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ON NONLINEAR WATER WAVES IN A CHANNEL

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Abstract: This paper is concerned with some approximate equations for the study of nonlinear water waves in a channel of variable cross section. A system of shallow water equations for finite amplitude waves is given and a Korteweg deVries (KdV) equation with variable coefficients for samll amplitude waves is also presented.

1. Introduction

One of the interesting problems of water waves in a sloping channel concerns the breaking of a wave moving toward a shoreline, the development of a bore, and the movement of the shoreline after the bore reaches it. For the two dimensional case corresponding to a rectangular channel of variable depth, the bore run-up problem was studied by Keller et al [1] on the basis of shallow water equations [2]. Later Gurtin [3] derived a criterion for the breaking of an acceleration wave in a two dimensional

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channel and his result was extended by Jeffery and Mvungi [4] to the case of a rectangular channel of variable width depth. We generalize Gurtin's result to predict the breaking point of an acceleration wave in a channel of variable cross section and review some existent results regarding the bore run-up problem for a rectangular channel with a uniformly sloping bottom. Up to date, the shallow water equation for a two dimensional channel with analytical initial data have been justified by Kano and Nashida [5] and for the three dimensional case with a priori assumptions on the free surface by Berger [6]. At present we may accept shallow water equations as model equations.

Another application of our results deals with the development of a solitary wave in a channel of variable cross section. Recently, there have been discussions on the so called infinite mass dilemma, which arises from the formation of a shelf behind the solitary wave. If the shelf were extended to infinity, then infinite mass would be created or annulled by a perturbation of the solitary wave. We shall establish a global existence theorem for the solution of the KdV equation for a general channel as a consequence of the existence results due to Kato [7]. It follows that the shelf, if formed behind the solitary wave in a general channel, can only be finite. A rigorous justification of the validity of the KdV equation here should be an important contribution to the theory of water waves.

2. Shallow Water Equations and the Breaking of a Wave

We consider the irrotational motion of an inviscid, incompressible fluid of constant density under gravity in a channel with a boundary defined by $h^*(x^*, y^*, z^*) = 0$, where z^* is positive upward and x^* is in the longitudinal direction (figure). The governing equations are

$$\nabla^*.\overline{q}^* = 0 \tag{2.1}$$

$$\nabla^* \wedge \overline{q}^* = 0 \tag{2.2}$$

$$\rho(\overline{q}^*_{t^*} + \overline{q}^*.\nabla^* q^*) = -\nabla^* p^* + \overline{g} \tag{2.3}$$

subject of the boundary conditions

$$n^*_{t^*} + \overline{g}^* + \nabla^* \xi^* = 0 \tag{2.4}$$

at

$$\xi^* = -\xi^* + \eta^*(t^*, x^*, y^*), \quad \rho^* = 0 \tag{2.5}$$

$$\overline{q}^* + \nabla^* h^* = 0 \quad \text{at} \quad h^* = 0 \tag{2.6}$$

Here

$$\nabla^* = \left(\frac{\partial}{\partial x^*}, \frac{\partial}{\partial y^*}, \frac{\partial}{\partial z^*} \right), \quad \overline{q}^* = (u^*, v^*, w^*)$$

is the velocity, t^* is the time, $\overline{g} = (0, 0, -g)$ is the constant gravitational acceleration, ρ is the constant density, $\rho*$ is the pressure, and $z^* = \eta^*$ is the equation of the free surface. To derive the shallow water equations, we make the following assumptions. The channel boundary is convex, sufficiently smooth, and varies slowly in the longitudinal direction; the magnitude of the transverse velocities is much smaller than that of the longitudinal velocity. As suggested by Friedrichs [8], we introduce non dimensional variables:

$$t = \frac{1}{\sqrt{B}} \frac{t^*}{\sqrt{h/g}}, \quad (x, y, z) = \left(\frac{1}{\sqrt{B}} \frac{x^*}{H}, \frac{y^*}{H}, \frac{z^*}{H} \right),$$

