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RIORDAN GROUP APPROACHES IN MATRIX FACTORIZATIONS

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Abstract. We consider an arbitrary binary polynomial sequence $\{A_n\}$ and then give a lower triangular matrix representation of the sequence. As a result, we obtain a factorization of the infinite generalized Pascal matrix in terms of this new matrix, using a Riordan group approach. Furthermore, some interesting results and applications are given.

1. **Introduction**

For $n > 0$, the $n \times n$ Pascal matrix $P_n = [p_{ij}]$ is defined as follows [7]:

$$p_{ij} = \begin{cases} \binom{i-1}{j-1}, & \text{if } i \geq j, \\ 0, & \text{otherwise.} \end{cases}$$

Matrix representations of the Pascal triangle are first given in [3]. In [15], for a nonzero real x, the Pascal matrices $P_n[x] = [P_n(x; i, j)]$ and $Q_n[x] = [Q_n(x; i, j)]$ are generalized as follows:

$$P_n(x; i, j) = \begin{cases} \binom{i-1}{j-1} x^{i-j}, & \text{if } i \geq j, \\ 0, & \text{otherwise,} \end{cases}$$

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and

$$Q_n(x;i,j)=\begin{cases}\binom{i-1}{j-1}x^{i+j-2}, & \text{if } i\geq j,\\ 0, & \text{otherwise.}\end{cases}$$

For more details about the Pascal matrices, see [1, 2, 12]. In [16], the Pascal matrices $P_n[x]$ and $Q_n[x]$ for two nonzero real numbers x and y are generalized as follows:

$$\varphi[x,y]_{ij}=\begin{cases}\binom{i-1}{j-1}x^{i-j}y^{i+j-2}, & \text{if } i\geq j,\\ 0, & \text{otherwise.}\end{cases}$$

The Fibonacci and Lucas sequences have been discussed in so many studies. Besides, various generalizations and matrix representations of these sequences have also been introduced and investigated. For nonnegative integers A and B such that $A^2+4B\neq 0$ and $n>0$, the generalized Fibonacci and Lucas type sequences $\{U_n(A,B)\}$ and $\{V_n(A,B)\}$ are defined by

$$U_{n+1}(A,B)=AU_n(A,B)+BU_{n-1}(A,B),$$

$$V_{n+1}(A,B)=AV_n(A,B)+BV_{n-1}(A,B),$$

where $U_0(A,B)=0$, $U_1(A,B)=1$ and $V_0(A,B)=2$, $V_1(A,B)=A$. For example, $U_n(1,1)=F_n$ (nth Fibonacci number) and $V_n(1,1)=L_n$ (nth Lucas number).

For the polynomial versions of generalized Fibonacci and Lucas numbers, we refer to [4]. Furthermore, general cases of these polynomials were considered in [6], where authors defined the polynomial sequence $\{A_n(a,b;p,q)(x)\}$ (briefly $\{A_n(x)\}$) satisfying

$$A_{n+1}(x)=p(x)A_n(x)-q(x)A_{n-1}(x),$$

with $A_0(x)=a(x)$ and $A_1(x)=b(x)$, where a, b, p,q are polynomials of x with real coefficients. In [6], it is shown that for $n>0$, any integer k and $n\equiv c\ (mod\ |k|)$, the sequence $\{A_n\}$ satisfies the following recursion:

$$A_{p(n+1,k,c)}=s_kA_{p(n,k,c)}-z_kA_{p(n-1,k,c)},$$

where $s_k=\alpha^k+\beta^k$, $z_k=q^k$, $p(n,k,c)=nk+c$ (c a constant) and $\alpha,\beta=\left(p\pm\sqrt{p^2-4q}\right)/2$.

Furthermore, in [8], the $n\times n$ Fibonacci matrix $\mathcal{F}_n=[f_{ij}]$ is defined in the form

$$[f_{ij}]=\begin{cases}F_{i-j+1}, & \text{if } i-j+1\geq 0,\\ 0, & \text{otherwise,}\end{cases}$$

where F_n is the nth Fibonacci number. This is generalized in [9], where the $n \times n$ generalized Fibonacci matrix $\mathcal{F}[x,y]_n = \left[f[x,y]_{ij}\right]$ is introduced as follows:

$$f[x,y]_{ij} = \begin{cases} F_{i-j+1} x^{i-j} y^{i+j-2}, & \text{if } i \geq j, \\ 0, & \text{otherwise.} \end{cases}$$

