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BIFLATNESS AND BIPROJECTIVITY OF TRIANGULAR BANACH ALGEBRAS

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Abstract. Let A and B be Banach algebras and M be a Banach (A, B)-module. Then, $T = \begin{bmatrix} A & M \\ 0 & B \end{bmatrix}$ equipped with the usual 2×2 matrix operations, obvious internal module actions and the Banach space norm $\| \begin{bmatrix} a & m \\ 0 & b \end{bmatrix} \| = \|a\|_A + \|m\|_M + \|b\|_B$ is a triangular Banach algebra. We show that T is biflat if and only if A and B are biflat and $M = 0$, where M is an essential (A, B)-module, that is, $\overline{AM} = M = \overline{MB}$. A similar result is obtained on biprojectivity.

1. **Introduction**

Forrest and Marcoux studied the n-weak amenability of triangular Banach algebras in [2]. In [5], we studied the amenability of $T = \begin{bmatrix} A & M \\ 0 & B \end{bmatrix}$ and show that T is amenable if and only if A and B are amenable and $M = 0$. Here, we investigate another interesting property of these algebras. We showed that if M is an essential (A, B)-module, then T is biflat if and only if A and B are biflat and $M = 0$. This extends the above result on the amenability of triangular Banach algebras. Every amenable Banach algebra has a bounded approximate identity, and so

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$\overline{AM} = M = \overline{MB}$. Similar conclusion is deduced for biprojectivity of this algebras. The hypothesis on M cannot be omitted; to see this, note that $\begin{bmatrix} 0 & \mathbb{C} \\ 0 & \mathbb{C} \end{bmatrix}$ is biprojective [7, page 3241].

A Banach algebra A is biprojective if $\Delta : A\hat{\otimes}A \to A$ has a bounded right inverse as an A-bimodule homomorphism, and A is biflat if $\Delta^* : A^* \to (A\hat{\otimes}A)^*$ has a bounded left inverse which is an A-bimodule homomorphism, where Δ is the diagonal operator on $A\hat{\otimes}A$ defined by $\Delta(a \otimes b) = ab$. As Runde pointed out in [6], biprojectivity and biflatness are notions that arise naturally in Helemskii's Banach homology [3]. The reader is referred to [3, 6] for notations and terminologies not defined here.

2. **Biflatness of triangular Banach algebras**

The aim of this section is to characterize the biflatness of triangular Banach algebras. First, we state the following theorem that is essential for our goal. We prove this theorem at the end of this section.

Theorem 2.1. *Let A be a biflat Banach algebra, L be a nilpotent closed ideal in A such that it is an essential ideal, i.e., $\overline{AL} = \overline{LA} = L$. Then, L is not complemented; in particular, it is not finite-dimensional.*

A very significant consequence of this theorem is the following result that resolves the biflatness of triangular Banach algebras.

Theorem 2.2. *Let A and B be Banach algebras and M be a Banach (A, B)-module such that $\overline{AM} = M = \overline{MB}$. Then, the triangular Banach algebra $T = \begin{bmatrix} A & M \\ 0 & B \end{bmatrix}$ is biflat if and only if A and B are biflat and $M = 0$.*

Proof. If A and B are biflat and $M = 0$, then T is the l^1-direct sum of A and B. Hence, T is biflat .

Conversely, let T be biflat. The closed ideal $L = \begin{bmatrix} 0 & M \\ 0 & 0 \end{bmatrix}$ of T is complemented closed ideal of T such that $0 = \begin{bmatrix} 0 & M \\ 0 & 0 \end{bmatrix}^2$, and

$$\begin{bmatrix} 0 & M \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \overline{MB} \\ 0 & 0 \end{bmatrix} = \left(\begin{bmatrix} 0 & M \\ 0 & 0 \end{bmatrix} \begin{bmatrix} A & M \\ 0 & B \end{bmatrix} \right)^{-},$$

$$\begin{bmatrix} 0 & M \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \overline{AM} \\ 0 & 0 \end{bmatrix} = \left(\begin{bmatrix} A & M \\ 0 & B \end{bmatrix} \begin{bmatrix} 0 & M \\ 0 & 0 \end{bmatrix} \right)^{-},$$

whence, using Theorem 2.1, we have $M = 0$.
Since T is the l^1-direct sum of A and B , by a result of Helemeskii on hereditary property of biflatness on ideals [4, Proposition 8], we conclude that A and B are biflat. □