$$\eta = \left(\frac{\eta^*}{H} \right), \; h = \left(\frac{h^*}{H} \right), \; (u, v, w) = \left(\frac{u^*}{\sqrt{gH}}, \frac{\sqrt{\beta} v^*}{\sqrt{gH}}, \frac{\sqrt{\beta} w^*}{\sqrt{gH}} \right)$$

where $\sqrt{B} = \left(\frac{L}{H}\right)$ and 'L' and 'H' are respectively the horizontal and transverse length scales. In terms of them (2.1) to (2.6) become

$$u_x + v_y + w_z = 0 \tag{2.7}$$

$$\beta u_y = v_x, \; u_z = w_x, \; v_z = w_y \tag{2.8}$$

$$u_t + u u_x + v u_y + w u_z + p_x = 0 \tag{2.9}$$

$$v_t + uv_x + vv_y + wv_z + \beta p_y = 0 \tag{2.10}$$

$$w_t + uw_x + vw_y + vw_z + \beta(p_z + 1) = 0 \tag{2.11}$$

$$\eta_t + u\eta_x + v\eta_y - w = 0, \quad \text{at} \quad z = \eta \tag{2.12}$$

$$p = 0 \tag{2.13}$$

$$uh_x + vh_y + wh_z = 0 \quad \text{at} \quad h = 0 \tag{2.14}$$

Assume that u, v, w and β possess an asymptotic expansion of the form

$$\phi \sim \phi_0 \beta^{-1} \phi_1 + \beta^{-2} \phi_2 + \ldots \tag{2.15}$$

Substiute (2.15) into (2.7) to (2.14). The equations for the zeroth order approximation are

$$u_{0x} + v_{0y} + w_{0z} = 0 \tag{2.16}$$

$$u_{0y} = u_{0z} = 0 \tag{2.17}$$

$$u_{0t} + u_0 u_{0x} + p_{0x} + v_0 u_{0y} + w_0 u_{0z} = 0 \tag{2.18}$$

$$p_0 y = 0, \quad p_0 z = -1 \tag{2.19}$$

$$\eta_{0t} + u_0 \eta_{0x} + v_0 \eta_{0y} - w_0 = 0 \quad \text{at} \quad z = 0 \tag{2.20}$$

$$p_0 = 0 \tag{2.21}$$

$$u_0 h_x + v_0 h_y + w_0 h_z = 0 \quad \text{at} \quad h = 0 \tag{2.22}$$

Seen from (2.17), (2.19) and (2.21), u_0 is a function of (t,x) only and

$$p_0 = (-z + \eta_0) \tag{2.23}$$

This implies η is also a function of (t,x) only. It follows from (2.17), (2.18) and (2.23) that

$$u_{0t} + u_0 u_{0x} + \eta_{0x} = 0 \tag{2.24}$$

Now we integrate (2.16) over a corss section D of the channel, apply the divergence theorem and make use of (2.20) and (2.22) to obtain:

$$\int_D (v_{0y}+w_{0z})dydz = -u_{0x}A(t,x) = -u_0 \int_\Gamma h_x(h_y^2+h_z^2)^{-1/2}ds+(\eta_{0t}+u_0\eta_{0x})B(t,x)$$

Rearranging the terms, we have

$$\eta_{0t} + u_0\eta_{0x} + u_{0x}\frac{A(t,x)}{B(t,x)} - \left[\frac{u_0}{B(t,x)}\right]\int_\Gamma h_x(\sqrt{h_y^2+h_z^2})^{-1}ds = 0 \tag{2.25}$$

where $A(t,x)$ is the area, $B(t,x)$ is the width and 'Γ' is the wetted boundary of the cross section D (figure). (2.24) and (2.25) form a system of nonlinear equations, which may be used to model bore formation and its subsequent devlopment in a channel of variable corss seciton.

In the following we extend Gurtin's method to the case of a general channel. The assumptions made are the following:

1) u_0, η_0 are continuous.

