

## **$z$ -IDEALS AND $z^\circ$ -IDEALS IN THE FACTOR RINGS OF $C(X)$**

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**ABSTRACT.** We characterize the  $z$ -ideals of the factor rings of  $C(X)$  via  $z$ -ideals of  $C(X)$ . We show that for pseudocompact spaces  $X$ ,  $J/I$  is a  $z$ -ideal in  $C(X)/I$  if and only if  $J$  is a  $z$ -ideal in  $C(X)$  containing the  $m$ -closure  $\bar{I}$  of the ideal  $I$ . Using this fact, it turns out that the sum of two  $m$ -closed ideals ( $e$ -ideals) in  $C(X)$ , whenever  $X$  is pseudocompact, is an  $m$ -closed ideal ( $e$ -ideal).  $z^\circ$ -ideals of factor rings of  $C(X)$  are also investigated and it is shown that for every two  $z^\circ$ -ideals  $I \subseteq J$  in  $C(X)$ ,  $J/I$  is a  $z^\circ$ -ideal in  $C(X)/I$  if and only if every prime  $z^\circ$ -ideal in  $C(X)$  is minimal.

### 1. Introduction

Here,  $C(X)$  will denote the ring of real valued continuous functions on a completely regular Hausdorff space  $X$ , all other rings are commutative with identity and “ideal” means “proper ideal”. For  $f \in C(X)$ ,  $Z(f)$  denotes the set of zeros of  $f$  and the collection of all zero-sets in  $X$  is denoted by  $Z(X)$ . Whenever  $I$  is an ideal in  $C(X)$  and  $Z[I] = \{Z(f) : f \in I\}$ , we call  $I$  a  $z$ -ideal in  $C(X)$  if  $g \in C(X)$  and  $Z(g) \in Z[I]$  imply that  $g \in I$ . Similarly, if we set  $Z^\circ[I] = \{\text{int}_X Z(f) : f \in I\}$ , then  $I$  is called a  $z^\circ$ -ideal if  $\text{int}_X Z(g) \in Z^\circ[I]$  implies that  $g \in I$ . Equivalently,

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$I$  is a  $z$ -ideal ( $z^\circ$ -ideal) in  $C(X)$  if  $f \in I$ ,  $g \in C(X)$  and  $Z(f) \subseteq Z(g)$  ( $\text{int}_X Z(f) \subseteq \text{int}_X Z(g)$ ) imply that  $g \in I$ . These ideals which are both algebraic and topological objects were first introduced in [15] and [13] respectively and play a fundamental role in studying the ideal theory of  $C(X)$  ( $z^\circ$ -ideals have been studied in [13] under the name of  $d$ -ideals). These ideals are also studied further by others; for instance, see [3-5, 6, 11, 13, 16-18]. Here, we study the  $z$ -ideals and  $z^\circ$ -ideals of  $C(X)/I$ , where  $I$  is an ideal of  $C(X)$ . In Section 2, the equivalent definitions of a  $z$ -ideal in  $C(X)/I$  are given and by these definitions, it turns out that whenever  $J/I$  is a  $z$ -ideal in  $C(X)/I$ , then  $J$  must contain  $\bar{I}$ , where  $\bar{I}$  is the intersection of all maximal ideals containing  $I$ . In case  $X$  is pseudocompact (compact) or  $I$  is a principal ideal of  $C(X)$ , the  $z$ -ideals of  $C(X)/I$  are characterized in terms of  $z$ -ideals of  $C(X)$ . Regularity of the ring  $C(X)/\bar{I}$ , where  $X$  is compact, is also characterized and we observe that if  $C(X)/I$  is regular, then every ideal containing  $I$  is closed (intersection of maximal ideals). In that section, we also obtain the  $z$ -ideals of the factor rings of  $C(X)$  under some special conditions. In Section 3,  $z^\circ$ -ideals of the factor rings of  $C(X)$  are investigated and several equivalent definitions for  $z^\circ$ -ideals of  $C(X)/I$  are given. We characterize the  $z^\circ$ -ideals of  $C(X)/I$ , where  $I$  is a  $z^\circ$ -ideal in  $C(X)$  and some important counterexamples concerning  $z^\circ$ -ideals in the factor rings of  $C(X)$  are given. For examples, we show that whenever  $J/I$  is a  $z^\circ$ -ideal in  $C(X)/I$  and  $I$  is a  $z$ -ideal in  $C(X)$ , then  $J$  is not necessarily a  $z$ -ideal. We also observe that whenever  $I \subseteq J$  are  $z^\circ$ -ideals in  $C(X)$ , then  $J/I$  need not be a  $z^\circ$ -ideal in  $C(X)/I$ . Finally, we give an example of two ideals  $I \subseteq J$  in  $C(X)$  such that  $I$  is semiprime,  $J$  and  $J/I$  are  $z^\circ$ -ideals but  $I$  is not a  $z$ -ideal.

We first recall some general information then [9]. If  $I$  is an ideal in  $C(X)$  and  $\bigcap Z[I] = \bigcap_{f \in I} Z(f)$  is nonempty,  $I$  is called fixed; otherwise, free. The fixed maximal ideals are the sets  $M_p = \{f \in C(X) : p \in Z(f)\}$ , for  $p \in X$  and free maximal ideals of  $C(X)$  are of the form  $M^p = \{f \in C(X) : p \in \text{cl}_{\beta X} Z(f), p \in \beta X \setminus X\}$ , where  $\beta X$  is the Stone-Ćech compactification of  $X$ . More generally, if  $A \subseteq \beta X$ , we let  $M^A = \{f \in C(X) : A \subseteq \text{cl}_{\beta X} Z(f)\}$ . The maximal ideals of  $C^*(X)$ , the ring of all bounded real-valued continuous functions on a completely regular Hausdorff space  $X$  are precisely the sets  $M^{*p} = \{f \in C^*(X) : f^\beta(p) = 0\}$ ,  $p \in \beta X$ , where  $f^\beta$  is the extension of  $f$  to  $\beta X$ ; see [9].  $M^{*p}$  is fixed or free accordingly as  $p \in X$  or  $p \in \beta X \setminus X$ . If  $M$  is a maximal ideal in  $C(X)$  and  $\frac{C(X)}{M} \cong R$ , then  $M$  is called real; otherwise, hyper-real.

$M^p \cap C^*(X)$  is always contained in  $M^{*p}$  and  $M^p \cap C^*(X) = M^{*p}$  if and only if  $M^p$  is real; see [9], 7.9(c). Thus  $M^p \cap C^*(X) = M^{*p}$ , for every  $p \in vX$ , where  $vX$  is the realcompactification of  $X$ . We recall that  $vX$  is the largest subspace of  $\beta X$  in which  $X$  is  $C$ -embedded. Thus, every  $f \in C(X)$  has an extension to a function  $f^v \in C(vX)$  and the mapping  $f \rightarrow f^v$  is an isomorphism of  $C(X)$  onto  $C(vX)$ ; see [9].

For any ideal  $I$  in  $C(X)$ , we recall  $\theta(I) = \{p \in \beta X : I \subseteq M^p\} = \bigcap_{f \in I} \text{cl}_{\beta X} Z(f)$ . Hence,  $M^{\theta(I)}$  is the intersection of all maximal ideals containing  $I$  and clearly  $I \subseteq M^{\theta(I)}$ ; see [9]. For every  $f \in C(X)$  and every positive unit  $\pi \in C(X)$ , let  $N_\pi(f)$  denote the set of all  $g \in C(X)$  for which  $|f - g| < \pi$ . The topology defined on  $C(X)$  by taking the family  $\{N_\pi(f) : f \in C(X) \text{ and } \pi \in C(X) \text{ is a positive unit}\}$  as a base of open sets is called  $m$ -topology; see [12], pp. 48-51. The closure of any subset  $A$  of  $C(X)$  with  $m$ -topology will be denoted by  $A^m$  or  $\bar{A}$ . The closure  $\bar{I}$  of every ideal  $I$  in  $C(X)$  is an ideal and it is known that  $\bar{I}$  is the intersection of all maximal ideals containing  $I$ ; i.e.,  $\bar{I} = M^{\theta(I)}$ ; see [10, 20].