Now, we first state and then prove the following lemma. The proof is modelled on the proof of [7, Lemma 2].

Lemma 2.3. *Let A be a biflat Banach algebra and E a closed ideal in A. Suppose that there exists a closed complemented ideal N in A, with $E \subseteq N$ and $EN = 0$. Then, $AE \cap \overline{EA} = 0$.*

Proof. Let $\iota : N \to A$ be the inclusion map, $q : A \to A/E$ be the quotient map, I_A, I_N and $I_{A/N}$ be the identity maps on A, N and A/N, respectively, and let $p : (A/E)\hat{\otimes}N \to N$ be the map determined by $p((a + E) \otimes c) = ac$ for any $a + E \in A/E$ and $c \in N$. Since $EN = 0$, then the map p is well defined.

Suppose towards a contradiction that $AE \cap \overline{EA} \neq 0$ and assume $0 \neq mc \in \overline{EA}$ for some $m \in A$, $c \in E$. From the assumption, $mc = \lim c_n a_n$, for some sequences $(c_n) \subset E$ and $(a_n) \subset A$. Since A is biflat, then there is a continuous bimodule homomorphism $S : (A\hat{\otimes}A)^* \to A^*$ such that $S \circ \pi^* = I_{A^*}$, where $\pi : (A\hat{\otimes}A) \to A$ is the diagonal operator on $(A\hat{\otimes}A)$ defined by $\pi(a \otimes b) = ab$. Let $\rho = S^*$, so that $\rho : A^{**} \to (A\hat{\otimes}A)^{**}$ is an A-bimodule homomorphism.

For $b \in N$, let $R_b(L_b) : A \to N$ be the map of right (resp. left) multiplication by b. Consider the operator $q \otimes R_c : (A\hat{\otimes}A) \to (A/E)\hat{\otimes}N$ and let $d = ((q \otimes R_c)^{**} \circ \rho)(m)$. Since $S \circ \pi^* = I_{A^*}$, by taking adjoint we have $\pi^{**} \circ \rho = I_{A^{**}}$, and so $(\pi^{**} \circ \rho(m)).c = (\pi^{**} \circ \rho)(mc) = mc \neq 0$. We have $p \circ (q \otimes R_c) = R_c \circ \pi$ and so $p^{**} \circ (q \otimes R_c)^{**} = R_c^{**} \circ \pi^{**}$. This implies that

$p^{**}(d) = (p^{**} \circ (q \otimes R_c)^{**} \circ \rho)(m) = ((R_c^{**} \circ \pi^{**}) \circ \rho)(m) = R_c^{**}((\pi^{**} \circ \rho)(m)) = R_c^{**}(m) = mc.$

Note that $R_c^{**}\mid_A = R_c$. Thus, $p^{**}(d) = mc \neq 0$, and hence $d \neq 0$. We then have,

$$\begin{aligned}(I_{A/E}\otimes\iota)^{**}(d) &= ((I_{A/E}\otimes\iota)^{**}\circ(q\otimes R_c)^{**}\circ\rho)(m)\\ &= ((I_{A/E}\otimes\iota)^{**}\circ((q\otimes I_N)\circ(I_A\otimes R_c))^{**}\circ\rho)(m)\\ &= (((I_{A/E}\otimes\iota)\circ(q\otimes I_N)\circ(I_A\otimes R_c))^{**}\circ\rho)(m).\end{aligned}$$

