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ESTIMATION OF THE MULTIVARIATE NORMAL MEAN UNDER THE EXTENDED REFLECTED NORMAL LOSS FUNCTION

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Abstract: This paper considers simultaneous estimation of multivariate normal mean vector using the extended reflected normal loss function (Spiring [9]). It is shown that the sample mean $\overline{X} = (\bar{X}_1, \dots, \bar{X}_p)'$ is admissible when $p \leq 2$, but for $p \geq 3$, we obtain a class of estimators similar to James-Stein estimators which dominate the sample mean in terms of risks.

1. Introduction

Let $X = (X_1, \dots, X_p)'$ be a normal vector with mean vector $\theta = (\theta_1, \dots, \theta_p)'$ and covariance matrix $\sigma^2 I$, where σ^2 is known. We use the notation
$X \sim N_p(\theta, \sigma^2 I)$, in this article. We consider the simultaneous estimation of $\theta = (\theta_1, \dots, \theta_p)'$ by using a random sample $X_1, \dots, X_N$ from $N_p(\theta, \sigma^2 I)$ under the extended reflected normal loss function, given by

$$L(\delta, \theta) = K \left[1 - exp\{-(\delta - \theta)'\Gamma^{-1}(\delta - \theta)\}\right] \tag{1.1}$$

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where $K > 0$, Γ is a constant positive definite matrix. In practice the maximum loss can be a function of many things (e.g., Production resources, cost of identification, rework and liabilities) but generally it is finite. As a result the quadratic loss function, with its infinite maximum loss, is often inadequate in describing the loss function associated with a product and has been criticized by some researchers (e.g., Tribus and Szonyi [13], Leon and Wu [8]). The bounded loss function (1.1) was introduced by Spiring [9] for the first time. This loss is a bounded and increasing function of the quadratic loss.

To estimate θ with $N = 1$ and $\sigma = 1$, Stein [10] showed that X is inadmissible when $p \geq 3$ under squared error loss. James and Stein [7] showed that the following estimator, known as J-S estimator,

$$\delta(X) = \left(1 - \frac{p-2}{\sum_{i=1}^{p} X_i^2}\right) X$$

has uniformly smaller risk than X, for all θ. Strawderman [12], Efron and Morris [6], and Casella and Hwang [4] studied the problem of estimating multivariate normal mean vector under quadratic loss function. Brandwein and Strawderman [3] provided minimax estimators for the mean of a spherically symmetric distribution with concave loss. Chung and Kim [5] investigated the admissibility of the sample mean $\bar{X}$ under balanced loss function. (see Zellner [14])

In section 2 of this paper, using the limiting Bayes method, we show that $\bar{X}$ is admissible when $p \leq 2$ under the loss (1.1). In section 3, we obtain an estimator similar to J-S estimator under the loss (1.1) when $p \geq 3$, in the following form

$$\delta^*(\bar{X}) = \left(1 - \frac{c^*}{\bar{X}'\Gamma^{-1}\bar{X}}\right) \bar{X}$$

and we show that δ^* dominates the usual estimator $\bar{X} = \frac{1}{N}\sum_{i=1}^{N} X_i = (\bar{X}_1, \ldots, \bar{X}_p)'$, where $X_i = (X_{i1}, \ldots, X_{ip})'$ and $\bar{X}_j = \frac{1}{N}\sum_{i=1}^{N} X_{ij}; j = 1, \ldots, p$.

2. Admissibility of $\bar{X}$ when $p \leq 2$

In this section, we consider the admissibility of $\bar{X}$ when $p = 1$ and 2. We show that $\bar{X}$ is admissibile, using the standard Blyth's technique [2].