For any  $a$  in a ring  $R$ , the intersection of all maximal (minimal prime) ideals containing  $a$  is denoted by  $M_a$  ( $P_a$ ). One can easily see that an ideal  $I$  in  $C(X)$  is a  $z$ -ideal ( $z^\circ$ -ideal) if and only if  $M_f \subseteq I$  ( $P_f \subseteq I$ ),  $\forall f \in I$ ; see 4A in [9] and [4, 18]. These algebraic definitions enable us to define  $z$ -ideals ( $z^\circ$ -ideals) in general rings. Whenever  $f \in C(X)$ , it is easy to see that  $M_f = M^{\text{cl}_{\beta X} Z(f)} = \overline{\langle f \rangle} = \{g \in C(X) : Z(f) \subseteq Z(g)\}$ . Since every maximal ideal is a  $z$ -ideal and any intersection of  $z$ -ideals is also a  $z$ -ideal,  $Jac(R)$ , the Jacobson radical of a ring  $R$  and every closed ideal in  $C(X)$  is a  $z$ -ideal. In fact,  $Jac(R)$  is the smallest  $z$ -ideal in  $R$  and any  $z$ -ideal in  $R$  contains  $Jac(R)$ . Minimal prime ideals in any commutative reduced ring  $R$  with  $Jac(R) = (0)$  are also  $z$ -ideals, see [4, 18]. Finally,  $\text{rad}(R)$ , the nilradical of the ring  $R$ , is a  $z$ -ideal if and only if  $\text{rad}(R) = Jac(R)$ . In  $C(X)$ , we have the following result which is proved in [6, 19].

**Proposition 1.1.** *An ideal  $I$  in  $C(X)$  is a  $z$ -ideal ( $z^\circ$ -ideal) if and only if every prime ideal minimal over  $I$  is a  $z$ -ideal ( $z^\circ$ -ideal).*

For every  $f \in C(X)$ ,  $P_f$  can be represented as an algebraic and a topological object; see the following proposition which is proved in [3].

**Proposition 1.2.** *For every  $f \in C(X)$ , we have  $P_f = \{g \in C(X) : \text{Ann}(f) \subseteq \text{Ann}(g)\} = \{g \in C(X) : \text{int}_X Z(f) \subseteq \text{int}_X Z(g)\}$ .*

This proposition immediately shows that an ideal  $I$  in  $C(X)$  is a  $z^\circ$ -ideal if and only if  $f \in I$ ,  $g \in C(X)$  and  $\text{Ann}(f) \subseteq \text{Ann}(g)$  imply that  $g \in I$ . Since for every  $f \in C(X)$ , we have  $M_f \subseteq P_f$ , every  $z^\circ$ -ideal in  $C(X)$  is a  $z$ -ideal but not conversely; see [3, 5] for more details of  $z^\circ$ -ideals. In a commutative ring, any ideal consisting entirely of zerodivisor is called a nonregular ideal. By Proposition 1.1, whenever  $f \in C(X)$  is not zerodivisor, i.e.,  $\text{int}Z(f) = \emptyset$ , then  $P_f = C(X)$  and hence every (proper)  $z^\circ$ -ideal in  $C(X)$  is a nonregular ideal.

Recall that if  $I$  and  $J$  are ideals in a commutative ring  $R$ , the ideal quotient  $(I : J)$  is defined by  $(I : J) = \{a \in R : aJ \subseteq I\}$ . In particular,  $(0 : J) = \text{Ann}(J)$ . Whenever  $J$  is a principal ideal generated by  $a \in R$ , then the ideal  $(I : (a))$  is sometimes denoted by  $(I : a)$ . It is clear that  $(I : a) = I, \forall a \notin I$ , if and only if  $I$  is prime and  $(I : a) = R$  if and only if  $a \in I$ . For more details of ideal quotients, see [1, 14] and for undefined terms and notations, the reader is referred to [1, 8, 9, 14].

## 2. $z$ -ideals in the factor rings of $C(X)$

Here, we identify all of  $z$ -ideals in  $C(X)/I$  for an arbitrary ideal  $I$  in  $C(X)$ . It is conjectured that an ideal  $J/I$  is a  $z$ -ideal in  $C(X)/I$  if and only if  $J$  is a  $z$ -ideal in  $C(X)$  containing  $\bar{I}$ . But, we have not completely succeeded to settle this conjecture. We give a positive answer to this conjecture in case  $X$  is pseudocompact and in some other cases. Regularity of the ring  $C(X)/\bar{I}$  is also characterized in this section. For compact spaces  $X$ , we show that  $C(X)/\bar{I}$  is a regular ring if and only if  $\theta(I)$  is finite. For an ideal  $I$  in  $C(X)$  and every  $f \in C(X)$ , we denote by  $M_{I,f}$  the intersection of all maximal ideals containing  $I$  and  $f$ . It is easy to see that  $M_{I,fg} = M_{I,f} \cap M_{I,g} = M_{I,f} M_{I,g}$  and  $M_{I,f} + M_{I,g} \subseteq M_{I,f^2+g^2}$ , for all  $f, g \in C(X)$  and for any ideal  $I$  in  $C(X)$ . Since for each ideal  $I$  in a ring  $R$ , the intersection of all maximal ideals in  $R/I$  containing  $I + f \in R/I$  is  $M_{I,f}/I$ , the definition of  $z$ -ideals in  $R/I$  may be abbreviated as follows.

**Definition 2.1.** Let  $I$  and  $J$  be two ideals in a ring  $R$  such that  $I \subseteq J$ .  $J/I$  is said to be a  $z$ -ideal in  $R/I$  if for every  $f \in J$ , we have  $M_{I,f} \subseteq J$ .

Since for every ideal  $I$  in  $C(X)$  we have  $M_{I,f} = \{g \in C(X) : \theta(I) \cap \text{cl}_{\beta X} Z(f) \subseteq \text{cl}_{\beta X} Z(g)\} = M^{\theta(I) \cap \text{cl}_{\beta X} Z(f)} = \overline{I + (f)} = \overline{(I, f)}$ ,  $\forall f \in C(X)$ , the proof of the following proposition is evident.

**Proposition 2.2.** *Let  $I \subseteq J$  be two ideals in  $C(X)$ . Then, the following statements are equivalent.*

- (a)  $J/I$  is a  $z$ -ideal in  $C(X)/I$ .
- (b) For every  $f \in J$  and  $g \in C(X)$ , whenever  $\theta(I) \cap cl_{\beta X} Z(f) \subseteq cl_{\beta X} Z(g)$ , then  $g \in J$ .
- (c) For every  $f \in J$ , we have  $M^{\theta(I) \cap cl_{\beta X} Z(f)} \subseteq J$ .
- (d) For every  $f \in J$ ,  $(\overline{I}, f) \subseteq J$ .
- (e)  $J = \sum_{f \in J} M_{I, f} = \sum_{f \in J} \overline{(I, f)}$ .