The infinite generalized Fibonacci matrix is defined in the form

$$\mathcal{F}[x,y] = \begin{bmatrix} 1 & 0 & 0 & ... \\ xy & y^2 & 0 & ... \\ 2x^2y^2 & xy^3 & y^4 & ... \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

and the infinite generalized Pell matrix is defined by

$$S[x,y] = \begin{bmatrix} 1 & 0 & 0 & ... \\ 2xy & y^2 & 0 & ... \\ 5x^2y^2 & 2xy^3 & y^4 & ... \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Similarly, the infinite matrices $L[x,y] = \left[l[x,y]_{ij}\right]$ and $M[x,y] = \left[m[x,y]_{ij}\right]$ are given as follows:

$$l[x,y]_{ij} = \left(\binom{i-1}{j-1} - \binom{i-2}{j-1} - \binom{i-3}{j-1}\right) x^{i-j} y^{j-i},$$

and

$$m[x,y]_{ij} = \left(\binom{i-1}{j-1} - 2\binom{i-2}{j-1} - \binom{i-3}{j-1}\right) x^{i-j} y^{j-i}.$$

It is also shown that the matrices $\mathcal{F}[x,y]$, $L[x,y]$, $S[x,y]$ and $M[x,y]$ satisfy $\Phi[x,y] = \mathcal{F}[x,y] * L[x,y]$ and $\Phi[x,y] = S[x,y] * M[x,y]$, where $\Phi[x,y]$ is the infinite generalized Pascal matrix defined by

$$\Phi[x,y] = \begin{bmatrix} 1 & 0 & 0 & ... \\ xy & y^2 & 0 & ... \\ x^2y^2 & 2xy^3 & y^4 & ... \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

The $n \times n$ matrix $R_n = [r_{i,j}]$ is given in [17], where

$$r_{ij} = \binom{i-1}{j-1} - \binom{i-1}{j} - \binom{i-1}{j+1},$$

which is used to show that $P_n = R_n \mathcal{F}_n$ and the following factorization

$$\begin{aligned}\binom{n-1}{r-1} &= F_{n-r+1} + (n-2) F_{n-r} + \tfrac{1}{2}\left(n^2 - 5n + 2\right) F_{n-r-1} \\ &\quad + \sum_{k=r}^{n-3} \binom{n-1}{k-1} \left[2 - \tfrac{n}{k} - \tfrac{(n-k)(n-k-1)}{k(k+1)}\right] F_{k-r+1},\end{aligned}$$

where $\mathcal{F}_n$ and P_n are defined as before. Stănică [11] looks at a more general case of the results of [8, 17]: he considers the $n \times n$ matrix $\mathcal{U}_n = (u_{ij})$ in terms of the sequence $\{U_n(A,B)\}$, where

$$u_{ij} = \begin{cases} U_{i-j+1}, & \text{if } i \geq j, \\ 0, & \text{otherwise.} \end{cases}$$

Then, he gives the factorization of *any* matrix in terms of the matrix $\mathcal{U}_n$. In [10], the Riordan group is defined as follows:
Let $R = [r_{ij}]_{i,j \geq 0}$ be an infinite matrix whose entries are complex numbers and $c_i(t) = \sum_{n\geq 0}^{\infty} r_{n,i} t^n$ is the generating function of the ith column of R. If $c_i(t) = g(t)[f(t)]^i$, where

$$g(t) = 1 + g_1 t + g_2 t^2 + g_3 t^3 + \cdots, \text{ and } f(t) = t + f_2 t^2 + f_3 t^3 + \cdots,$$

then R is a Riordan matrix. When $\Re$ denotes the set of Riordan matrices, the set $\Re$ is a group under matrix multiplication $*$, with the following properties:

- (R_1) $(g(t), f(t)) * (h(t), l(t)) = (g(t) h(f(t)), l(f(t)))$.
- (R_2) $I = (1, t)$ is the identity element.
- (R_3) The inverse of R is given by $R^{-1} = \left(\frac{1}{g(\overline{f}(t))}, \overline{f}(t)\right)$, where $\overline{f}(t)$ is the compositional inverse of $f(t)$, i.e., $f\left(\overline{f}(t)\right) = \overline{f}(f(t)) = t$.
- (R_4) If $(a_0, a_1, a_2, ...)^T$ is a column vector with generating function $A(t)$, then multiplying $R = (g(t), f(t))$ on the right by this column vector yields a column vector with generating function $B(t) = g(t) A(f(t))$.