2) The first and the second derivatives of u_0 and η_0 possess at most jump discontinuities.

3) $u_0 = \eta_0 = 0$ ahead of the wave.

Denote the value of a function 'f' immediately behind the wave front by $\overline{f}$. Thereafter we also drop the subscript '0', from assumptions (1), (2), we have

$$\overline{u} = \overline{\eta} = 0 \tag{2.26}$$

By total differentiation

$$\overline{u}_t = -c\overline{u}_x, \quad \overline{\eta}_1 = -c\overline{\eta}_x \tag{2.27}$$

Where 'c' is the speed of the wave front. From (2.24), (2.25) and (2..26) it follows that:

$$overline\overline{u}_1 + \overline{\eta}_x = 0, \quad \overline{\eta}_t + \frac{\overline{u}_x\ A}{\overline{B}} = 0 \tag{2.28}$$

Comparing (2.27) and (2.28), we have:

$$c = \sqrt{\frac{\overline{A}}{\overline{B}\overline{u}_t = c^{-1}\overline{\eta}_x,}} \tag{2.29}$$

Now we differentiate (2.24) with respect to 't' and (2.25) with respect to x, and evaluate the equations behind the wave front. Then we eliminate $\overline{\eta}_{tx}$ and make use of the expression

$$c^2\overline{u}_{xt} - \overline{u}_{tt} = c^2\frac{d}{dx}(\overline{u}_x) - \frac{d}{dx}(\overline{u}_t)$$

to obtainq

$$-2c\frac{d}{dx}(\overline{\eta}_x)^{-1} + (\overline{\eta}_x)^{-1}[c^t - \overline{I}_1(\overline{B}c)] + \frac{3}{c} = 0$$

where

$$\overline{I}_1 = \int_\Gamma \frac{h_x}{\sqrt{h_y^2 + h_z^2}}ds$$

Hence

$$\overline{\eta}_x = \frac{a_0}{\sqrt{c}}\left[(\frac{3}{2}a_0\int_{x_0}^{x} c^{-5/2}\exp\int_{x_0}^{x'}\overline{I}_1(2\overline{A})^{-1}dx'dx + 1\right]^{-1} x\exp\int_{x_0}^{x}\overline{I}_1(2\overline{A})^{-1}dx \tag{2.30}$$

where a_0 is the initial value of η_x at $x = x_0$. We call $x = l$ a shoreline of $\overline{A}(l) = 0$ but $\overline{B}(l) \neq 0$, and let

$$I(x) = \frac{3}{2}\int_{x_0}^{x}\exp(-5/2)\exp\int_{x_0}^{x'} I_1(2\overline{A})^{-1}dx'dx$$

Suppose $a_0 < 0$. If $I(l) = \infty$, then $\overline{\eta}_x$ and the wave breaks before it reaches the shoreline. If $I(l) \neq \infty$, then either the wave breaks before it reaches the shoreline or it breaks at the shoreline. Next suppose

$a_0 > 0, I(l) \neq \infty$, then the wave breaks at the shoreline. Otherwise if $I(l) = \infty$, evaluate the limit of $\overline{\eta}_x$ given by (2.30) as $x \longrightarrow l$ and obtain

$$\lim_{x \longrightarrow l} \overline{\eta}_x = \frac{2}{3} \int - \left(\frac{(\overline{d})'}{4} + \frac{\overline{I_1}}{2\overline{B}} \right)_{x=l} \tag{2.31}$$

Here $\overline{d} = (\overline{A}/\overline{B})$, hence the wave will never break if $(\overline{d})'$ is finite at $x = l$. However for the channels of variable cross section the equilibrium for surface may converge to a point and this case is also of interest. Assume again $(a_0 > 0)$, if $I(l) = \infty$. If $\overline{B}(l) = \overline{d}(l)$ and $(\overline{d})'$ is finite at $x = l$, we assume $h(x, y, z) = -z + g(x, y)$

$$\overline{I}_1 = \int_\Gamma h_x (h_y^2 + h_z^2)^{-1/2} ds = \int_{-b_1}^{b_2} g_x dy$$

Here $y = -b_1, b_2$ are the end points of the width $\overline{B}(x)$. It follows from (2.31) that

$$\lim \overline{\eta}_x = (\frac{2}{3}) \left[- \left(\frac{(\overline{d})'}{4} + \frac{gx}{2} \right) \right]_{x=l}$$

then the wave will never break.

