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A NEW FAMILY IN THE STABLE HOMOTOPY GROUPS OF SPHERES

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Abstract. Let p be a prime number greater than three. Here, we prove the existence of a new family of homotopy elements in the stable homotopy groups of spheres $\pi_*(S)$ which is represented by $h_n h_m \tilde{\beta}_{s+2} \in \mathrm{Ext}_A^{s+4,q[p^n+p^m+(s+2)p+(s+1)]+s}(\mathbb{Z}_p,\mathbb{Z}_p)$ up to a nonzero scalar in the Adams spectral sequence, where $n \geq m+2 > 5$, $0 \leq s < p-2$, $q = 2(p-1)$ and $\tilde{\beta}_{s+2} \in \mathrm{Ext}_A^{s+2,q[(s+2)p+(s+1)]+s}(\mathbb{Z}_p,\mathbb{Z}_p)$ as defined by Wang and Zheng.

1. Introduction

Throughout the paper, we fix a prime $p \geq 5$, and put $q = 2(p-1)$. Let M be the Moore spectrum modulo the prime p given by the cofibration

$$S \xrightarrow{p} S \xrightarrow{i} M \xrightarrow{j} \Sigma S,$$

where S is the sphere spectrum localized at the prime p. Let $\alpha : \Sigma^q M \to M$ be the Adams map and $V(1)$ be its cofibre given by the cofibration

$$\Sigma^q M \xrightarrow{\alpha} M \xrightarrow{i'} V(1) \xrightarrow{j'} \Sigma^{q+1} M.$$

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This spectrum $V(1)$ is known to be the Toda-Smith spectrum. Let $V(2)$ be the cofibre of the v_2-map $\beta : \Sigma^{(p+1)q}V(1) \to V(1)$, given by the cofibration

$$\Sigma^{(p+1)q}V(1) \xrightarrow{\beta} V(1) \xrightarrow{\bar{i}} V(2) \xrightarrow{\bar{j}} \Sigma^{(p+1)q+1}V(1).$$

Recall that there exists the β-family $\beta_t \in \pi_{q(tp+t-1)-2}(S)$, for $t \geq 1$, where $\beta_t = jj'\beta^t i'i$. The β-family β_t was defined by Smith [7], and was detected in the Adams-Novikov spectral sequence for BP-cohomology in [5].

In [8], the second Greek letter family, denoted by $\tilde{\beta}_s$, was first defined. Wang and Zheng proved the following results.

Theorem 1.1. (1)[8, Theorem 2] *When $p \geq 5$, $2 \leq s < p$, $\tilde{\beta}_s$ represents the β-family β_s in the Adams spectral sequence (ASS).*

(2)[8, Theorem 3] *For $p \geq 5$, $2 \leq s < p$, $k \geq 2$, $\tilde{\beta}_s h_0 h_{k+1}$ survives to E_∞.*

Recently, Liu [2, 3] got some results about $\tilde{\beta}_s$.

Theorem 1.2. (1)[2, Theorem 1.1] *For $p \geq 5$, $n \geq 3$, $2 \leq s < p-1$, $b_0 h_n \tilde{\beta}_s$ represents a nontrivial homotopy element in $\pi_* S$ in the Adams spectral sequence.*

(1)[3, Theorem 1.4] *For $p \geq 5$, $n \geq 3$, $2 \leq s < p-2$, $b_0 h_0 h_n \tilde{\beta}_s$ represents a nontrivial homotopy element in $\pi_* S$ in the Adams spectral sequence.*

In [1], Lin detected a new element in the stable homotopy groups of $V(1)$.

Theorem 1.3. [1, Theorem 4.1] *Let $p \geq 5$, $n \geq 0$ and $h_n \in \mathrm{Ext}_A^{1,p^n q}(\mathbb{Z}_p, \mathbb{Z}_p)$. Then, $(i'i)_*(h_n) \in \mathrm{Ext}_A^{1,p^n q}(H^*V(1), \mathbb{Z}_p)$ is a permanent cycle in the ASS and converges to a nontrivial element $\varrho_n \in \pi_{p^n q-1}(V(1))$.*

Here, we will make use of the above theorem to detect a ϱ_n-related homotopy element in $\pi_* S$.