In [8], the infinite Pascal, Fibonacci and Pell matrices are generalized and the factorizations of the infinite generalized Pascal matrix are given by using the Riordan method. Let $R_n = [r_{i,j}]$ be the $n \times n$ matrix given as before. In [13], using the equations $P_n = R_n \mathcal{F}_n$ and $P_n E_n = R_n \mathcal{F}_n E_n$ for the $n \times n$ Fibonacci matrix $\mathcal{F}_n = [f_{ij}]$, the $n \times n$ Pascal matrix

$P_n = [p_{ij}]$ and the $n \times 1$ matrix $E_n = (1,1,...,1)^T$, it is shown that

$$n+1=\sum_{l=1}^{n}\tfrac{(n-1)!}{(l+1)!(n-l)!}\left[l^2+(n+1)\,l-n^2\right]F_{l+2},$$

where $1 \leq i,j \leq n$ and F_n is the nth Fibonacci number. Here, we consider the arbitrary binary polynomial sequence $\{A_n\}$ and then give a lower triangular matrix representation of this sequence. By the definition of Riordan matrices, we obtain a factorization of the infinite generalized Pascal matrix in terms of this new matrix. Furthermore, some interesting results and applications are given.

2. A Factorization of the generalized Pascal matrix

For any two nonzero real variables x and y, an infinite matrix $H[x,y] = \left[h[x,y]_{ij}\right]$ is defined as follows:

$$h[x,y]_{ij}=\begin{cases} A_{p(i-j+1,k,c)}x^{i-j}y^{i+j-2}, & \text{if } i\geq j,\\ 0, & \text{otherwise,}\end{cases}$$

where $\left\{A_{p(n+1,k,c)}\right\}$ and $p(n+1,k,c)$ are defined as before. Clearly, the matrix $H[x,y]$ is of the form

$$H[x,y]=\begin{bmatrix} A_{p(1,k,c)} & 0 & 0 & ...\\ A_{p(2,k,c)}xy & A_{p(1,k,c)}y^2 & 0 & ...\\ A_{p(3,k,c)}x^2y^2 & A_{p(2,k,c)}xy^3 & A_{p(1,k,c)}y^4 & ...\\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Now, we give the Riordan representation of the infinite matrix $H[x,y]$. Let the Riordan representation of $H[x,y]$ be $(g_H(t), f_H(t))$. Here, the generating function of the jth column of $H[x,y]$ is $c_j(t) = g_H(t)[f_H(t)]^j$ Since the first column vector of $H[x,y]$ is

$$\left(A_{p(1,k,c)}, A_{p(2,k,c)}xy, A_{p(3,k,c)}x^2y^2, ...\right)^T,$$

we can write

$$\begin{aligned}
&g_H(t) = A_{p(1,k,c)}+A_{p(2,k,c)}xyt + A_{p(3,k,c)}x^2y^2t^2+\dots\\
&-s_kxytg_H(t) = -s_kA_{p(1,k,c)}xyt - s_kA_{p(2,k,c)}x^2y^2t^2-s_kA_{p(3,k,c)}x^3y^3t^3\\
&-\dots\\
&z_kx^2y^2t^2g_H(t) = z_kA_{p(1,k,c)}x^2y^2t^2+z_kA_{p(2,k,c)}x^3y^3t^3+z_kA_{p(3,k,c)}x^3y^3t^3\\
&+\dots
\end{aligned}$$

By summing the above equalities side by side, we get

$$g_H(t) = \frac{A_{p(1,k,c)} - z_k A_{p(0,k,c)} xyt}{1 - s_k xyt + z_k (xyt)^2}.$$

Since $h[x,y]_{ij} = y^2 h[x,y]_{i-1,j-1}$, for $j \geq 2$, we have that $c_j(t) = y^2 t c_{j-1}(t)$ and $g_H(t)[f_H(t)]^j = y^2 t g_H(t)[f_H(t)]^{j-1}$. Hence, we get $f_H(t) = y^2 t$. Consequently, the Riordan representation of $H[x,y]$ is given by

$$H[x,y] = \left(\frac{A_{p(1,k,c)} - z_k A_{p(0,k,c)} xyt}{1 - s_k xyt + z_k (xyt)^2}, y^2 t \right).$$