Since $(I_{A/E}\otimes\iota)\circ(q\otimes I_N) = (q\otimes I_A)\circ(I_A\otimes\iota)$, then we have,

$$\begin{aligned}(I_{A/E}\otimes\iota)^{**}(d) &= (((I_{A/E}\otimes\iota)\circ(q\otimes I_N)\circ(I_A\otimes R_c))^{**}\circ\rho)(m)\\ &= (((q\otimes I_A)\circ(I_A\otimes\iota)\circ(I_A\otimes R_c))^{**}\circ\rho)(m)\\ &= ((q\otimes I_A)^{**}\circ(I_A\otimes(\iota\circ R_c))^{**}\circ\rho)(m)\\ &= (q\otimes I_A)^{**}((\rho(m)).c)\\ &= (q\otimes I_A)^{**}(\rho(mc))\\ &= (q\otimes I_A)^{**}(\lim_n c_n\rho(a_n))\\ &= \lim_n (q\otimes I_A)^{**}(c_n\rho(a_n))\\ &= \lim_n((q\otimes I_A)^{**}\circ((\iota\circ L_{c_n})\otimes I_A)^{**})(\rho(a_n))\\ &= \lim_n((q\otimes I_A)^{**}\circ((\iota\otimes I_A)\circ(L_{c_n}\otimes I_A))^{**})(\rho(a_n))\\ &= \lim_n((q\otimes I_A)\circ(\iota\otimes I_A)\circ(L_{c_n}\otimes I_A))^{**}(\rho(a_n))\\ &= \lim_n((\underbrace{(q\circ\iota\circ L_{c_n})}_{0}\otimes I_A))^{**}(\rho(a_n))\\ &= 0.\end{aligned}$$

Thus, $(I_{A/E}\otimes\iota)^{**}(d) = 0$. Since N is a complemented closed ideal in A, then the map $I_{A/E}\otimes\iota$ is injective and has closed range, and hence $(I_{A/E}\otimes\iota)^{**}$ is injective by [1, A.3.48]. This contradicts $d \neq 0$ and $(I_{A/E}\otimes\iota)^{**}(d) = 0$. Therefore, $AE \cap \overline{EA} = 0$. □

Now we are ready to prove Theorem 2.1.

Proof of Theorem 2.1. Let n be a positive integer such that $L^n \neq 0$ and $L^{n+1} = 0$. Let $E = \overline{L^n}$ and $N = L$. Now, we have $EN = 0$ and $\overline{AE} = \overline{EA} = E$. Since $AE \subset E \subset \overline{EA}$, then Lemma 2.3 implies that $E = 0$, giving a contradiction. □

3. **Biprojectivity of triangular Banach algebras**

The following lemma shows a hereditary property of biprojectivity. The proof is straightforward and we omit it.

Lemma 3.1. *Let A and B be Banach algebras. Then, $A \underset{l^1}{\oplus} B$ is biprojective if and only if A and B are biprojective.*

Now we characterize the biprojectivity of a triangular Banach algebra.

Theorem 3.2. *Let A and B be Banach algebras and M be a Banach (A, B)-module such that $\overline{AM} = M = \overline{MB}$. Then, the triangular Banach algebra $T = \begin{bmatrix} A & M \\ 0 & B \end{bmatrix}$ is biprojective if and only if A and B are biprojective and $M = 0$.*

Proof. If A and B are biprojective and $M = 0$, then T is the l^1-direct sum of A and B. It follows from Lemma 3.1 that T is biprojective.
Conversely, let T be biprojective. Thus, T is biflat. Utilizing Theorem 2.2, we get $M = 0$. Since T is the l^1-direct sum of A and B, then Lemma 3.1 implies that A and B are biprojective. $\square$

Corollary 3.3. *Let A be a non-zero Banach algebra. Then, the triangular Banach algebra $T = \begin{bmatrix} A & A \\ 0 & A \end{bmatrix}$ is never biprojective.*

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