If  $I \subseteq J$  are two ideals in a ring  $R$ , then clearly the necessary condition for  $J/I$  to be a  $z$ -ideal in  $R/I$  is that  $J$  be a  $z$ -ideal in  $R$  containing  $M_I$ . In particular, if  $J/I$  is a  $z$ -ideal in  $C(X)/I$ , then  $J$  is a  $z$ -ideal in  $C(X)$  containing  $\overline{I}$ .

**Example 2.3.** Let  $I$  be an ideal in a ring  $R$ . Since  $Jac(R/I) = M_I/I$ ,  $M_I/I$  is the smallest  $z$ -ideal in  $R/I$ . In particular, if  $I$  is an ideal in  $C(X)$ , then  $\overline{I}/I$  is the smallest  $z$ -ideal in  $C(X)/I$ . On the other hand, if  $I \subseteq J$  are two ideals in a ring  $R$  and  $M_I = M_J$ , then  $J/I$  is a  $z$ -ideal in  $R/I$  if and only if  $J/I$  is the smallest  $z$ -ideal in  $R/I$ ; i.e., if and only if  $J = M_I = M_J$ . It is easy to see that  $J/I$  is a  $z$ -ideal in  $R/I$  if and only if  $J/M_I$  is a  $z$ -ideal in  $R/M_I$ . It is also clear that if  $I, J \subseteq K$  are three ideals in  $R$  ( $C(X)$ ) and  $M_J \subseteq M_I$  ( $\theta(I) \subseteq \theta(J)$ ), then  $K/I$  is a  $z$ -ideal in  $R/I$  ( $C(X)/I$ ) if  $K/J$  is a  $z$ -ideal in  $R/J$  ( $C(X)/J$ ).

By the following lemma and corollary, the answer to our conjecture for the rings  $C(X)$  modulo principal ideals (or the closure of principal ideals) is positive.

**Lemma 2.4.** *Suppose that  $I$  is an ideal and  $J$  is a  $z$ -ideal in a ring  $R$  ( $C(X)$ ) such that  $I \subseteq M_f$  ( $cl_{\beta X} Z(f) \subseteq \theta(I)$ ) for some  $f \in J$ . Then,  $J/I$  is a  $z$ -ideal in  $R/I$  ( $C(X)/I$ ).*

**Proof.** Let  $M$  be a maximal ideal in  $R$  such that  $f, g \in M$ . Since  $M$  is a  $z$ -ideal,  $M_f \subseteq M$  and by our hypothesis,  $I \subseteq M$ . This means that  $M_{I, g} \subseteq M_{f, g}$ ,  $\forall g \in J$ . Now, we have  $M_{f, g} = M_{f^2+g^2} \subseteq J$  for  $J$  is a  $z$ -ideal and  $f^2 + g^2 \in J$ . Hence,  $M_{I, g} \subseteq J$ ;  $\forall g \in J$ , i.e.,  $J/I$  is a  $z$ -ideal in  $R/I$ .  $\square$

**Corollary 2.5.** *For each non-unit  $f \in C(X)$ , the ideal  $J$  in  $C(X)$  containing  $f$  is a  $z$ -ideal if and only if  $J/(f)$  ( $J/M_f$ ) is a  $z$ -ideal in  $C(X)/(f)$  ( $C(X)/M_f$ ).*

It is known that a Hausdorff space  $X$  is normal if and only if every closed subset of  $X$  is  $C$ -embedded. By the following proposition, whenever  $X$  is a normal Hausdorff space and  $f \in C(X)$ , then every ideal of  $C(X)/M_f$  is a  $z$ -ideal if and only if  $Z(f)$  is a  $P$ -space (a space in which every  $G_\delta$ -set or every zero-set is open).

**Proposition 2.6.** (a) *For  $f \in C(X)$ , every ideal of  $C(X)/(f)$  is a  $z$ -ideal if and only if  $Z(f)$  is an open  $P$ -space.*  
 (b) *If  $f \in C(X)$  and  $Z(f)$  is  $C$ -embedded in  $X$ , then every ideal of  $C(X)/M_f$  is a  $z$ -ideal if and only if  $Z(f)$  is a  $P$ -space.*

**Proof.** First, we show that if  $Z(f)$  is  $C$ -embedded in  $X$ , then  $\frac{C(X)}{M_f} \cong C(Z(f))$ . To see this, we define  $\varphi : C(X) \rightarrow C(Z(f))$  so that  $\varphi(g) = g|_{Z(f)}$ ,  $\forall g \in C(X)$ . Since  $Z(f)$  is  $C$ -embedded, clearly  $\varphi$  is an onto homomorphism and  $\text{Ker}(\varphi) = M_f$ . This shows that  $\frac{C(X)}{M_f} \cong C(Z(f))$ . Now, if every ideal of  $C(X)/(f)$  is a  $z$ -ideal, then  $(f^{\frac{1}{3}})/(f)$  is a  $z$ -ideal in  $C(X)/(f)$ , and so  $(f^{\frac{1}{3}})$  is a  $z$ -ideal in  $C(X)$ . This implies that  $Z(f^{\frac{1}{3}}) = Z(f)$  is open. Therefore,  $M_f = (f)$  and  $Z(f)$  is  $C$ -embedded, for  $Z(f)$  is an open set. But  $\frac{C(X)}{M_f} = \frac{C(X)}{(f)} \cong C(Z(f))$  implies that every ideal of  $C(Z(f))$  is a  $z$ -ideal; i.e.,  $Z(f)$  is an open  $P$ -space. Conversely, if  $Z(f)$  is an open  $P$ -space, then it is  $C$ -embedded and hence  $\frac{C(X)}{M_f} = \frac{C(X)}{(f)} \cong C(Z(f))$ . Now, since every ideal of  $C(Z(f))$  is a  $z$ -ideal, every ideal of  $C(X)/(f)$  is a  $z$ -ideal and this proves part (a). To prove part (b), we have  $\frac{C(X)}{M_f} \cong C(Z(f))$ , and clearly every ideal of  $C(X)/M_f$  is a  $z$ -ideal if and only if every ideal of  $C(Z(f))$  is a  $z$ -ideal if and only if  $Z(f)$  is a  $P$ -space.  $\square$

By the above proposition and Problem 4K in [9], whenever  $X$  is compact (more generally, whenever  $X$  is locally compact or pseudocompact) and  $I$  is a principal ideal  $(f)$ , then every ideal of  $C(X)/I$  is a  $z$ -ideal if and only if  $Z(f)$  consists only a finite number of isolated points. If  $X$  is a compact space and  $I$  is any arbitrary ideal in  $C(X)$ , then we have the following result. Our proof of the following result shows that the equivalence of parts (b) and (h) is true for any completely regular

Hausdorff space  $X$ . Moreover, this equivalence shows that if  $C(X)/I$  is regular, then every ideal containing  $I$  is closed.