3. Run-up Problem

We consider a bore propagating towards a shoreline in a rectangular channel with a uniformly slopping bottom. On the basis of the shallow water equations, we can find a fairly complete solution of the bore run up problem. The bore path at the point of breaking to the shoreline may be approximately determined by Whitham's rule [10]. Here we shall consider the movement of shoreline after the bore reaches to shore. The shallow water wave equations for a rectangular channel of variable depth are obtained from (2.24) and (2.25) as

$$u_t + u u_x + \eta_x = 0 \tag{3.1}$$

$$\eta_x + [u(\eta + d_0)]_x = 0 \tag{3.2}$$

Here we also drop the superscripts of u and η and $d_0 = -\gamma x, \gamma > 0$. We assume $t = 0$ when the bore reaches the shoreline $x = 0$. Let

$$c^2 = \eta + d_0 \tag{3.3}$$

$$\alpha = 2c + u + yt = u^0, \quad \beta = 2c - u - \gamma t + u^0 \tag{3.4}$$

In terms of 'a' and 'B', (3.1) and (3.2) can be expressed as

$$x_\alpha = (u - c)t_a, \quad x_\beta = (u + c)t_\beta \tag{3.5}$$

By cross differentiation of equation (3.5) and making use of (3.4), we have

$$t_{\alpha\beta} + \frac{3(t_\alpha + t_\beta)}{[2(\alpha + \beta)]} = 0 \tag{3.6}$$

If we introduce the canonical variables

$$a = (\alpha + \beta)^{3/2} t_\alpha, \quad b = (\alpha + \beta)^{3/2} t_\beta \tag{3.7}$$

(3.6) yields as system of equations

$$(\alpha + \beta)a_\beta = -\frac{3b}{2}, \quad (\alpha + \beta)b_\alpha = -\frac{3a}{2} \tag{3.8}$$

Let

$$Y = a + b, \quad Z = a - b \tag{3.9}$$

If follows from (3.8) that

$$Y_{\alpha\beta} = \frac{15Y}{[4(\alpha + \beta)^2]}, \quad Z_{\alpha\beta} = \frac{3z}{[4(\alpha + \beta)^2]} \tag{3.10}$$

In the $\alpha\beta$-plane we prescribe sufficiently smooth data. However, the precise nature of the data is immaterial. We require only $t_\alpha(\alpha, \beta^*) > 0$, $t_\beta(\theta, \beta) < 0$ and that as $\beta \longrightarrow 0^+$ along $\alpha = 0$.

$$\lim a = a^0 > 0, \ \lim y = u^0 > 0 \tag{3.11}$$

$$\lim x = \lim t = 0, \quad b(0, \beta) = 0(\beta^{a/2}) \tag{3.12}$$

where the existence of the positive limit a^0 and the behavior of $b(0, \beta)$ for small β were established by Ho and Mayer [9].

4. KdV Equation and the Development of a Solitary Wave

We only sketch the derivation of KdV equation for a channel of variable corss section; the details may be found in Shen and Zhong [10]. We introduce the non dimensional variables

$$t = \beta^{-3/2}\frac{t^*}{\sqrt{H/g}}, \quad (x, y, z) = (\beta^{-3/2}\frac{x^*}{H}, \frac{y^*}{H}, \frac{z^*}{H})$$