For two nonzero real numbers x and y, let us define the infinite matrix $C[x,y] = \left[c[x,y]_{ij} \right]$ by

$$\begin{aligned} c[x,y]_{ij} \quad = \quad & \left(\tfrac{1}{A_{p(1,k,c)}} \binom{i-1}{j-1} - \frac{A_{p(2,k,c)}}{A^2_{p(1,k,c)}} \binom{i-2}{j-1} \right. \\ & - z_k \left(\tfrac{A_{p(0,k,c)} A_{p(2,k,c)} - A^2_{p(1,k,c)}}{A^3_{p(1,k,c)}} \right) \binom{i-3}{j-1} \\ & - z_k \left(\tfrac{A_{p(0,k,c)} A_{p(2,k,c)} - A^2_{p(1,k,c)}}{A^3_{p(1,k,c)}} \right) \\ & \left. \times \left(\sum_{m=1}^{i-3} \binom{i-m-3}{j-1} \left(\frac{z_k A_{p(0,k,c)}}{A_{p(1,k,c)}} \right)^m \right) \right) x^{i-j} y^{j-i}, \end{aligned}$$

if $i \geq j$, and 0 otherwise. We now give the following result.

Theorem 2.1.

$$\Phi[x,y] = H[x,y] * C[x,y].$$

Proof. Since $C[x,y]$ is a Riordan matrix, we write $C[x,y]=(g_C(t), f_C(t))$. Considering the first column vector of $C[x,y]$, we get

$$\begin{aligned}
& g_C(t) \\
= \; & \tfrac{1}{A_{p(1,k,c)}} + \left(\tfrac{1}{A_{p(1,k,c)}} - \tfrac{A_{p(2,k,c)}}{A^2_{p(1,k,c)}}\right) xy^{-1}t + \left(\tfrac{1}{A_{p(1,k,c)}} - \tfrac{A_{p(2,k,c)}}{A^2_{p(1,k,c)}}\right. \\
& \left. - z_k \left(\tfrac{A_{p(0,k,c)}A_{p(2,k,c)} - A^2_{p(1,k,c)}}{A^3_{p(1,k,c)}}\right)\right) \left(xy^{-1}t\right)^2 + \\
& \left(\tfrac{1}{A_{p(1,k,c)}} - \tfrac{A_{p(2,k,c)}}{A^2_{p(1,k,c)}} - z_k \left(\tfrac{A_{p(0,k,c)}A_{p(2,k,c)} - A^2_{p(1,k,c)}}{A^3_{p(1,k,c)}}\right)\right) \left(xy^{-1}t\right)^3 + \cdots \\
= \; & \left(1 + xy^{-1}t + \left(xy^{-1}t\right)^2 + \cdots\right) \left(\tfrac{1}{A_{p(1,k,c)}} - \tfrac{A_{p(2,k,c)}}{A^2_{p(1,k,c)}} xy^{-1}t\right) - \\
& z_k \left(1 + xy^{-1}t + \left(xy^{-1}t\right)^2 + \cdots\right) \left(\tfrac{A_{p(0,k,c)}A_{p(2,k,c)} - A^2_{p(1,k,c)}}{A^3_{p(1,k,c)}}\right) \\
& \times \left(xy^{-1}t\right)^2 \left(1 + \left(\tfrac{z_k A_{p(0,k,c)}}{A_{p(1,k,c)}}\right) xy^{-1}t + \left(\tfrac{z_k^2 A^2_{p(0,k,c)}}{A^2_{p(1,k,c)}}\right) \left(xy^{-1}t\right)^2 + \cdots\right) \\
= \; & \left(\tfrac{1}{1-xy^{-1}t}\right) \left(\tfrac{1 - s_k xy^{-1}t + z_k \left(xy^{-1}t\right)^2}{\left(A_{p(1,k,c)} - z_k A_{p(0,k,c)} xy^{-1}t\right)}\right).
\end{aligned}$$

Let the generating function of the jth column of $C[x,y]$ be

$$c_j(t) = g_C(t)\left[f_C(t)\right]^j.$$

Considering

$$c[x,y]_{ij} = c[x,y]_{i-1,j-1} + xy^{-1}c[x,y]_{i-1,j},$$

for $j \geq 2$, we obtain

$$c_j(t) = tc_{j-1}(t) + xy^{-1}tc_j(t)$$

and

$$g_C(t)\left[f_C(t)\right]^j = tg_C(t)\left[f_C(t)\right]^{j-1} + xy^{-1}tg_C(t)\left[f_C(t)\right]^j.$$