**Theorem 2.7.** *Let  $X$  be a compact space and  $I$  be an ideal in  $C(X)$ . Then, the following statements are equivalent.*

- (a) *For every ideal  $J$  containing  $\bar{I}$ ,  $J/I$  is a  $z$ -ideal in  $C(X)/I$ .*
- (b) *Every ideal of  $C(X)/\bar{I}$  is a  $z$ -ideal.*
- (c) *Every principal ideal of  $C(X)/\bar{I}$  is a  $z$ -ideal.*
- (d)  *$\bar{I} + (f) = \bar{I} + M_f = \bar{I} + \overline{(f)}$ ,  $\forall f \in C(X)$ .*
- (e) *For every  $f, g \in C(X)$ , if  $Z(f) \subseteq Z(g)$ , then there exists  $h \in C(X)$  such that  $\theta(I) \subseteq Z(g - hf)$ .*
- (f) *For every  $f, g \in C(X)$ , whenever  $Z(f) \subseteq Z(g)$ , then  $g|_{\theta(I)}$  is a multiple of  $f|_{\theta(I)}$  in  $C(\theta(I))$ .*
- (g)  *$\theta(I)$  is finite.*
- (h)  *$C(X)/\bar{I}$  is a regular ring; i.e., every prime ideal of  $C(X)/\bar{I}$  is maximal.*

**Proof.** (a)  $\Leftrightarrow$  (b)  $\Rightarrow$  (c) are evident.

(c)  $\Rightarrow$  (d). Clearly,  $\bar{I} + (f) \subseteq \bar{I} + \overline{(f)}$ . Now, let  $h + g \in \bar{I} + \overline{(f)} = \bar{I} + M_f$ , where  $h \in \bar{I}$  and  $g \in M_f$ . Then,  $\theta(I) \cap Z(f) \subseteq Z(h + g)$ . But,  $\frac{\bar{I} + (f)}{\bar{I}}$  is a  $z$ -ideal by part (c) and hence  $h + g \in \bar{I} + (f)$ .

(d)  $\Rightarrow$  (e). If  $Z(f) \subseteq Z(g)$ , then  $g \in \bar{I} + \overline{(f)}$ , for  $\bar{I} + \overline{(f)}$  is a  $z$ -ideal and hence  $g \in \bar{I} + (f)$  by part (d). Therefore,  $\exists h \in C(X)$  such that  $g - hf \in \bar{I}$ ; i.e.,  $\theta(I) \subseteq Z(g - hf)$ .

(e)  $\Rightarrow$  (f). If  $Z(f) \subseteq Z(g)$ , then  $g - hf \in \bar{I}$  implies that  $g|_{\theta(I)} = h|_{\theta(I)} f|_{\theta(I)}$ .

(f)  $\Rightarrow$  (g). We claim that  $\theta(I)$  is a  $P$ -space. Since  $\theta(I)$  is compact, if we prove our claim, then we are through, for in that case  $\theta(I)$  will be finite; see [9], 4K. So, let  $f, g \in C(\theta(I))$ . We show that  $(f, g) \subseteq (f^2 + g^2)$ ; see 4J in [9]. Since  $\theta(I)$  is  $C$ -embedded in  $X$ , there exist extensions  $\bar{f}$  and  $\bar{g}$  in  $C(X)$  of  $f$  and  $g$ , respectively. Now,  $Z(\bar{f}^2 + \bar{g}^2) = Z(\bar{f}) \cap Z(\bar{g})$  and part (f) imply that  $f = \bar{f}|_{\theta(I)}$  and  $g = \bar{g}|_{\theta(I)}$  are in principal ideal  $(\bar{f}^2|_{\theta(I)} + \bar{g}^2|_{\theta(I)}) = (f^2 + g^2)$ ; i.e.,  $(f, g) = (f^2 + g^2)$ .

(g)  $\Rightarrow$  (h). Let  $P$  be a prime ideal in  $C(X)$  containing  $\bar{I}$  and  $\theta(I) = \{x_1, \dots, x_n\}$ . Hence,  $\bar{I} = M_{\theta(I)} = \bigcap_{i=1}^n M_{x_i} \subseteq P$  implies that  $M_{x_j} \subseteq P$  for some  $1 \leq j \leq n$ ; i.e.,  $P$  is maximal.

(h)  $\Rightarrow$  (b). Since every prime ideal containing  $\bar{I}$  is maximal, every prime

ideal containing  $J$  is also maximal and this implies that  $J/\bar{I}$  is an intersection of maximal ideals.

Now, we answer the question concerning the characterization of  $z$ -ideals in the factor rings of  $C(X)$ , where  $X$  is pseudocompact. First, we need the following lemma.

**Lemma 2.8.** (a) *Let  $A$  and  $B$  be two compact subsets of the space  $X$  and  $G$  be a  $G_\delta$ -set in  $X$ . If  $A \cap B \subseteq G$ , then there exist two zero-sets  $E$  and  $F$  in  $X$  such that  $A \subseteq E$ ,  $B \subseteq F$  and  $E \cap F \subseteq G$ .*

(b)  *$X$  is normal if and only if  $M_A + M_B = M_{A \cap B}$ , for every pair of closed sets  $A$  and  $B$  in  $X$ .*

**Proof.** (a) Let  $G = \bigcap_{n=1}^{\infty} U_n$ , where the  $U_n$  are open sets. Given  $n \in \mathbb{N}$ , for every  $a \in A \cap B$ , there exists a zero-set  $Z_{n,a}$  such that  $a \in \text{int} Z_{n,a} \subseteq Z_{n,a} \subseteq U_n$ , and for every  $a \in A \setminus B$ , there exists a zero-set  $Z_{n,a}$  such that  $Z_{n,a} \cap B = \emptyset$  and  $Z_{n,a}$  is a neighborhood of  $a$ . since  $A \subseteq \bigcup_{a \in A} Z_{n,a}$  and  $A$  is compact, there are  $a_1, \dots, a_k \in A$  such that  $A \subseteq \bigcup_{i=1}^k Z_{n,a_i}$ . Take the zero-set  $Z_n = \bigcup_{i=1}^k Z_{n,a_i}$ . Then,  $Z_n \cap B = \left( \bigcup_{i=1}^k Z_{n,a_i} \right) \cap B = \left[ \left( \bigcup_{a_i \in A \cap B} Z_{n,a_i} \right) \cup \left( \bigcup_{a_i \in A \setminus B} Z_{n,a_i} \right) \right] \cap B \subseteq U_n$ , for  $\left( \bigcup_{a_i \in A \setminus B} Z_{n,a_i} \right) \cap B = \emptyset$ . Now, consider the zero-set  $E = \bigcap_{n=1}^{\infty} Z_n$ . We have  $A \subseteq E$  and  $E \cap B \subseteq G$ . Again, we will employ the same procedure for the compact set  $B$  and we find a zero-set  $F$  such that  $E \cap F \subseteq G$  and  $B \subseteq F$ . This completes the proof.

(b) Let  $X$  be a normal space. If  $f \in M_{A \cap B}$ , then the function  $g$ , defined by  $g = f$  on  $A$  and  $g = 0$  on  $B$ , is continuous on the closed set  $A \cup B$ . By Tietze's extension Theorem,  $g$  can be extended to a function  $h \in C(X)$ . Thus,  $f = (f - h) + h \in M_A + M_B$ . Conversely, suppose that  $M_A + M_B = M_{A \cap B}$ , for every pair of closed sets  $A$  and  $B$  in  $X$ . If  $A \cap B = \emptyset$ , then  $M_A + M_B = M_{A \cap B} = C(X)$ , and therefore, there exist functions  $f \in M_A$  and  $g \in M_B$  such that  $f + g = 1$ .  $\square$

The following corollary and remark show that the sum of two closed ideals in  $C(X)$ , where  $X$  is compact (pseudocompact), is a closed ideal.