Theorem 1.4. *Let $p \geq 5$, and $n \geq m+2 > 5$, and $0 \leq s < p-2$. Then, the product $h_n h_m \tilde{\beta}_{s+2} \in \mathrm{Ext}_A^{s+4,t(s)}(\mathbb{Z}_p, \mathbb{Z}_p)$ is a permanent cycle in the ASS and converges to a nontrivial homotopy element in $\pi_{t(s)-s-4}(S)$, where $t(s) = q[p^n + p^m + (s+2)p + (s+1)] + s$.*

The proof of Theorem 1.4 is similar to that of Theorem 1.2. From the conditions on the indices n, m and s in Theorem 1.4, we easily see that Theorem 1.4 can not be obtained from published papers.

The reminder of our work is arranged as follows: after giving some results on the May spectral sequence (MSS) in Section 2, we will make use of the MSS and the ASS to prove Theorem 1.4 in Section 3.

2. **Some results on the MSS**

For computing the stable homotopy groups of spheres $\pi_*(S)$ with the ASS, we must compute the Adams E_2-term, $\mathrm{Ext}_A^{*,*}(\mathbb{Z}_p,\mathbb{Z}_p)$. From [4], $\mathrm{Ext}_A^{1,*}(\mathbb{Z}_p,\mathbb{Z}_p)$, has $\mathbb{Z}_p$-basis consisting of $a_0 \in \mathrm{Ext}_A^{1,1}(\mathbb{Z}_p,\mathbb{Z}_p)$, $h_i \in \mathrm{Ext}_A^{1,p^iq}(\mathbb{Z}_p,\mathbb{Z}_p)$, for all $i \geq 0$, and $\mathrm{Ext}_A^{2,*}(\mathbb{Z}_p,\mathbb{Z}_p)$ has $\mathbb{Z}_p$-basis consisting of α_2, a_0^2, $a_0h_i(i > 0)$, $g_i(i \geq 0)$, $k_i(i \geq 0)$, $b_i(i \geq 0)$, and $h_ih_j(j \geq i+2, i \geq 0)$, whose internal degrees are $2q+1$, 2, $p^iq+1, p^{i+1}q+2p^iq$, $2p^{i+1}q+p^iq$, $p^{i+1}q$ and p^iq+p^jq, respectively.

As we know, the most successful method for computing $\mathrm{Ext}_A^{*,*}(\mathbb{Z}_p,\mathbb{Z}_p)$ is the MSS. From [6, Theorem 3.2.5], there is a May spectral sequence (MSS) $\{E_r^{s,t,*}, d_r\}$ which converges to $\mathrm{Ext}_A^{s,t}(\mathbb{Z}_p,\mathbb{Z}_p)$ with E_1-term,

(2.1)
$$E_1^{*,*,*} = E(h_{m,i}|m > 0, i \geq 0) \otimes P(b_{m,i}|m > 0, i \geq 0) \otimes P(a_n|n \geq 0),$$

where E is the exterior algebra, P is the polynomial algebra, and

(2.2)
$$h_{m,i} \in E_1^{1,2(p^m-1)p^i,2m-1}, b_{m,i} \in E_1^{2,2(p^m-1)p^{i+1},p(2m-1)}, a_n \in E_1^{1,2p^n-1,2n+1}.$$

By the knowledge on the p-adic expression in number theory, each integer $t \geq 0$ can be expressed uniquely as

$$t = q(c_np^n + c_{n-1}p^{n-1} + \cdots + c_1p + c_0) + e,$$

where $0 \leq c_i < p$ $(0 \leq i < n)$, $0 < c_n < p$, $0 \leq e < q$. Let s be a given positive integer. Suppose that a generator of the May E_1-term $E_1^{s,t,*}$ is of the form $h = x_1 \cdots x_m$, where x_i is one of a_k, $h_{r,j}$ or $b_{u,z}$, $0 \leq k \leq n+1$, $0 < r+j \leq n+1$, $0 < u+z \leq n$, $r > 0$, $j \geq 0$, $u > 0$, $z \geq 0$. Assume that, for any $1 \leq i \leq m$, deg $x_i = q(c_{i,n}p^n + \cdots + c_{i,1}p + c_{i,0}) + e_i$, where $c_{i,j} = 0$ or 1, for $0 \leq j \leq n$, $e_i = 1$ if $x_i = a_{k_i}$, or $e_i = 0$. Then,