Hence, we have $f_C(t) = \frac{t}{1-xy^{-1}t}$. Finally, the Riordan representation of the matrix $C[x,y]$ is

$$C[x,y] = \left(\tfrac{1 - s_k xy^{-1}t + z_k \left(xy^{-1}t\right)^2}{\left(A_{p(1,k,c)} - z_k A_{p(0,k,c)} xy^{-1}t\right)\left(1-xy^{-1}t\right)}, \tfrac{t}{1-xy^{-1}t}\right).$$

From [9], we have that $\Phi[x,y]=\left(\frac{1}{1-xyt},\frac{y^2t}{1-xyt}\right)$. Moreover,

$$\begin{aligned}
&H[x,y]*C[x,y]\\
=\;&\left(\frac{A_{p(1,k,c)}-z_kA_{p(0,k,c)}xyt}{1-s_kxyt+z_k(xyt)^2},y^2t\right)\\
&*\left(\frac{1-s_kxy^{-1}t+z_k\left(xy^{-1}t\right)^2}{\left(A_{p(1,k,c)}-z_kA_{p(0,k,c)}xy^{-1}t\right)(1-xy^{-1}t)},\frac{t}{1-xy^{-1}t}\right)\\
=\;&\left(\frac{\left(A_{p(1,k,c)}-z_kA_{p(0,k,c)}xyt\right)\left(1-s_kxy^{-1}y^2t+z_k\left(xy^{-1}y^2t\right)^2\right)}{\left(1-s_kxyt+z_k(xyt)^2\right)\left(A_{p(1,k,c)}-z_kA_{p(0,k,c)}xy^{-1}y^2t\right)(1-xy^{-1}y^2t)},\right.\\
&\left.\frac{y^2t}{1-xy^{-1}y^2t}\right)\\
=\;&\left(\frac{1}{1-xyt},\frac{y^2t}{1-xyt}\right)=\Phi[x,y].
\end{aligned}$$

Thus, the proof is complete. □

Now, we consider some special cases. When $k=1$, $p=1$, $q=-1$ and $c=0$, the matrix $H[x,y]$ is reduced to the Fibonacci matrix $\mathcal{F}[x,y]$. In Theorem 2.1, taking $\mathcal{F}[x,y]$ instead of $H[x,y]$, we find the matrix $L[x,y]$ such that $\Phi[x,y]=\mathcal{F}[x,y]*L[x,y]$, from [9]. Thus, the matrix $L[x,y]$ is a special case of $C[x,y]$. When $k=1$, $p=2$, $q=-1$ and $c=0$, the matrix $H[x,y]$ is reduced to the Pell matrix $S[x,y]$, defined in [9]. Also, taking $S[x,y]$, instead of $H[x,y]$, we get the matrix $M[x,y]$ such that $\Phi[x,y]=S[x,y]*M[x,y]$, given in [9]. The matrix $M[x,y]$ is a special case of the matrix $C[x,y]$.

Corollary 2.2. *For $i,j=1,2,...,n$, we have*

$$\sum_{r=1}^{i}\binom{i-1}{r-1}x^{i-r}y^{i+r-2}=\sum_{j=1}^{i}\left(A_{p(i-j+1,k,c)}x^{i-j}y^{i+j-2}\left(\sum_{m=1}^{j}c_{im}\right)\right),$$

where c_{im} is the (i,m) th element of $C_n[x,y]$.

Proof. Considering the $n\times n$ Pascal matrix $\Phi_n[x,y]$ and $\Phi[x,y]=H[x,y]*C[x,y]$ in Theorem 2.1, we have $\Phi_n[x,y]=H_n[x,y]\,C_n[x,y]$ and $\Phi_n[x,y]\,E_n=H_n[x,y]\,C_n[x,y]\,E_n$, where $E_n=(1,1,...,1)^T$. Therefore, we obtain the desired result. □

