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THE COMPLEMENT OF A D-TREE IS PURE SHELLABLE

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Abstract. Let G be a simple undirected graph and let Δ_G be a simplicial complex whose faces correspond to the independent sets of G. A graph G is called shellable if Δ_G is a shellable simplicial complex. We prove that the complement of a d-tree is a pure shellable graph. This generalizes a recent result of Ferrarello who used a theorem due to R. Fröberg to prove that the complement of a d-tree is a Cohen-Macaulay graph.

1. **Introduction**

Let G be a finite simple graph on n vertices $V(G) = \{x_1, ..., x_n\}$. One can then associate to G a quadratic square-free monomial ideal $I(G)$ in $R = k[x_1, ..., x_n]$ by setting $I(G) = (x_ix_j|\{x_i, x_j\} \in E(G))$, where $E(G)$ is the edge set of G. Besides to this algebraic object, a simplicial complex associates to a graph G which reflects many nice properties of the graph. The simplicial complex Δ_G of a graph G is defined by

$$\Delta_G = \{A \subseteq V | A \text{ is an independent set in } G\},$$

where A is an independent set in G if none of its elements are adjacent.

A simplicial complex Δ is called shellable if the facets (maximal faces) of Δ can be ordered as $F_1, \ldots, F_s$ such that for all $1 \leq i < j \leq s$, there

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exist some $v \in F_j \setminus F_i$ and some $l \in \{1, \ldots, j-1\}$ with $F_j \setminus F_l = \{v\}$; cf. [2]. We call $F_1, \ldots, F_s$ a shelling of Δ when the facets have been ordered with respect to the definition of shellable. An important property of a pure shellable complex is that it is Cohen-Macaulay over every field which is due to Hochster [6] (see also [1, Theorem 5.1.13]).
A graph G is called shellable, if the simplicial complex Δ_G is a shellable simplicial complex.

A recent theme in commutative algebra is to understand how the properties of G appear within the properties of $R/I(G)$, and vice versa. Cohen-Macaulay rings are of great interest to commutative algebraists. As a consequence, one particular stream of research has focused on the question of what graphs G have the property that $R/I(G)$ is Cohen-Macaulay. Although a solution to the general problem is probably intractable, Herzog and Hibi [5] gave a combinatorial description of those graphs G that are bipartite and Cohen-Macaulay (recall that a graph G on the vertex set $[n]$ is bipartite if there exists a partition $[n] = X \cup Y$, with $X \cap Y = \emptyset$, such that each edge of G is of the form $\{i, j\}$ with $i \in X$ and $j \in Y$).

We denote the complete graph on t vertices by K_t. The following inductive definition of d-trees is customary:

(i) K_{d+1} is a d-tree.
(ii) If G is a d-tree and v a new vertex, and v is adjoined to G via a sub-K_d of G, then $\{v\} \cup G$ is a d-tree.

Note that a 0-tree is a graph that has a set of vertices but without any edge, and 1-tree is a usual tree.

Recently, Ferrarello [3] showed that the complement of a d-tree is Cohen-Macaulay. To do this, she uses a theorem due to R. Fröberg [4] that establishes a condition for a Stanley-Reisner ring of a simplicial complex to be Cohen-Macaulay.

Here, we prove a generalization of Ferrarello's result by showing that the complement of a d-tree is a pure shellable graph and so it is Cohen-Macaulay.

2. Main result

First, we recall the definitions we use throughout the paper.

Definition 2.1. the complement of a graph G is a graph H on the same vertices such that two vertices of H are adjacent if and only if they are not adjacent in G.

Definition 2.2. A simplicial complex Δ over a set of vertices $V = \{v_1, ..., v_n\}$ is a collection of subsets of V, with the property that $\{v_i\} \in \Delta$ for all i, and if $F \in \Delta$, then all subsets of F are also in Δ (including the empty set). An element of Δ is called a face of Δ, and the dimension of a face F of Δ is defined as $|F| - 1$, where $|F|$ is the number of vertices of F. The faces of dimensions 0 and 1 are called vertices and edges, respectively, and dim $\emptyset = -1$.
The maximal faces of Δ under inclusion are called facets of Δ. We denote the simplicial complex Δ with facets $F_1, ..., F_t$ by

$$\Delta = \langle F_1, ..., F_t \rangle,$$

and we call $\{F_1, ..., F_t\}$ the facet set of Δ. A simplicial complex is called pure if all its facets have the same cardinality. A pure simplicial complex Δ is called shellable if the facets of Δ can be ordered as $F_1, \ldots, F_s$ such that for all $1 \leq i < j \leq s$, there exist some $v \in F_j \setminus F_i$ and some $l \in \{1, \ldots, j-1\}$ with $F_j \setminus F_l = \{v\}$. We call $F_1, \ldots, F_s$ a shelling of Δ when the facets are been ordered with respect to the definition of shellable.

Lemma 2.3. *Let G be a simple graph and $F \subseteq V(G)$. Then, F is a facet of $\Delta_{\overline{G}}$ if and only if the induced subgraph of G on the vertex set F is a maximal complete subgraph of G.*

Proof. First, let F be a facet of $\Delta_{\overline{G}}$; i.e., F is a maximal independent subset of $V(\overline{G})$. For any $v, v' \in F$, we have $\{v, v'\} \notin E(\overline{G})$, and thus $\{v, v'\} \in E(G)$. So, the induced subgraph of G on the vertex set F is complete. The maximality of this induced subgraph follows from the maximality of F in $\overline{G}$.
Conversely, if F is a maximal complete subgraph of G, then no two vertices of F are adjacent in $\overline{G}$; i.e., F is an independent subset of $V(\overline{G})$ and hence is a face of $\Delta_{\overline{G}}$. The maximality again follows from the maximality of F as a maximal complete subgraph of G. □

Now, we are ready to state and prove our main result.