$\dim h = \sum_{i=1}^{m} \dim x_i = s$ and

(2.1)

$$\deg h = \sum_{i=1}^{m} \deg x_i = q[(\sum_{i=1}^{m} c_{i,n})p^n + \cdots + (\sum_{i=1}^{m} c_{i,1})p + (\sum_{i=1}^{m} c_{i,0})] + (\sum_{i=1}^{m} e_i).$$

For convenience, we denote $\sum_{i=1}^{m} c_{i,j}$ and $\sum_{i=1}^{m} e_i$ by $\tilde{c}_j$ and $\tilde{e}$, respectively.

Proposition 2.1. *With the notation as above, we have the following consequences.*

(1) *Suppose that there exist three integers $0 \leq n_1 < n_2 < n_3 \leq n$ such that $m = \tilde{c}_{n_3} \geq \tilde{c}_{n_1} > \tilde{c}_{n_2}$ or $m = \tilde{c}_{n_1} \geq \tilde{c}_{n_3} > \tilde{c}_{n_2}$. Then, the generator h of the form cannot exist.*

(2) *Suppose that there exist two integers $0 \leq n_1 < n_2 \leq n$ such that $m = \tilde{c}_{n_2} \geq \tilde{e} > \tilde{c}_{n_1}$ or $m = \tilde{e} \geq \tilde{c}_{n_2} > \tilde{c}_{n_1}$. Then, the generator h of the form cannot exist.*

Proof. The second part is [3, Lemma 2.3]. The proof of (1) is similar to that of (2). □

Proposition 2.2. *With the notation as above, let s' be a given positive integer. Then, we have the following consequences.*

(1) *If there exists an integer $0 \leq i' \leq n$ such that $c_{i'} > s'$, then the May E_1-term $E_1^{s',t,*} = 0$.*

(2) *If $e > s'$, then, the May E_1-term $E_1^{s',t,*} = 0$.*

Proof. The first part is [2, Lemma 2.1]. The proof of (2) is similar to that of (1). □

3. **Two Adams E_2-terms and proof of Theorem 1.4**

In this section, we make use of the MSS to determine two Adams E_2-terms which will be used in the proof of Theorem 1.4.

Lemma 3.1. *Let $p \geq 5$, $n \geq m+2 > 5$, $0 \leq s < p-2$ and $r \geq 1$. Then, the May E_1-term*

$$E_1^{s+4-r,t(s)+1-r,*} = 0,$$

where $t(s) = q[p^n + p^m + (s+2)p + (s+1)] + s$.

Proof. Obviously, when $r \geq s+4$, the May E_1-term $E_1^{s+4-r,t(s)+1-r,*} = 0$. Thus, in the rest of the proof, we assume that $1 \leq r < s+4$.

Consider $h = x_1x_2\cdots x_l \in E_1^{s+4-r,t(s)-r+1,*}$ in the MSS, where x_i is one of a_k, $h_{r,j}$ or $b_{u,z}$, $0 \le k \le n+1$, $0 \le r+j \le n+1$, $0 \le u+z \le n$, $r > 0$, $j \ge 0$, $u > 0$, $z \ge 0$. Assume that, for any $1 \le i \le l$, deg $x_i = q(c_{i,n}p^n + \cdots + c_{i,1}p + c_{i,0}) + e_i$, where $c_{i,j} = 0$ or 1, for $0 \le j \le n$, $e_i = 1$ if $x_i = a_{k_i}$, or $e_i = 0$. It follows that $dim\ h = \sum\limits_{i=1}^{l} dim\ x_i = s+4-r$ and

$$
\begin{aligned}
(3.1) \quad deg\ h &= \sum_{i=1}^{l} deg\ x_i \\
&= q[(\sum_{i=1}^{l} c_{i,n})p^n + \cdots + (\sum_{i=1}^{l} c_{i,1})p + (\sum_{i=1}^{l} c_{i,0})] + (\sum_{i=1}^{l} e_i) \\
&= q[p^n + p^m + (s+2)p + (s+1)] + (s+1-r).
\end{aligned}
$$

Note that $dim\ h_{i,j} = dim\ a_i = 1$, $dim\ b_{i,j} = 2$ and $0 \le s < p-2$. From $dim\ h = \sum\limits_{i=1}^{l} dim\ x_i = s+4-r$, we can have $l \le s+4-r < p-2+4-r = p+2-r \le p+1$.