**Corollary 2.9.** *If  $A$  and  $B$  are two compact subsets of  $X$ , then the sum of two closed ideals  $M_A$  and  $M_B$  in  $C(X)$  is a closed ideal. In particular, If  $X$  is compact, then the sum of every two closed ideals in  $C(X)$  is a closed ideal.*

**Proof.** We show that  $M_A + M_B = M_{A \cap B}$ . Clearly,  $M_A + M_B \subseteq M_{A \cap B}$ . Now, suppose that  $h \in M_{A \cap B}$ . Then,  $A \cap B \subseteq Z(h)$ . Since  $Z(h)$  is a  $G_\delta$ -set and  $A$  and  $B$  are compact sets, by Lemma 2.7 there are  $f, g \in C(X)$  such that  $A \subseteq Z(f)$ ,  $B \subseteq Z(g)$  and  $Z(f) \cap Z(g) \subseteq Z(h)$ . Now,  $f^2 + g^2 \in M_A + M_B$  and  $M_A + M_B$  is a  $z$ -ideal, and hence  $h \in M_A + M_B$ . In particular, if  $X$  is compact, then  $X$  is normal and by part (b) of Lemma 2.7, the proof is evident.  $\square$

**Remark 2.10.** It is easy to see that for any two closed subsets  $A$  and  $B$  of the space  $\beta X$ , we have  $M^A M^B = M^A \cap M^B = M^{A \cup B}$ . But, the equality  $M^A + M^B = M^{A \cap B}$  is equivalent to our conjecture stated at the beginning of this section. By Lemma 2.8, this equality also holds in  $C(X)$  for a pseudocompact space  $X$ . For if  $X$  is pseudocompact, then  $C(X) = C^*(X) \cong C(\beta X)$ . Since the sum of two closed ideals in  $C(\beta X)$  is a closed ideal, this fact also holds in  $C(X)$ ; i.e., if  $A$  and  $B$  are two closed sets in  $\beta X$ , then  $M^A + M^B$  is a closed ideal  $M^C$  in  $C(X)$ , where  $C \subseteq \beta X$  is a closed set. Moreover,  $C = A \cap B$ , for  $M^A, M^B \subseteq M^C$  imply that  $C \subseteq A \cap B$ ; see [7], Lemma 1.6. Now, let  $x \in A \cap B$  and  $x \notin C$ . Then,  $\exists f \in C(X)$  such that  $C \subseteq \text{cl}_{\beta X} Z(f)$  and  $x \notin \text{cl}_{\beta X} Z(f)$ . Hence,  $f \in M^C = M^A + M^B$ , and so there exist  $f_1 \in M^A$  and  $f_2 \in M^B$  such that  $f = f_1 + f_2$ . Thus,  $x \in A \cap B \subseteq \text{cl}_{\beta X} Z(f_1) \cap \text{cl}_{\beta X} Z(f_2) \subseteq \text{cl}_{\beta X} Z(f)$ , which gives a contradiction.  $\square$

**Corollary 2.11.** *Suppose that  $X$  is a pseudocompact space and  $I$  is an ideal in  $C(X)$ .  $J$  is a  $z$ -ideal in  $C(X)$  containing  $\bar{I}$  if and only if  $J/I$  is a  $z$ -ideal in  $C(X)/I$ .*

**Proof.** Let  $J$  be a  $z$ -ideal in  $C(X)$ ,  $\bar{I} \subseteq J$  and  $f \in J$ . By Remark 2.9,  $M^{\theta(I)} \cap \text{cl}_{\beta X} Z(f) = M^{\theta(I)} + M^{\text{cl}_{\beta X} Z(f)} = \bar{I} + M_f$ . Since  $J$  is a  $z$ -ideal and  $f \in J$ ,  $M_f \subseteq J$ , and hence  $M^{\theta(I)} \cap \text{cl}_{\beta X} Z(f) \subseteq J$  implies that  $J/I$  is a  $z$ -ideal in  $C(X)/I$ .  $\square$

Since  $e$ -ideals in  $C^*(X)$  are precisely closed ideals in  $C^*(X)$  with uniform norm topology (see [9], 2L and 2M for details of  $e$ -ideals in  $C^*(X)$  and uniform norm topology on  $C^*(X)$ ) and relative  $m$ -topology on  $C^*(X)$  coincides with uniform norm topology, in case  $X$  is pseudocompact (see [9], 2N), the following corollary is evident.

**Corollary 2.12.** *If  $X$  is a pseudocompact space, then the sum of two  $e$ -ideals in  $C^*(X)$  is an  $e$ -ideal.*

The rest of this section is devoted to some results supporting our conjecture under some special conditions. First, let  $A \subseteq \beta X$  be a closed set in  $\beta X$ . Whenever for any two zero-sets  $Z_1$  and  $Z_2$  with  $A \cap \text{cl}_{\beta X} Z_1 \subseteq \text{cl}_{\beta X} Z_2$ , there exists a zero-set  $Z$  such that  $A \subseteq \text{cl}_{\beta X} Z$  and  $\text{cl}_{\beta X} Z \cap \text{cl}_{\beta X} Z_1 \subseteq \text{cl}_{\beta X} Z_2$ , then we say that  $A$  is an  $\epsilon$ -set. In fact,  $A \subseteq \beta X$  is an  $\epsilon$ -set if and only if  $M^{A \cap \text{cl}_{\beta X} Z(f)} = M^A + M_f$ ,  $\forall f \in C(X)$ . To see this, if  $A$  is an  $\epsilon$ -set and  $g \in M^{A \cap \text{cl}_{\beta X} Z(f)}$ , then  $A \cap \text{cl}_{\beta X} Z(f) \subseteq \text{cl}_{\beta X} Z(g)$ . Since  $A$  is an  $\epsilon$ -set,  $\exists h \in C(X)$  such that  $A \subseteq \text{cl}_{\beta X} Z(h)$  and  $\text{cl}_{\beta X} Z(h) \cap \text{cl}_{\beta X} Z(f) \subseteq \text{cl}_{\beta X} Z(g)$ . Hence,  $\text{cl}_{\beta X} Z(f^2 + h^2) \subseteq \text{cl}_{\beta X} Z(g)$  and  $f^2 + h^2 \in M^A + M_f$  imply that  $g \in M^A + M_f$ , for  $M^A + M_f$  is a  $z$ -ideal and therefore  $M^{A \cap \text{cl}_{\beta X} Z(f)} = M^A + M_f$  (note, the inclusion  $M^{A \cap \text{cl}_{\beta X} Z(f)} \supseteq M^A + M_f$  is always true). Conversely,  $A \cap \text{cl}_{\beta X} Z(f) \subseteq \text{cl}_{\beta X} Z(g)$  implies that  $g \in M^{A \cap \text{cl}_{\beta X} Z(f)} = M^A + M_f$ , and hence there exist  $h \in M^A$  and  $k \in M_f$  such that  $g = h + k$ . Now,  $\text{cl}_{\beta X} Z(h) \cap \text{cl}_{\beta X} Z(f) \subseteq \text{cl}_{\beta X} Z(h) \cap \text{cl}_{\beta X} Z(k) \subseteq \text{cl}_{\beta X} Z(g)$  and  $A \subseteq \text{cl}_{\beta X} Z(h)$ ; i.e.,  $A$  is an  $\epsilon$ -set.

If  $\theta(I)$  is an  $\epsilon$ -set and  $J$  is a  $z$ -ideal in  $C(X)$  containing  $\bar{I}$ , then  $J/I$  is a  $z$ -ideal in  $C(X)/I$ . In fact, if  $g \in J$  and  $h \in C(X)$  such that  $\theta(I) \cap \text{cl}_{\beta X} Z(g) \subseteq \text{cl}_{\beta X} Z(h)$ , then  $\exists f \in C(X)$  such that  $\theta(I) \subseteq \text{cl}_{\beta X} Z(f)$  and  $\text{cl}_{\beta X} Z(f) \cap \text{cl}_{\beta X} Z(g) \subseteq \text{cl}_{\beta X} Z(h)$  or  $\text{cl}_{\beta X} Z(f^2 + g^2) \subseteq \text{cl}_{\beta X} Z(h)$ . Now,  $\theta(I) \subseteq \text{cl}_{\beta X} Z(f)$  implies that  $f \in \bar{I} \subseteq J$ , and hence  $f^2 + g^2 \in J$ . But,  $J$  is a  $z$ -ideal, and thus  $h \in J$ ; i.e.,  $J/I$  is a  $z$ -ideal in  $C(X)/I$ .