Corollary 2.3. *For $n > 0$ and $j = 1, 2, ..., n$, we have*

$$\binom{n-1}{r-1} = \sum_{j=r}^{n} \left(\frac{A_{p(n-j+1,k,c)}}{A_{p(1,k,c)}}\right) \left(\binom{j-1}{r-1} - \frac{A_{p(2,k,c)}}{A_{p(1,k,c)}} \binom{j-2}{r-1} \right.$$
$$-z_k \left(\frac{A_{p(0,k,c)}A_{p(2,k,c)} - A^2_{p(1,k,c)}}{A^2_{p(1,k,c)}}\right) \binom{j-3}{r-1}$$
$$-z_k \left(\frac{A_{p(0,k,c)}A_{p(2,k,c)} - A^2_{p(1,k,c)}}{A^2_{p(1,k,c)}}\right)$$
$$\left. \times \left(\sum_{m=1}^{n-3} \binom{n-m-3}{r-1} \left(\frac{z_k A_{p(0,k,c)}}{A_{p(1,k,c)}}\right)^m \right) \right).$$

Proof. We take $x = y = 1$ in the equality $\Phi_n [x, y] = H_n [x, y] * C_n [x, y]$. □

If we take $r = 1$ in the previous corollary, we have

$$\sum_{j=1}^{n} \left(\frac{A_{p(n-j+1,k,c)}}{A_{p(1,k,c)}}\right) 1 - \frac{A_{p(2,k,c)}}{A_{p(1,k,c)}} - z_k \left(\frac{A_{p(0,k,c)}A_{p(2,k,c)} - A^2_{p(1,k,c)}}{A^2_{p(1,k,c)}}\right)$$
$$-z_k \left(\frac{A_{p(0,k,c)}A_{p(2,k,c)} - A^2_{p(1,k,c)}}{A^2_{p(1,k,c)}}\right) \left(\sum_{m=1}^{n-3} \left(\frac{z_k A_{p(0,k,c)}}{A_{p(1,k,c)}}\right)^m \right) = 1.$$

For example, when $k = 3$, $p = 1$, $q = -1$ and $c = 0$, the sequence $\left\{A_{p(n,k,c)}\right\}$ is reduced to the Fibonacci subsequence $\{F_{3n}\}$. By Corollary 2.3, we obtain

$$\binom{n-1}{r-1} = \sum_{j=r}^{n} \left(\frac{F_{3(n-j+1)}}{F_3}\right) \left(\binom{j-1}{r-1} - \frac{F_6}{F_3} \binom{j-2}{r-1} - \binom{j-3}{r-1} \right).$$

Now, we give another factorization of the generalized Pascal matrix with a matrix associated with the sequence $\left\{A_{p(n,k,c)}\right\}$. First, for two nonzero real numbers x and y, we define the infinite matrix $C' [x, y] = \left[c' [x, y]_{ij}\right]$

with

$$\begin{aligned} c'[x,y]_{ij} &= \Bigg(\tfrac{1}{A_{p(1,k,c)}} \binom{i-1}{j-1} - \frac{A_{p(2,k,c)}}{A^2_{p(1,k,c)}} \binom{i-1}{j} \\ &\quad - z_k \left(\tfrac{A_{p(0,k,c)} A_{p(2,k,c)} - A^2_{p(1,k,c)}}{A^3_{p(1,k,c)}} \right) \binom{i-1}{j+1} \\ &\quad - z_k \left(\tfrac{A_{p(0,k,c)} A_{p(2,k,c)} - A^2_{p(1,k,c)}}{A^3_{p(1,k,c)}} \right) \\ &\quad \times \left(\sum_{m=1}^{i-3} \binom{i-1}{j+m+1} \left(\frac{z_k A_{p(0,k,c)}}{A_{p(1,k,c)}} \right)^m \right) \Bigg) x^{i-j} y^{i+j-2}, \end{aligned}$$

if $i \geq j$, and 0 otherwise. Secondly, we define the infinite matrix $H'[x,y] = \left[h'[x,y]_{ij} \right]$ with $h'[x,y]_{ij} = A_{p(i-j+1,k,c)} x^{i-j} y^{j-i}$, if $i \geq j$, and 0 otherwise. Then, we can give the following result.