We claim that $s+1-r \ge 0$. Otherwise, by $1 \le r < s+4$ and $p \ge 5$, we would have that $\sum\limits_{i=1}^{l} e_i = (s-r+1)+q > q-3 \ge p$. Meanwhile, by $e_i = 0$ or 1, we would also have that $\sum\limits_{i=1}^{l} e_i \le l \le p$. This yields a contradiction. Thus, the claim is proved.

Using $0 \le s+2, s+1, s+1-r < p$ and the knowledge in number theory, from (3.1), we have

$$\left\{\begin{array}{ll}
\sum_{i=1}^{l} e_i = s+1-r+\lambda_{-1}q, & \lambda_{-1} \ge 0;\\
\sum_{i=1}^{l} c_{i,0} + \lambda_{-1} = s+1+\lambda_0 p, & \lambda_0 \ge 0;\\
\sum_{i=1}^{l} c_{i,1} + \lambda_0 = s+2+\lambda_1 p, & \lambda_1 \ge 0;\\
\sum_{i=1}^{l} c_{i,2} + \lambda_1 = 0+\lambda_2 p, & \lambda_2 \ge 0;\\
\sum_{i=1}^{l} c_{i,3} + \lambda_2 = 0+\lambda_3 p, & \lambda_3 \ge 0;\\
\cdots & \cdots\\
\sum_{i=1}^{l} c_{i,m-1} + \lambda_{m-2} = 0+\lambda_{m-1} p, & \lambda_{m-1} \ge 0;\\
\sum_{i=1}^{l} c_{i,m} + \lambda_{m-1} = 1+\lambda_m p, & \lambda_m \ge 0;\\
\sum_{i=1}^{l} c_{i,m+1} + \lambda_m = 0+\lambda_{m+1} p, & \lambda_{m+1} \ge 0;\\
\cdots & \cdots\\
\sum_{i=1}^{l} c_{i,n-1} + \lambda_{n-2} = 0+\lambda_{n-1} p, & \lambda_{n-1} \ge 0;\\
\sum_{i=1}^{l} c_{i,n} + \lambda_{n-1} = 1. &
\end{array}\right. \tag{3.2}$$

From $e_i = 0$ or 1, $c_{i,j} = 0$ or 1, and $l \le p$, we easily have that $(\lambda_{-1}, \lambda_0, \lambda_1) = (0,0,0)$. Consider the fourth equality of (3.2), $\sum_{i=1}^{l} c_{i,2} = \lambda_2 p$. By $c_{i,2} = 0$ or 1 and $l \le p$, λ_2 may equal 0 or 1.

Case 1 $\lambda_2 = 0$. We claim that in this case λ_3 must equal 0. Otherwise, $\lambda_3 = 1$, and then we would have $\sum_{i=1}^{l} c_{i,2} = 0$ and $\sum_{i=1}^{l} c_{i,3} = p$. Noting that $l \le p$ and $c_{i,3} = 0$ or 1, it would follow that l must equal p. By Proposition 2.1, h is impossible to exist. Hence, λ_3=0. The claim is proved.

By induction on j, we have that

$$\lambda_j = 0 \ (3 \le j \le m-1).$$

Consider the $(m+2)$-th equality $\sum_{i=1}^{l} c_{i,m} = 1 + \lambda_m p$. By $l \le p$ and $c_{i,m} = 0$ or 1, we get

$$\lambda_m = 0.$$

Similarly, it is easy to obtain that $\lambda_j = 0$ $(m+1 \le j \le n-1)$. By $\lambda_j = 0$, for $-1 \le j \le n-1$, from (3.2) we have the following four possibilities.