**Proposition 2.13.** *Let  $A \subseteq \beta X$  be a closed set in  $\beta X$  and either*

- (a)  $\text{cl}_{\beta X}(A \cap X) = A$ , or
- (b)  $A = Z(f^\beta) \subseteq \beta X \setminus X$  for some  $f \in C^*(X)$  and  $X$  is normal, or
- (c)  $A \subseteq vX$ .

*Then,  $A$  is an  $\epsilon$ -set.*

**Proof.** Let  $A \cap \text{cl}_{\beta X} Z_0 \subseteq \text{cl}_{\beta X} Z_1$ , where  $Z_0$  and  $Z_1$  are two zero-sets. First, suppose that (a) holds. Since  $\beta X$  is compact, by Lemma 2.7 there exists  $f \in C^*(X)$  such that  $A \subseteq Z(f^\beta)$  and  $Z(f^\beta) \cap \text{cl}_{\beta X} Z_0 \subseteq \text{cl}_{\beta X} Z_1 \subseteq Z(g^\beta)$ , where  $Z_1 = Z(g)$ , for some  $g \in C^*(X)$ . Now, we have  $A \cap X \subseteq Z(f)$ , and hence  $A = \text{cl}_{\beta X}(A \cap X) \subseteq \text{cl}_{\beta X} Z(f)$ . Thus,  $\text{cl}_{\beta X} Z(f) \cap \text{cl}_{\beta X} Z_0 \subseteq Z(f^\beta) \cap \text{cl}_{\beta X} Z_0 \subseteq \text{cl}_{\beta X} Z_1$ ; i.e.,  $A$  is an  $\epsilon$ -set. Next, we suppose that (b) holds and  $A$  is not an  $\epsilon$ -set. Letting  $f^\beta = F$ ,  $F^{-1}([-\frac{1}{n}, \frac{1}{n}]) \cap \text{cl}_{\beta X} Z_0 \not\subseteq \text{cl}_{\beta X} Z_1$ ,  $\forall n \in \mathbb{N}$ . Therefore, for

$n \in \mathbb{N}$ , there exists  $x_n \in X$  such that  $F(x_n) \leq \frac{1}{n}$  and  $x_n \in Z_0 \setminus Z_1$ . Take  $B = \{x_n : n \in \mathbb{N}\}$ . Clearly,  $B$  is closed in  $X$ , the set of limit points  $B'$  of  $B$  is nonempty,  $B' \subseteq Z(F)$ ,  $B \cap Z_1 = \emptyset$  and  $B \subseteq \text{cl}_{\beta X} Z_0$ . Now,  $X$  is normal. Then,  $Z \in Z(X)$  exists such that  $B \subseteq Z$  and  $Z \cap Z_1 = \emptyset$ , and hence  $\text{cl}_{\beta X} Z_0 \cap \text{cl}_{\beta X} Z_1 = \emptyset$  and  $B' \subseteq \text{cl}_{\beta X} Z$ . But  $B' \subseteq Z(F) \cap \text{cl}_{\beta X} Z_0 \subseteq \text{cl}_{\beta X} Z_1$ , and hence  $B' \subseteq \text{cl}_{\beta X} Z_1$ . So,  $\emptyset \neq B' \subseteq \text{cl}_{\beta X} Z_0 \cap \text{cl}_{\beta X} Z_1$ , which is a contradiction. Finally, suppose that the condition (c) holds. Then,  $A \cap vX = A$  implies that  $\text{cl}_{\beta X}(A \cap vX) = A$  and by part (a), there exists  $f^v \in C(vX)$  such that  $A \subseteq \text{cl}_{\beta X} Z(f^v)$  and  $\text{cl}_{\beta X} Z(f^v) \cap \text{cl}_{\beta X} Z_0 \subseteq \text{cl}_{\beta X} Z_1$ . But,  $\text{cl}_{\beta X} Z(f^v) = \text{cl}_{\beta X} Z(f)$  implies that  $A$  is an  $\epsilon$ -set.  $\square$

**Corollary 2.14.** *Let  $I \subseteq J$  be two ideals in  $C(X)$  and  $A = \theta(I)$ . If one of the parts (a), (b) or (c) of the above proposition holds, then  $J/I$  is a  $z$ -ideal in  $C(X)/I$  if and only if  $J$  is a  $z$ -ideal in  $C(X)$  containing  $\bar{I}$ .*

### 3. $z^\circ$ -ideals in the factor rings

We note that if  $I$  is a  $z^\circ$ -ideal in a commutative ring  $R$ , then for every  $a \in I$ ,  $P_a \neq R$ . This means that every member of a  $z^\circ$ -ideal  $I$  is zero divisor. Since every minimal prime ideal is a  $z^\circ$ -ideal and any intersection of  $z^\circ$ -ideals is also a  $z^\circ$ -ideal, the nilradical  $\text{rad}(R)$  of  $R$  is a  $z^\circ$ -ideal which is the smallest  $z^\circ$ -ideal in  $R$ ; i.e., every  $z^\circ$ -ideal of  $R$  contains  $\text{rad}(R)$ . Therefore, the  $z^\circ$ -ideals structure of  $R$  is equivalent to that of  $R/\text{rad}(R)$ . Thus, we may assume that  $\text{rad}(R) = (0)$ ; i.e.,  $R$  is a reduced ring. Whenever  $I$  is an ideal of a ring  $R$ , in order for  $R/I$  to be a reduced ring, we should consider  $I$  to be a semiprime ideal. In this section, we study the  $z^\circ$ -ideals of factor rings  $R/I$  and  $C(X)/I$ , where  $I$  is a semiprime ideal. First, we need the following two useful lemmas which are proved in [4, 18].

**Lemma 3.1.** *Let  $R$  be a reduced ring and  $a \in R$ . Then, we have,*

$$P_a = \{b \in R : \text{Ann}(a) \subseteq \text{Ann}(b)\} = \text{Ann}(\text{Ann}(a)).$$

**Lemma 3.2.** *Let  $R$  be a reduced ring and  $I$  be an ideal in  $R$ . Then, the following statements are equivalent.*

- (a)  $I$  is a  $z^\circ$ -ideal in  $R$  (i.e.,  $a \in I$  implies that  $P_a \subseteq I$ ).
- (b)  $P_b \subseteq P_a$  and  $a \in I$  imply that  $b \in I$ .

- (c)  $\text{Ann}(a) \subseteq \text{Ann}(b)$  and  $a \in I$  imply that  $b \in I$ .  
 (d)  $a \in I$  implies that  $\text{Ann}(\text{Ann}(a)) \subseteq I$ .

The following is also known. The parts (a) and (c) of the following lemma are proved in [16] and [4] respectively and the part (b) is evident.

**Lemma 3.3.** *Let  $I$  and  $J$  be two ideals in a commutative ring  $R$ . Then, the following statements hold.*

- (a) *If  $I$  is a  $z$ -ideal, then so is  $[I : J]$ .*  
 (b) *If  $I$  is semiprime, then so is  $[I : J]$ .*  
 (c) *If  $R$  is a reduced ring and  $I$  is  $z^\circ$ -ideal, then so is  $[I : J]$ .*

Next, we show that the converse of the above results are not true in general.

