, t)$ be defined by

$$P(z, t) = G(z) + \frac{1 + t}{a} zG'(z) \qquad (z \in \mathbb{U}).$$

Since $\Re(a) > 0$, and G is convex, we can prove as in Theorem 2.1 that $P(z, t)$ is a subordination chain. Therefore, by Lemma 1.6, we conclude $G \prec F$. Furthermore, since the differential equation,

$$\psi(z) = G(z) + \frac{1}{a} zG'(z) := \phi\left(G(z), zG'(z)\right), \tag{2.15}$$

has a univalent solution G, it is the best subordination. The proof of Theorem 2.5 is evidently complete. □

By using similar arguments as in the proof of Theorem 2.5, we get the following superordination result.

Corollary 2.6. *Let $f,\ g \in \mathcal{A}_0,\ \beta \in \mathbb{C} \setminus \{0\}$ and $b > 0$. If*

$$\Re\left(1 + \frac{z\varphi''(z)}{\varphi'(z)}\right) > -\eta \qquad \left(z \in \mathbb{U};\ \varphi(z) := \mathcal{L}_a^{\lambda}(b + 1, c; \beta)g(z)\right),$$

where η is given by (2.12), and if $\mathcal{L}_a^{\lambda}(b + 1, c; \beta)f$ is univalent in $\mathbb{U}$ and $\mathcal{L}_a^{\lambda}(b, c; \beta)f \in Q$, then the following subordination relationship,

$$\mathcal{L}_a^{\lambda}(b + 1, c; \beta)g(z) \prec \mathcal{L}_a^{\lambda}(b + 1, c; \beta)f(z) \qquad (z \in \mathbb{U})$$

implies that:

$$\mathcal{L}_a^\lambda(b,c;\beta)g(z) \prec \mathcal{L}_a^\lambda(b,c;\beta)f(z) \qquad (z \in \mathbb{U}).$$

Moreover, the function $\mathcal{L}_a^\lambda(b,c;\beta)g$ *is the best subordinant.*

Combining Theorems 2.1 and 2.5, we easily get the following sandwich-type result.

Corollary 2.7. *Let* $f,\ g_k \in \mathcal{A}_0$ $(k=1,2)$, $\beta \in \mathbb{C}\setminus\{0\}$, $\lambda \geq 0$ *and* $\Re(a) > 0$. *If*

$$\Re\left(1+\frac{z\psi_k''(z)}{\psi_k'(z)}\right) > -\delta \quad \Big(z \in \mathbb{U};\ \psi_k(z) := \mathcal{L}_a^\lambda(b,c;\beta)g(z)\ (k=1,2)\Big),$$

where δ *is given by* (2.2), *and if let the function* $\mathcal{L}_a^\lambda(b,c;\beta)f$ *is univalent in* $\mathbb{U}$ *and* $\mathcal{L}_a^{\lambda+1}(b,c;\beta)f \in Q$, *then the following subordination relationship,*

$$\mathcal{L}_a^\lambda(b,c;\beta)g_1(z) \prec \mathcal{L}_a^\lambda(b,c;\beta)f(z) \prec \mathcal{L}_a^\lambda(b,c;\beta)g_2(z) \qquad (z \in \mathbb{U})$$

implies:

$$\mathcal{L}_a^{\lambda+1}(b,c;\beta)g_1(z) \prec \mathcal{L}_a^{\lambda+1}(b,c;\beta)f(z) \prec \mathcal{L}_a^{\lambda+1}(b,c;\beta)g_2(z) \qquad (z \in \mathbb{U}).$$

Moreover, the functions $\mathcal{L}_a^{\lambda+1}(b,c;\beta)g_1$ *and* $\mathcal{L}_a^{\lambda+1}(b,c;\beta)g_2$ *are, respectively, the best subordinant and the best dominant.*

Combining of Corollaries 2.3 and 2.6, we readily get the following sandwich-type result.

Corollary 2.8. *Let* $f,\ g_k \in \mathcal{A}_0$ $(k=1,2)$, $\beta \in \mathbb{C}\setminus\{0\}$, $\lambda \geq 0$ *and* $b > 0$. *If*

$$\Re\left(1+\frac{z\varphi_k''(z)}{\varphi_k'(z)}\right) > -\eta \quad \Big(z \in \mathbb{U};\ \varphi_k(z) := \mathcal{L}_a^\lambda(b+1,c;\beta)g_k(z)\ (k=1,2)\Big),$$

where η *is given by* (2.12), *and if* $\mathcal{L}_a^\lambda(b+1,c;\beta)f$ *is univalent in* $\mathbb{U}$ *and* $\mathcal{L}_a^\lambda(b,c;\beta)f \in Q$, *then the following subordination relationship,*

$$\mathcal{L}_a^\lambda(b+1,c;\beta)g_1(z) \prec \mathcal{L}_a^\lambda(b+1,c;\beta)f(z) \prec \mathcal{L}_a^\lambda(b+1,c;\beta)g_2(z) \qquad (z \in \mathbb{U})$$

implies:

$$\mathcal{L}_a^\lambda(b,c;\beta)g_1(z) \prec \mathcal{L}_a^\lambda(b,c;\beta)f(z) \prec \mathcal{L}_a^\lambda(b,c;\beta)g_2(z) \qquad (z \in \mathbb{U}).$$

Moreover, the functions $\mathcal{L}_a^\lambda(b,c;\beta)g_1$ *and* $\mathcal{L}_a^\lambda(b,c;\beta)g_2$ *are, respectively, the best subordinant and the best dominant.*

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