Subcase 1.1 There are two factors $h_{1,n}$ and $h_{1,m}$ in h. Then, up to a sign, $h = h_{1,n}h_{1,m}h'$ with $h' \in E_1^{s+2-r,q[(s+2)p+(s+1)]+(s+1-r),*}$. By Proposition 2.2, $E_1^{s+2-r,q[(s+2)p+(s+1)]+(s+1-r),*} = 0$. Thus, in this case h cannot exist.

Similarly, by Proposition 2.2 we can show the followings:

Subcase 1.2 There cannot exist two factors $h_{1,n}$ and $b_{1,m-1}$ in h.

Subcase 1.3 There cannot exist two factors $b_{1,n-1}$ and $h_{1,m}$ in h.

Subcase 1.4 There cannot exist two factors $b_{1,n-1}$ and $b_{1,m-1}$ in h.

Case 2 $\lambda_2 = 1$. In this case, we easily get that l must equal p. It follows that in this case $s = p-3$ and $r = 1$. Thus, we have that $h = x_1 \cdots x_p \in E_1^{p,t(p-3),*}$. From the fifth equality of (3.2), $\sum\limits_{i=1}^{p} c_{i,3} + 1 = 0 + \lambda_3 p$ and $0 \leq \sum\limits_{i=1}^{p} c_{i,3} \leq p$, one can easily deduce that

$$\lambda_3 = 1.$$

By induction on j, we have

$$\lambda_j = 1 \ (4 \leq j \leq m-1).$$

Now consider the (m+2)-th equality of (3.2), $\sum\limits_{i=1}^{p} c_{i,m} + 1 = 1 + \lambda_m p$. Noting that $0 \leq \sum\limits_{i=1}^{p} c_{i,m} \leq p$, we have that $\lambda_m = 0$ or 1.

Subcase 2.1 $\lambda_m = 1$. It follows that $\sum\limits_{i=1}^{p} c_{i,m} = p$. Note that in this case $\sum\limits_{i=1}^{p} c_{i,2} = p$ and $\sum\limits_{i=1}^{p} c_{i,j} = p-1$ for $3 \leq j \leq m-1$. By Proposition 2.1, h is impossible to exist.

Subcase 2.2 $\lambda_m = 0$. It follows that $\sum\limits_{i=1}^{p} c_{i,m} = 0$. By Proposition 2.1, we can easily prove that $\lambda_{m+1} = 0$, i.e., $\sum\limits_{i=1}^{p} c_{i,m+1} = 0$. Otherwise,

$\lambda_{m+1} = 1$, then $\sum_{i=1}^{p} c_{i,m+1} = p$. From $\sum_{i=1}^{p} c_{i,2} = p$, $\sum_{i=1}^{p} c_{i,m} = 0$ and $\sum_{i=1}^{p} c_{i,m+1} = p$, we have h cannot exist by Proposition 2.1.

By induction on j, we have $\lambda_j = 0$ $(m+1 \leq j \leq n-1)$. Hence, we have $\sum_{i=1}^{p} c_{i,n} = 1$. Note that $\sum_{i=1}^{p} c_{i,2} = p$ and $\sum_{i=1}^{p} c_{i,j} = 0$, for $m \leq j \leq n-1$. By Proposition 2.1, h cannot exist either.

Combining cases 1 and 2, we complete the proof of Lemma 3.1. $\square$

Theorem 3.2. *Let $p \geq 5$, $n \geq m+2 > 5$, $0 \leq s < p-2$. Then,*

$$h_n h_m \tilde{\beta}_{s+2} \neq 0 \in \mathrm{Ext}_A^{s+4,t(s)}(\mathbb{Z}_p, \mathbb{Z}_p),$$

where $t(s) = q[p^n + p^m + (s+3)p^2 + (s+2)p + (s+2)] + s$.