Theorem 2.4.

$$\Phi[x,y] = C'[x,y] * H'[x,y].$$

Proof. From Theorem 2.1, the Riordan representation of the matrix $C[x,y]$ is known. Thus, we get the Riordan representations of $C'[x,y]$ and $H'[x,y]$ as follows:

$$C'[x,y] = \left(\tfrac{1-(2+s_k)xyt+(1+s_k+z_k)(xyt)^2}{\left(A_{p(1,k,c)} - \left(A_{p(1,k,c)} + z_k A_{p(0,k,c)}\right)xyt\right)(1-xyt)^2}, \tfrac{y^2 t}{1-xyt} \right)$$

and

$$H'[x,y] = \left(\frac{A_{p(1,k,c)} - z_k A_{p(0,k,c)} x y^{-1} t}{1 - s_k x y^{-1} t + z_k \left(x y^{-1} t\right)^2}, t \right).$$

From property (R_1), we have

$$
\begin{aligned}
& C'[x,y] * H'[x,y] \\
= \; & \left(\tfrac{1-(2+s_k)xyt+(1+s_k+z_k)(xyt)^2}{\left(A_{p(1,k,c)}-\left(A_{p(1,k,c)}+z_kA_{p(0,k,c)}\right)xyt\right)(1-xyt)^2}, \tfrac{y^2t}{1-xyt}\right) \\
& * \left(\tfrac{A_{p(1,k,c)}-z_kA_{p(0,k,c)}xy^{-1}t}{1-s_kxy^{-1}t+z_k(xy^{-1}t)^2}, t\right) \\
= \; & \left(\frac{\begin{array}{c}\left(1-(2+s_k)xyt+(1+s_k+z_k)(xyt)^2\right) \\ \left(A_{p(1,k,c)}-z_kA_{p(0,k,c)}xy^{-1}\frac{y^2t}{1-xyt}\right)\end{array}}{\begin{array}{c}\left(A_{p(1,k,c)}-\left(A_{p(1,k,c)}+z_kA_{p(0,k,c)}\right)xyt\right)(1-xyt)^2 \\ \left(1-s_kxy^{-1}\frac{y^2t}{1-xyt}+z_k\left(xy^{-1}\frac{y^2t}{1-xyt}\right)^2\right)\end{array}}, \frac{y^2t}{1-xyt}\right) \\
= \; & \left(\tfrac{1}{1-xyt}, \tfrac{y^2t}{1-xyt}\right) = \Phi[x,y].
\end{aligned}
$$

Thus, the proof is complete. □

Corollary 2.5. *For $i, j = 1, 2, ..., n,$ we have*

$$
\sum_{r=1}^{i} \binom{i-1}{r-1} x^{i-r} y^{i+r-2} = \sum_{j=1}^{i} \left(c'_{ij}\left(\sum_{m=1}^{j} A_{p(m,k,c)} x^{m-1} y^{1-m}\right)\right),
$$

where c'_{ij} is the (i,j)th element of $C'_n[x,y]$.

Proof. Since $\Phi[x,y] = C'[x,y] * H'[x,y]$ in Theorem 2.4, we have

$$
\Phi_n[x,y] = C'_n[x,y] H'_n[x,y], \Phi_n[x,y] E_n = C'_n[x,y] H'_n[x,y] E_n,
$$

where $E_n = (1,1,...,1)^T$. Thus, we obtain the desired result. □

Corollary 2.6. *For $n > 0$ and $i, j = 1, 2, ..., n,$ we have*

$$
\begin{aligned}
\binom{n-1}{r-1} \;=\; & \sum_{j=r}^{n} \left(\binom{n-1}{j-1} - \frac{A_{p(2,k,c)}}{A_{p(1,k,c)}}\binom{n-1}{j}\right. \\
& -z_k\left(\tfrac{A_{p(0,k,c)}A_{p(2,k,c)}-A^2_{p(1,k,c)}}{A^2_{p(1,k,c)}}\right) \times \\
& \binom{n-1}{j+1} - z_k\left(\tfrac{A_{p(0,k,c)}A_{p(2,k,c)}-A^2_{p(1,k,c)}}{A^2_{p(1,k,c)}}\right) \times \\
& \left.\left(\sum_{m=1}^{n-3}\binom{n-1}{j+m+1}\left(\frac{z_kA_{p(0,k,c)}}{A_{p(1,k,c)}}\right)^m\right)\right) \frac{A_{p(j-r+1,k,c)}}{A_{p(1,k,c)}}.
\end{aligned}
$$

Proof. By taking $x=y=1$ in $\Phi[x,y]=C'[x,y]*H'[x,y]$, we have the result. $\square$