Let $X_1, \ldots, X_N$ be a random sample from $N_p(\theta, I)$ with the prior normal distribution $\pi_a(\theta)$, where θ has the mean vector zero and covariance matrix $\frac{1}{a}I$. It is easy to show that the Bayes estimator of θ w.r.t. $\pi_a(\theta)$ under the extended reflected normal loss function is

$$\delta_a(\bar{X}) = \frac{N\bar{X}}{N+a}$$

with the risk function,

$$\begin{aligned} R(\theta, \delta_a) &= K\left[1 - E\left[exp\left\{-(\frac{N\bar{X}}{N+a} - \theta)'\Gamma^{-1}(\frac{N\bar{X}}{N+a} - \theta)\right\}\right]\right] \\ &= K[1 - (\frac{2\pi}{N})^{-\frac{p}{2}} \int exp\{-(\frac{Nx}{N+a} - \theta)'\Gamma^{-1}(\frac{Nx}{N+a} - \theta) \\ &- \frac{N}{2}(x-\theta)'(x-\theta)\}dx] \end{aligned}$$

Now using the fact that for any matrices C_1 and C_2 of appropriate dimensions,

$$(C_1 + C_2)^{-1} = C_1^{-1} - C_1^{-1}(C_1^{-1} + C_2^{-1})^{-1}C_1^{-1} \tag{2.1}$$

it follows that the risk function of the estimator δ_a is equal to

$$K[1 - (\frac{2\pi}{N})^{-\frac{p}{2}}(\frac{N+a}{N})^p \int exp[-\frac{1}{2}\{(y-\eta)'(2\Gamma^{-1} + \frac{(N+a)^2}{N}I)(y-\eta)$$
$$+(\frac{a}{N+a})^2\theta'(\frac{1}{2}\Gamma + \frac{N}{(N+a)^2}I)^{-1}\theta\}]dy]$$

or

$$K\left[1 - \frac{(N+a)^p}{N^{p/2}}|2\Gamma^{-1} + \frac{(N+a)^2}{N}I|^{-\frac{1}{2}} exp\left\{-\frac{1}{2}(\frac{a}{N+a})^2\theta'(\frac{1}{2}\Gamma + \frac{N}{(N+a)^2}I)^{-1}\theta\right\}\right] \tag{2.2}$$

where η is a function of θ.

Theorem 2.1: *$\bar{X} = (\bar{X}_1, \ldots, \bar{X}_p)'$ is admissible under the loss (1.1) when $p = 1, 2$, where $\bar{X}_j = \frac{1}{N}\sum_{i=1}^{N} X_{ij}, j = 1, \ldots, p$.*

Proof: Suppose $\bar{X}$ is dominated by some estimator $\delta(\bar{X})$ of θ. Using the continuity of the risk function in θ for an estimator $\delta(\bar{X})$, it follows that there exists some $\theta_0, \epsilon > 0$ and $\xi > 0$ such that

all $R(\theta, \delta) < R(\theta, \bar{X}) - \epsilon$ for $\theta_0 - \xi 1 < \theta < \theta_0 + \xi 1$

where $1 = (1, 1, \ldots, 1)'$.

Let r_a, r_a^*, r_a^{**} be defined as follows:

r_a = Bayes risk of the Bayes solution δ_a w.r.t. π_a.

r_a^* = Bayes risk of $\bar{X}$ w.r.t. π_a.

r_a^{**} = Bayes risk of δ w.r.t. π_a.

Then the difference of Bayes risks of $\bar{X}$ and δ is

$$\begin{aligned} r_a^* - r_a^{**} &\geq \int_{\theta_0 - \xi 1}^{\theta_0 + \xi 1} \left[R(\theta, \bar{X}) - R(\theta, \delta)\right] \pi_a(\theta) d\theta \\ &\geq \int_{\theta_0 - \xi 1}^{\theta_0 + \xi 1} \epsilon (2\pi)^{-\frac{p}{2}} |\frac{1}{a} I|^{-\frac{1}{2}} exp(-\frac{a}{2}\theta'\theta) d\theta \\ &\geq c a^{\frac{p}{2}} \end{aligned}$$

The last inequality holds for all $a < 1$, where c is a positive constant not depending on a.