Proof. Since $h_{1,i}, a_2^s h_{2,0} h_{1,1} \in E_1^{*,*,*}$ are permanent cycles in the MSS and converge nontrivially to h_i, $\tilde{\beta}_{s+2} \in \mathrm{Ext}_A^{*,*}(\mathbb{Z}_p, \mathbb{Z}_p)$, for $i \geq 0$ and $0 \leq s < p-2$, respectively (cf. [2, Theorem 2.2]),

$$h_{1,n} h_{1,m} a_2^s h_{2,0} h_{1,1} \in E_1^{s+4,t(s),*}$$

is a permanent cycle in the MSS and converges to

$$h_n h_m \tilde{\beta}_{s+2} \in \mathrm{Ext}_A^{s+4,t(s)}(\mathbb{Z}_p, \mathbb{Z}_p).$$

From the case when $r = 1$ in Lemma 3.1, we have the May E_1-term

$$E_1^{s+3,t(s),*} = 0.$$

Then,

$$E_r^{s+3,t(s),*} = 0,$$

for $r \geq 1$. Thus, $h_{1,n} h_{1,m} a_2^s h_{2,0} h_{1,1} \in E_r^{s+4,t(s),*}$ does not bound and converges nontrivially to $h_n h_m \tilde{\beta}_{s+2} \in \mathrm{Ext}_A^{s+4,t(s)}(\mathbb{Z}_p, \mathbb{Z}_p)$ in the MSS. Then, the desired result follows. $\square$

Theorem 3.3. *Let $p \geq 5$, $n \geq m+2 > 5$, $0 \leq s < p-2$ and $r \geq 2$. Then,*

$$\mathrm{Ext}_A^{s+4-r,t(s)-r+1}(\mathbb{Z}_p, \mathbb{Z}_p) = 0.$$

Proof. By Lemma 3.1, in this case the May E_1-term

$$E_1^{s+4-r,t(s)-r+1,*} = 0.$$

By the MSS, the desired result follows. $\square$

Now, we give the proof of Theorem 1.4.

Proof of Theorem 1.4. From Theorem 1.3,

$$(i'i)_*(h_n) \in \mathrm{Ext}_A^{1,p^nq}(H^*V(1), \mathbb{Z}_p)$$

is a permanent cycle in the ASS and converges to a nontrivial element $\varrho_n \in \pi_{p^nq-1}(V(1))$. Since $V(1)$ is a ring spectrum,

$$(i'i)_*(h_n h_m) \in \mathrm{Ext}_A^{2,q(p^n+p^m)}(H^*V(1), \mathbb{Z}_p)$$

is also a permanent cycle in the ASS and converges to a homotopy element in $\pi_{q(p^n+p^m)-2}(V(1))$, denoted by $\rho_{n,m}$.

Consider the composite

$$\varphi_{m,n,s} = jj'\beta^{s+2}\rho_{m,n}.$$

Since $\rho_{m,n}$ is represented by $(i'i)_*(h_n h_m) \in \mathrm{Ext}_A^{2,q(p^n+p^m)}(H^*V(1), \mathbb{Z}_p)$ in the ASS, the above $\varphi_{m,n,s}$ is represented in the ASS by $\bar{c} = (jj'\beta^{s+2}i'i)_*$ $(h_n h_m)$. By Theorem 1.1 and the knowledge of Yoneda products, we know that the composite

$$\begin{aligned}(j_0 j_1 \beta^{s+2} i_1 i_0)_* : \mathrm{Ext}_A^{0,*}(\mathbb{Z}_p, \mathbb{Z}_p) &\xrightarrow{(i_1 i_0)_*} \mathrm{Ext}_A^{0,*}(H^*V(1), \mathbb{Z}_p) \\ &\xrightarrow{(j_0 j_1 \beta^{s+2})_*} \mathrm{Ext}_A^{s+2,*+(s+2)pq+(s+1)q+s}(\mathbb{Z}_p, \mathbb{Z}_p)\end{aligned}$$

is a multiplication up to a nonzero scalar by

$$\tilde{\beta}_{s+2} \in \mathrm{Ext}_A^{s+2,q[(s+2)p+(s+1)]+s}(\mathbb{Z}_p, \mathbb{Z}_p).$$

Hence, $\varphi_{m,n,s}$ is represented up to a nonzero scalar by $h_n h_m \tilde{\beta}_{s+2}$ in the ASS (cf. Theorem 3.2). Moreover, from Theorem 3.3, $h_n h_m \tilde{\beta}_{s+2}$ cannot be hit by any differential in the ASS. Consequently, the corresponding homotopy element $\varphi_{m,n,s}$ is nontrivial. This completes the proof. □

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