**Remark 3.4.** If  $[I : J]$  is a  $z$ -ideal, then  $I$  may not be even semiprime. To see this, let  $P$  be a prime  $z$ -ideal in  $C(X)$  and  $f \notin P$  be a non-zero-divisor; i.e.,  $\text{int}_X Z(f) = \emptyset$ . Since  $(f)$  is essential (see [2]),  $I = (f) \cap P \neq (0)$ .  $I$  is not a  $z$ -ideal, for if  $g \in P \setminus (f^{\frac{1}{3}})$ , then  $f^{\frac{1}{3}}g \notin (f) \cap P = I$ , otherwise  $f^{\frac{1}{3}}g = fk$ , for some  $k \in C(X)$ . Hence,  $f^{\frac{1}{3}}(g - f^{\frac{2}{3}}k) = 0$  implies that  $g = f^{\frac{1}{3}}k$ , for  $f$  is not a zero-divisor. This contradicts  $g \notin (f^{\frac{2}{3}})$ . So,  $f^{\frac{2}{3}}g \notin I$ , but  $fg^3 \in I$  and hence  $I$  is not a semiprime ideal. Now, we show that  $(I : f)$  is a  $z$ -ideal. Suppose that  $Z(h) = Z(k)$  and  $h \in (I : f)$ . Hence,  $hf \in I = (f) \cap P$  implies that  $h \in P$ , for  $f \notin P$ . Since  $P$  is a  $z$ -ideal,  $k \in P$  and therefore  $kf \in (f) \cap P = I$ ; i.e.,  $k \in (I : f)$ , and hence  $(I : f)$  is a  $z$ -ideal. This shows that the converse of parts (a) and (b) of preceding proposition is not true. To show that the converse of part (c) of this proposition is not true either, take the ideal  $I = M_{[0,1] \cup \{2\}}$  in  $C(\mathbb{R})$  and  $f \in C(\mathbb{R})$  with  $Z(f) = \{2\}$ . Clearly,  $(I : f) = M_{[0,1]}$  is a  $z^\circ$ -ideal but  $I$  is not a  $z^\circ$ -ideal.  $\square$

Now, let  $I$  be a semiprime ideal in a ring  $R$ . For any  $a \in R$ , we denote by  $P_{a+I}$ , as before, the intersection of all minimal prime ideals of  $R/I$  containing  $a + I$ . If we take  $P_{I,a} = \bigcap_{a \in P \in \text{Min}(I)} P$ , where  $\text{Min}(I)$  denotes the set of all prime ideals minimal over  $I$ , then we have  $P_{a+I} = \frac{P_{I,a}}{I}$ . On the other hand, by Lemma 3.1, we have,

$$P_{a+I} = \left\{ b + I : \text{Ann}_{\frac{R}{I}}(a + I) \subseteq \text{Ann}_{\frac{R}{I}}(b + I) \right\}.$$

But it is easy to see that  $\text{Ann}_{\frac{R}{I}}(a + I) \subseteq \text{Ann}_{\frac{R}{I}}(b + I)$  if and only if  $(I : a) \subseteq (I : b)$ , and so  $P_{I,a} = \{b \in R : (I : a) \subseteq (I : b)\}$ . By this argument and Lemma 3.2, the following result is evident.

**Proposition 3.5.** *Let  $I \subseteq J$  be two ideals in a ring  $R$  and  $I$  be a semiprime ideal. Then, the following statements are equivalent.*

- (a)  $J/I$  is a  $z^\circ$ -ideal in  $R/I$ ; i.e.,  $\forall a \in J, P_{a+I} \subseteq J/I$ , (or  $P_{I,a} \subseteq J$ ).
- (b)  $P_{I,b} \subseteq P_{I,a}$  and  $a \in J$  imply that  $b \in J$ .
- (c) If  $(I : a) \subseteq (I : b)$  and  $a \in J$ , then  $b \in J$ .
- (d)  $\forall a \in J, P_{I,a} \subseteq J$ .
- (e)  $\forall a \in J, (I : (I : a)) \subseteq J$ .
- (f)  $J = \sum_{a \in J} (I : (I : a))$ .

**Example 3.6.** (1) It is easy to show that every minimal prime ideal and the annihilator of any subset of a reduced ring is a  $z^\circ$ -ideal; see [4]. It is also clear that the intersection of  $z^\circ$ -ideals is also a  $z^\circ$ -ideal. Now, we let  $I$  be a semiprime ideal in a ring  $R$ . Then,  $R/I$  is a reduced ring, and hence  $(I : a)/I = \text{Ann}_{R/I}(I + a)$  is a  $z^\circ$ -ideal in  $R/I, \forall a \in R$ . Similarly,  $(I : ab)/(I : a)$  is also a  $z^\circ$ -ideal in  $R/(I : a)$ .  $P_{I,a}/I$  is also a  $z^\circ$ -ideal in  $R/I, \forall a \in R$ , for it is an intersection of minimal prime ideals in the reduced ring  $R/I$ .

(2) If  $R$  is a reduced ring, then for every  $a \in R$ ,  $\text{Ann}(a)$  is a semiprime ideal. In this case, for any  $a, b \in R$ , if  $\text{Ann}(a) \subseteq \text{Ann}(b)$ , then  $\text{Ann}(b)/\text{Ann}(a)$  is a  $z^\circ$ -ideal in  $\frac{R}{\text{Ann}(a)}$ . To see this, let  $c \in \text{Ann}(b)$ ,  $d \in R$  and  $(\text{Ann}(a) : c) \subseteq (\text{Ann}(a) : d)$ . Since  $bc = 0$ ,  $b \in (\text{Ann}(a) : c) \subseteq (\text{Ann}(a) : d)$ , and hence  $bd \in \text{Ann}(a) \subseteq \text{Ann}(b)$  implies that  $db = 0$ , or  $d \in \text{Ann}(b)$  and we are through. More generally, if  $I$  is a semiprime ideal in a ring  $R$  and  $b \in R$  such that  $I \subseteq \text{Ann}(b)$ , then  $\frac{\text{Ann}(b)}{I}$  is a  $z^\circ$ -ideal in  $R/I$ .

(3) Let  $f \in C(\mathbb{R})$ ,  $Z(f) = [0, \infty]$  and  $J$  be a  $z^\circ$ -ideal containing  $\text{Ann}(f)$ . We show that  $\frac{J}{\text{Ann}(f)}$  is a  $z^\circ$ -ideal. Since  $J$  is an intersection of prime  $z^\circ$ -ideals (see [5]), it is enough to show that for every prime  $z^\circ$ -ideal  $P$  containing  $\text{Ann}(f)$ ,  $\frac{P}{\text{Ann}(f)}$  is a  $z^\circ$ -ideal. Clearly,  $f \notin P$ , for otherwise take  $g \in C(\mathbb{R})$  with  $Z(g) = [-\infty, 0]$ . Then,  $g \in \text{Ann}(f) \subseteq P$  implies that  $f^2 + g^2 \in P$ . But,  $\text{int}Z(f^2 + g^2) = \emptyset$ , which is a contradiction. Now, suppose that  $h \in P$ ,  $k \in C(\mathbb{R})$  and  $(\text{Ann}(f) : h) \subseteq (\text{Ann}(f) : k)$ . Hence,  $\text{Ann}(fh) \subseteq \text{Ann}(fk)$  implies that  $fk \in P$ , by Proposition 1.2, for  $P$  is a  $z^\circ$ -ideal and  $fh \in P$ . Now,  $f \notin P$  implies that  $k \in P$ ; i.e.,

$\frac{P}{\text{Ann}(f)}$  is a  $z^\circ$ -ideal. In Counter-example 3.9 below, we will observe that if  $P$  is a prime  $z^\circ$ -ideal in  $C(X)$  containing  $\text{Ann}(f)$ , for some  $f \in C(X)$ , then  $\frac{P}{\text{Ann}(f)}$  is not always a  $z^\circ$ -ideal in  $\frac{C(X)}{\text{Ann}(f)}$ .