Also, using (2.2), the difference of Bayes risks of $\bar{X}$ and δ_a is

$$\begin{aligned} r_a^* - r_a &= K\{\frac{(N+a)^p}{N^{p/2}}[|2\Gamma^{-1} + \frac{(N+a)^2}{N} I|| \frac{a}{(N+a)^2}(\frac{1}{2}\Gamma + \frac{N}{(N+a)^2} I)^{-1} + I|]^{-\frac{1}{2}} \\ &- N^{\frac{p}{2}} |NI + 2\Gamma^{-1}|^{-\frac{1}{2}}\} \\ &= K\{\frac{(N+a)^p}{N^{p/2}} |(\frac{a}{N} + 1)(2\Gamma^{-1} + \frac{(N+a)^2}{N} I) - \frac{a(N+a)^2}{N^2} I|^{-\frac{1}{2}} \\ &- N^{\frac{p}{2}} |NI + 2\Gamma^{-1}|^{-\frac{1}{2}}\} \\ &= K\left\{(N+a)^{p/2} |2\Gamma^{-1} + (N+a) I|^{-\frac{1}{2}} - N^{p/2} |NI + 2\Gamma^{-1}|^{-\frac{1}{2}}\right\} \end{aligned}$$

The second equality is carried out by using the relation (2.1). It can easily be verified that for $p = 1$, the ratio $\frac{r_a^* - r_a^{**}}{r_a^* - r_a}$ tends to infinity as

$a \to 0$ and for $p = 2$, this ratio tends to a positive constant as $a \to 0$. Hence, there exists an $a > 0$ such that $r_a^{**} < r_a$ which contradicts the fact that δ_a is a Bayes solution with respect to π_a. Therefore $\bar{X}$ is for $p = 1, 2$. ■

3. Inadmissibility of $\bar{X}$ for $p \geq 3$

In this section, we consider estimation of $\theta = (\theta_1, \ldots, \theta_p)'$ from the model of section 1 under the loss (1.1) and find a class of estimators which have uniformly smaller risk than $\bar{X}$ for $p \geq 3$.

Lemma 3.1: *Let $X = (X_1, \ldots, X_p)'$ be distributed as $N_p(\theta, I)$. If $h : \Re^p \to \Re$ is an almost differentiable function with $E\|\nabla h(X)\| < \infty$, then*

$$E[\nabla h(X)] = E[(X - \theta)h(X)]$$

, where $\nabla h(x) = \left(\frac{\partial h(x)}{\partial x_1}, \ldots, \frac{\partial h(x)}{\partial x_p}\right)'$.

Proof: See Stein [11]. ■

Theorem 3.1: *Let the positive values $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_p$ be the eigenvalues of the matrix Γ. If the estimator δ^c is defined as*

$$\delta^c(\bar{X}) = \left(1 - \frac{c}{\bar{X}'\Gamma^{-1}\bar{X}}\right)\bar{X}$$

where $0 < c < c^*$,$c^* = 2\left[\sum_{i=2}^{p} \frac{1}{2+N\lambda_i} - \frac{1}{2+N\lambda_1}\right]$,then $\delta^c(\bar{X})$ dominates $\bar{X}$ in terms of risks under the extended reflected normal loss function (1.1) for $p \geq 3$, when $c^* > 0$.

bf Proof: For any estimator $\delta(\bar{X})$, we define a function g as

$$g(\theta, \delta) = E\left[exp\left\{-(\delta(\bar{X}) - \theta)'\Gamma^{-1}(\delta(\bar{X}) - \theta)\right\}\right]$$

and show that for all $\theta, g(\theta,\delta^c)\geq g(\theta,\bar{X})$. We observe that

$$\begin{aligned} g(\theta,\delta^c) &= E\left[e^{-(\bar{X}-\theta)'\Gamma^{-1}(\bar{X}-\theta)}e^{-\frac{c^2}{\bar{X}'\Gamma^{-1}\bar{X}}+2c(\bar{X}-\theta)'\frac{\Gamma^{-1}\bar{X}}{\bar{X}'\Gamma^{-1}\bar{X}}}\right] \\ &\geq E\left[e^{-(\bar{X}-\theta)'\Gamma^{-1}(\bar{X}-\theta)}\left\{1-\frac{c^2}{\bar{X}'\Gamma^{-1}\bar{X}}+2c(\bar{X}-\theta)'\frac{\Gamma^{-1}\bar{X}}{\bar{X}'\Gamma^{-1}\bar{X}}\right\}\right] \end{aligned} \tag{3.1}$$