(η, h, p) and (u, v, w) are the same as before. The method used here is the specialization of the procedure developed by Shen [2] and Keller [1]. We assume that u, v, w, p, η depend explicitly upon a new variable, $\xi = \beta S(t, x)$, where S is a function of t and x only, will be called a phase function. Then we assume that they possess an asymptotic expansion of the form:

$$\phi(\xi, t, x, y, z, \beta) \sim \phi_0 + \beta^{-1}\phi_1 + \beta^{-2}\phi_2 + \cdots$$

and we assume that the zeroth order approximation is given by

$$(u_0, v_0, w_0) = 0, \quad p_0 = -z_0, \quad \eta_0$$

The equation for the first approximation determines a Hamilton-Jacobi equation for S. Let $k = S_x, w = -S_t$. Then

$$w = kG(x), G(x) = \pm\sqrt{\frac{a(x)}{b(x)}} \tag{4.1}$$

where $a(x)$ is the area of the corss section D_0, and $b(x)$ is the width of D_0 of water wave at rest (figure). (4.1) may be solved by the method of characteristics and the corresponding characteristic equation are

$$\frac{dt}{d\sigma} = \mu, \ \frac{dx}{d\sigma} = \mu G(x), \ \frac{dk}{d\sigma} = -k\mu G'(x), \ \frac{d\mu}{d\sigma} = \frac{dS}{d\sigma} = 0 \tag{4.2}$$

where 'μ' is the proportionality factor. We choose $\mu = 1$, so that $\sigma = t$. The equation of (4.2) determine a one parameter family of bicharacteristics, called rays.

$$x = x(t, \sigma_1)$$

where σ_1 is constant along a ray. The equations for the second approximation determine a KdV equation with variable coefficients:

$$m_0\eta_{1t} + m_1\eta_{1x} + m_2\eta_1 + m_3\eta_1\eta_{1\xi} + m_4\eta_{1\xi\xi\xi} = 0 \tag{4.3}$$

where

$$m_0 = 2b(x) \tag{4.4}$$

$$m_1 = \frac{2a(x)}{G(x)} \tag{4.5}$$

$$m_2 = -[G(x)]^{-1}\int_{\Gamma_0}\frac{h_x}{\sqrt{h_y^2 + h_z^2}}ds - G^{-2}(x)G'(x)a(x) \tag{4.6}$$

$$m_3 = 3k[G(x)]^{-1}b(x) - \frac{1}{w}[\phi_y(t,x,y_2,0) - \phi_y(t,x,y_1,0)] \tag{4.7}$$

$$m_4 = w^{-1}\int\int_{D_0}(\nabla\phi)^2 dydz \tag{4.8}$$

'Γ_0' is the wetted boundary of D_0; $y = y_1,\ y_2$ are the endpoints of the width of D_0; and ϕ_0 is a solution of the Neumann problem.

$$\begin{aligned}
&\nabla^2\phi = k^2 \ \text{ in } D_0\\
&\phi_z = w^2 \ \text{ at } \ z = 0\\
&\phi_y h_y + \phi_z h_z = 0 \ \text{ at } \ \Gamma_0
\end{aligned}$$

Since from equation (4.2)

$$(\frac{d}{d\sigma}) = \partial_t + G(x)\partial_x,\ (\frac{dx}{d\sigma}) = G(x)$$

along a ray, we may express (4.3) in terms of σ and ξ.

$$m_0\eta_{1\sigma} + m_2\eta_1 + m_3\eta_1\eta_{1\xi} + m_4\eta_{1\xi\xi\xi} = 0$$

or in terms of x and ξ

$$G(x) = +\sqrt{\frac{a(x)}{b(x)}}, \quad S = -t + \int_0^x \frac{1}{G(x)} dz$$

which is a solution of equation (4.2) and it follows that

$$w = 1, \quad k = G^{-1}(x)$$

For rectangular and triangular channels, the coefficients given in (4.4) to (4.8) can be explicitly evaluated, Shen and Zhong [10]. It is also remarked in passing that (4.3) has been used to study the fission of solutions in channel of variable cross section [10] and a justification of the asymptotic method used should also be of interest.

Figure: A cross section of the channel

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