(4) Finally, if  $P$  is a prime ideal of a reduced ring  $R$ , then  $R/P$  has no nonzero  $z^\circ$ -ideal. In fact, every member of a  $z^\circ$ -ideal should be zerodivisor, whereas  $R/P$  has no nonzero zerodivisor.

In [18], it is shown that  $\text{Jac}(R) = 0$  if and only if every  $z^\circ$ -ideal of a reduced ring  $R$  is a  $z$ -ideal. This fact proves part (a) of the following proposition.

**Proposition 3.7.** (a) *If  $I$  is a semiprime ideal in  $C(X)$ , then every  $z^\circ$ -ideal of  $C(X)/I$  is a  $z$ -ideal if and only if  $I$  is closed.*

(b) *If  $I \subseteq J$  are ideals in a ring  $R$ ,  $I$  is a  $z$ -ideal ( $z^\circ$ -ideal) and  $J/I$  is a  $z^\circ$ -ideal in  $R/I$ , then  $J$  is a  $z$ -ideal ( $z^\circ$ -ideal).*

**Proof.** (b) Let  $M_b \subseteq M_a$  ( $P_a \subseteq P_b$ ), where  $a \in J$  and  $b \in R$ . We show that  $(I : a) \subseteq (I : b)$ . Let  $c \in (I : a)$ . Since  $M_{bc} \subseteq M_{ac}$  ( $P_{bc} \subseteq P_{ac}$ ),  $ac \in I$  and  $I$  is a  $z$ -ideal ( $z^\circ$ -ideal),  $bc \in I$ ; i.e.,  $c \in (I : b)$ . Now,  $J/I$  is a  $z^\circ$ -ideal. Then,  $b \in J$ . This shows that  $J$  is a  $z$ -ideal ( $z^\circ$ -ideal).  $\square$

In case  $I = P_f$ , for some  $f \in C(X)$ , the set of  $z^\circ$ -ideals of  $\frac{C(X)}{I}$  is exactly the set of all ideals of the form  $J/I$ , where  $J$  is a  $z^\circ$ -ideal in  $C(X)$  containing  $I$ .

**Proposition 3.8.** *If  $f \in C(X)$  is a zerodivisor, then  $J$  is a  $z^\circ$ -ideal in  $C(X)$  containing  $P_f$  if and only if  $J/P_f$  is a  $z^\circ$ -ideal in  $C(X)/P_f$ .*

**Proof.** If  $J/P_f$  is a  $z^\circ$ -ideal in  $C(X)/P_f$ , then by Proposition 3.7 (b),  $J$  is a  $z^\circ$ -ideal in  $C(X)$ . Conversely, let  $J$  be a  $z^\circ$ -ideal in  $C(X)$  containing  $P_f$ . Suppose that  $g \in J$ ,  $h \in C(X)$  and  $(P_f : g) \subseteq (P_f : h)$ . We must show that  $h \in J$ . Let  $h \notin J$ . Since  $f \in P_f \subseteq J$ ,  $f^2 + g^2 \in J$  and by Proposition 1.2,  $\text{int}Z(f) \cap \text{int}Z(g) \not\subseteq Z(h)$ . Take  $x \in \text{int}Z(f) \cap \text{int}Z(g) \setminus Z(h)$  and define  $t \in C(X)$  such that  $t(x) = 1$  and  $t(X \setminus \text{int}Z(g)) = \{0\}$ . Clearly,  $tg = 0$  implies that  $t \in (P_f : g)$ , but  $\text{int}Z(f) \not\subseteq Z(th)$  implies that  $t \notin (P_f : h)$ , which is a contradiction.  $\square$

**Counter-example 3.9.** (a) An example of two ideals  $I$  and  $J$  such that  $I$  is semiprime (a  $z$ -ideal),  $J/I$  is a  $z^\circ$ -ideal but  $J$  is not a  $z$ -ideal (a  $z^\circ$ -ideal): Consider a semiprime ideal (a  $z$ -ideal)  $I$  which is not a  $z$ -ideal (a  $z^\circ$ -ideal). By Proposition 1.1, there exists a minimal prime ideal  $P$

over  $I$  which is not a  $z$ -ideal (a  $z^\circ$ -ideal). Now,  $P/I$  is a  $z^\circ$ -ideal, but  $P$  is not a  $z$ -ideal ( $z^\circ$ -ideal).

(b) An example of two ideals  $I$  and  $J$  such that  $I$  is semiprime and  $J$  and  $J/I$  are  $z^\circ$ -ideals, but  $I$  is not a  $z$ -ideal: Let  $P_1$  and  $P_2$  be two prime ideals such that  $P_1$  is a  $z^\circ$ -ideal and  $P_2$  is not a  $z$ -ideal. Hence,  $I = P_1 \cap P_2$  is a semiprime ideal which is not a  $z$ -ideal. Since  $P_1$  is minimal over  $I$ ,  $\frac{P_1}{I}$  is a  $z^\circ$ -ideal, and so  $P_1$  and  $\frac{P_1}{I}$  are  $z^\circ$ -ideal, but  $I$  is not a  $z$ -ideal.

(c) An example of two  $z^\circ$ -ideal  $I$  and  $J$  such that  $J/I$  is not a  $z^\circ$ -ideal: Take a prime  $z^\circ$ -ideal  $P$  in  $C(X)$  which is not minimal (see [4], Proposition 1.26). Then, there exists  $f \in P$  such that  $\text{Ann}(f) \subseteq P$  (note that the prime ideal  $P$  is minimal if and only if  $\forall f \in P, \exists g \notin P$  such that  $fg = 0$ ). Now,  $\frac{P}{\text{Ann}(f)}$  is not a  $z^\circ$ -ideal in  $\frac{C(X)}{\text{Ann}(f)}$ . In fact,  $(\text{Ann}(f) : f) \subseteq (\text{Ann}(f) : k), \forall k \in C(X)$ , and  $f \in P$  imply that  $\frac{P}{\text{Ann}(f)}$  is not a  $z^\circ$ -ideal

Now, by Counter-example 3.9(c), we have the following result.

**Proposition 3.10.** *For every two  $z^\circ$ -ideals  $I \subseteq J$  in  $C(X)$ ,  $J/I$  is a  $z^\circ$ -ideal in  $C(X)/I$  if and only if every prime  $z^\circ$ -ideal in  $C(X)$  is minimal.*

**Proof.** First, suppose that every prime  $z^\circ$ -ideal of  $C(X)$  is minimal. Then,  $J$  is an intersection of minimal prime ideals. Since for every minimal prime ideal  $P$  containing  $I$ ,  $P/I$  is a  $z^\circ$ -ideal, it follows that  $J/I$  is a  $z^\circ$ -ideal. Next, suppose that for every two  $z^\circ$ -ideals  $I \subseteq J$ ,  $J/I$  is a  $z^\circ$ -ideal and let  $P$  be a prime  $z^\circ$ -ideal. If  $P$  is not minimal, then  $\exists f \in P$  such that  $\text{Ann}(f) \subseteq P$ . Now,  $\frac{P}{\text{Ann}(f)}$  is not a  $z^\circ$ -ideal, by Counter-example 3.9(c), but  $P$  and  $\text{Ann}(f)$  are both  $z^\circ$ -ideals in  $C(X)$ , which is a contradiction.  $\square$

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