This inequality follows using the fact that

$e^{-x}\geq 1-x \qquad \forall x\in\Re$

Now by defining $\Sigma^{-1}=2\Gamma^{-1}+NI, A=[a_{ij}]_{p\times p}=\Sigma^{1/2}\Gamma^{-1}\Sigma^{1/2}, Y=(Y_1,\ldots,Y_p)'=\Sigma^{-\frac{1}{2}}\bar{X}$ and $\beta=\Sigma^{-\frac{1}{2}}\theta$, the inequality (3.1) reduces to

$$g(\theta,\delta^c)\geq g(\theta,\bar{X})-N^{\frac{p}{2}}|\Sigma|^{\frac{1}{2}}\left\{E\left[\frac{c^2}{Y'AY}\right]-2cE\left[(Y-\beta)'\frac{AY}{Y'AY}\right]\right\} \tag{3.2}$$

where Y is distributed as $N_p(\beta,I)$.

Note that by using lemma 3.1, it follows that

$$\begin{aligned} E\left[(Y-\beta)'\frac{AY}{Y'AY}\right] &= E\left[\sum_{i=1}^{p}\frac{\partial}{\partial Y_i}\frac{\sum_{j=1}^{p}a_{ij}Y_j}{\sum_i\sum_j a_{ij}Y_iY_j}\right] \\ &= E\left[\frac{(\sum_i a_{ii})(\sum_i\sum_j a_{ij}Y_iY_j)-2\sum_i(\sum_j a_{ij}Y_j)^2}{(\sum_i\sum_j a_{ij}Y_iY_j)^2}\right] \\ &= E\left[\frac{tr(A)}{Y'AY}-\frac{2Y'A^2Y}{(Y'AY)^2}\right] \end{aligned}$$

and

$$\begin{aligned} &-E\left[\tfrac{c^2}{Y'AY}\right]+2cE\left[(Y-\beta)'\tfrac{AY}{Y'AY}\right] \\ &\qquad = E\left\{\frac{Y'[(-c^2+2ctr(A))A-4cA^2]Y}{(Y'AY)^2}\right\} \end{aligned} \tag{3.3}$$

We know that A is a positive definite matrix and is diagonable as $U'AU=T=diag\{t_1,\ldots,t_p\}$, where the positive values $t_1,\ldots,t_p$ are the eigenvalues of **A**. Now, we have $U'A^2U=T^2=diag\{t_1^2,\ldots,t_p^2\}$ and

therefore (3.3) reduces to
$-E\left[\frac{c^2}{Y'AY}\right] + 2cE\left[(Y-\beta)'\frac{AY}{Y'AY}\right]$

$$= E\left\{\frac{Y'U[(-c^2+2ctr(A))T-4cT^2]U'Y}{(Y'AY)^2}\right\}$$

According to (3.2), we complete the proof by showing that the matrix

$$(-c^2+2ctr(A))T-4cT^2 = diag\{ct_1(-c+2tr(A)-4t_1),\dots,ct_p(-c+2tr(A)-4t_p)\} \tag{3.4}$$

is positive definite when $0 < c < c^*$.
It can be verified that $t_i = \frac{1}{N\lambda_i+2}; i = 1,\dots,p$, where the values $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_p$ are the eigenvalues of Γ. Hence, the diagonal elements of the diagonal matrix (3.4) is positive when

$$0 < c < 2tr(A) - \frac{4}{N\lambda_i+2} \quad i = 1,\dots,p$$

This condition is equivalent to $0 < c < c^*$ with $c^* = 2\left[\sum_{i=2}^{p}\frac{1}{2+N\lambda_i} - \frac{1}{2+N\lambda_1}\right]$.
■

Corollary 3.1: *Let the estimator $\delta^*(\bar{X})$ be given as*

$$\delta^*(\bar{X}) = \left(1 - \frac{p-2}{(N+2)\bar{X}'\bar{X}}\right)\bar{X}$$

Now, $\delta^*(\bar{X})$ dominates $\bar{X}$ under the loss function (1.1) with $\Gamma = I$, for $p > 2$. This estimator is similar to J-S estimator.

Conclusions 3.1: Let the estimator $\delta^*(\bar{X})$ be given as

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