Theorem 2.4. *The complement of a d-tree is pure shellable.*

Proof. Let G be a d-tree with the vertex set $V(G) = \{v_1, ..., v_n\}$. By the definition of d-tree, all the maximal complete subgraphs of G have $d+1$ vertices. It follows from Lemma 2.3 that $\Delta_{\overline{G}}$ is pure with $\dim \Delta_{\overline{G}} = d$.

By the inductive structure of d-tree, we may assume

$$G = \bigcup_{i=1}^{s} G_i,$$

where G_i is a complete subgraph of G isomorphic to K_{d+1}(the complete graph over $d+1$ vertices), for all $i = 1, ..., s$, with the following properties:

(1) $V(G_1) = \{v_1, ..., v_{d+1}\}$,

(2) $V(G_i) = V(G'_i) \cup \{v_{d+i}\}$, where G'_i is a complete subgraph of $\bigcup_{t=1}^{i-1} G_t$ isomorphic to K_d.

It is easy to see that $s = n - d$. Let $F_i = V(G_i)$ for all $i = 1, ..., s$. It follows from Lemma 2.3 that

$$\Delta_{\overline{G}} = \langle F_1, ..., F_s \rangle.$$

For all $1 \leq i < j \leq s$, we have $v_{d+j} \in F_j \setminus F_i$. Note that $V(G_j) = V(G'_j) \cup \{v_{d+j}\}$ and G'_j is a complete subgraph of $\bigcup_{t=1}^{j-1} G_t$ isomorphic to K_d. So, there exists a maximal complete subgraph of $\bigcup_{t=1}^{j-1} G_t$, say H, such that G'_j is a subgraph of H. But the only maximal complete subgraphs of $\bigcup_{t=1}^{j-1} G_t$ are G_t for $t = 1, ..., j-1$. Therefore, G'_j is a subgraph of G_l for some $l < j$, and hence $V(G'_j) \subseteq V(G_l) = F_l$. It is clear that

$$F_j \setminus F_l = V(G_j) \setminus V(G_l) = (V(G'_j) \cup \{v_{d+j}\}) \setminus V(G_l) = \{v_{d+j}\}.$$

□

It is known that any pure shellable simplicial complex is Cohen-Macaulay; i.e., I_Δ is a Cohen-Macaulay ideal; cf. [1, Theorem 5.1.13]. This fact together with Theorem 2.4 implies the next result.

Corollary 2.5. ([3, Theorem 3.3]) *The complement of a d-tree is Cohen-Macaulay.*

Proof. Let G be a d-tree. Notice that $I(\overline{G}) = I_{\Delta_{\overline{G}}}$ and $\Delta_{\overline{G}}$ is pure shellable, and so $I_{\Delta_{\overline{G}}}$ is Cohen-Macaulay. □

It is natural to ask about the converse of Corollary 2.5. In the following we give an answer to this question in the case of chordal graphs. Recall that a graph G is called chordal if every cycle of length greater than 3 of G has a chord.

Proposition 2.6. *Let G be a chordal graph. The followings are equivalent.*

(i) *G is a d-tree.*
(ii) *$\overline{G}$ is a pure shellable graph.*
(iii) *$\overline{G}$ is a Cohen-Macaulay graph.*

Proof. (i)⇒(ii): this follows from Theorem 2.4.
(ii)⇒(iii): this follows from [1, Theorem 5.1.13].
(iii)⇒(i): this follows from [4, Corollary Page 63]. □

Example 2.7. The star graph S_d of order d is a tree on d vertices with one vertex having vertex degree $d-1$ and the other $d-1$ vertices having vertex degree 1. The complement of S_d has two connected components; a complete graph K_{d-1} and an isolated vertex. It is easy to see that a graph is pure shellable if and only if its connected components are pure shellable. Thus, a complete graph is pure shellable. (Note that we can use 0-tree to show that a complete graph is pure shellable. Let G be a 0-tree with d vertices. Then, the complement of G is the complete graph K_d.)

Example 2.8. Let G be the complete graph K_{d+1}. Take H as a graph obtained by connecting a new vertex of degree 1 to each vertex of G. Since the complement of H is a d-tree, we have that H is a pure shellabe graph.

References

[1] W. Bruns and J. Herzog, *Cohen-Macaulay rings*, Cambridge Studies in Advanced Mathematics **39**, revised edition, 1998.

[2] A. Björner and M. Wachs, *Shellable nonpure complexes and posets I*, Trans. Amer. Math. Soc. **348** (1996) 1299-1327.

[3] D. Ferrarello, *The complement of a d-tree is Cohen-Macaulay*, Math. Scand. **99**(2) (2006) 161-167.

[4] R. Fröberg, *On Stanley-Reisner rings*, Topics in algebra, (2) (Warsaw, 1988) 57–70, Banach Center Publ. **26**(2), PWN, Warsaw, 1990.

[5] J. Herzog and T. Hibi, *Distributive lattices, bipartite graphs and Alexander duality*, J. Algebraic Combin. **22** (2005) 289-302.
[6] M. Hochster, *Rings of invariants of tori, Cohen-Macaulay rings generated by monomials, and polytopes*, Ann. of Math. **96** (1972) 318–337.

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