Particularly, if we take $r=1$ in Corollary 2.6, we get

$$\sum_{j=1}^{n}\left(\binom{n-1}{j-1}-\frac{A_{p(2,k,c)}}{A_{p(1,k,c)}}\binom{n-1}{j}-z_k\left(\frac{A_{p(0,k,c)}A_{p(2,k,c)}-A^2_{p(1,k,c)}}{A^2_{p(1,k,c)}}\right)\right.$$
$$\times\binom{n-1}{j+1}-z_k\left(\frac{A_{p(0,k,c)}A_{p(2,k,c)}-A^2_{p(1,k,c)}}{A^2_{p(1,k,c)}}\right)\times$$
$$\left.\sum_{m=1}^{n-3}\binom{n-1}{j+m+1}\left(\frac{z_kA_{p(0,k,c)}}{A_{p(1,k,c)}}\right)^{m-1}\right)\frac{A_{p(j,k,c)}}{A_{p(1,k,c)}}=1.$$

As an example, when $k=2$, $p=2$, $q=-1$ and $c=0$, the sequence $\left\{A_{p(n,k,c)}\right\}$ is reduced to the Pell subsequence $\{P_{2n}\}$. By Corollary 2.6, we obtain

$$\begin{aligned}
&\binom{n-1}{r-1}\\
=\;&\sum_{j=r}^{n}\binom{n-1}{j-1}-\frac{A_{p(2,2,0)}}{A_{p(1,2,0)}}\binom{n-1}{j}\\
&-z_2\left(\frac{A_{p(0,2,0)}A_{p(2,2,0)}-A^2_{p(1,2,0)}}{A^2_{p(1,2,0)}}\right)\times\binom{n-1}{j+1}\\
&-z_2\left(\sum_{m=1}^{n-3}\binom{n-1}{j+m+1}\left(\frac{z_2A_{p(0,2,0)}}{A_{p(1,2,2)}}\right)^{m}\right)\frac{A_{p(j-r+1,2,0)}}{A_{p(1,2,0)}}\\
=\;&\sum_{j=r}^{n}\left(\binom{n-1}{j-1}-\frac{P_4}{P_2}\binom{n-1}{j}+\binom{n-1}{j+1}\right)\frac{P_{2(j-r+1)}}{P_2}.
\end{aligned}$$

From property (R_3), we can find the inverses of $H[x,y]$, $C[x,y]$ and $C'[x,y]$. Using the computation of the inverse of $\Phi[x,y]$ in [9], we can give the next two results.

Lemma 2.7. *The inverses of matrices $H[x,y]$, $C[x,y]$, $C'[x,y]$ and $H'[x,y]$ are respectively given by*

$$H[x,y]^{-1}=\left(\frac{1-s_kxy^{-1}t+z_k\left(xy^{-1}t\right)^2}{\left(A_{p(1,k,c)}-z_kA_{p(0,k,c)}xy^{-1}t\right)},y^{-2}t\right),$$

$$C[x,y]^{-1} = \left(\frac{A_{p(1,k,c)} + \left(A_{p(1,k,c)} - z_k A_{p(0,k,c)}\right) xy^{-1}t}{1 + (2 - s_k) xy^{-1}t + (1 - s_k + z_k) (xy^{-1}t)^2}, \frac{t}{1 + xy^{-1}t}\right),$$

$$C'[x,y]^{-1} = \left(\frac{A_{p(1,k,c)} - z_k A_{p(0,k,c)} xy^{-1}t}{\left(1 - s_k xy^{-1}t + z_k (xy^{-1}t)^2\right)(1 + xy^{-1}t)}, \frac{t}{y^2 + xyt}\right),$$

and

$$H'[x,y]^{-1} = \left(\frac{1 - s_k xy^{-1}t + z_k \left(xy^{-1}t\right)^2}{\left(A_{p(1,k,c)} - z_k A_{p(0,k,c)} xy^{-1}t\right)}, t\right).$$

Proof. Firstly, we cosider the matrix $H[x,y]$. Since $f_H(t) = y^2 t$, we get $\overline{f}_H(t) = y^{-2}t$. Substituting $\overline{f}_H(t)$ in $\left(g_H\left(\overline{f}_H(t)\right)\right)^{-1}$, we obtain

$$\frac{1}{g_H\left(\overline{f}_H(t)\right)} = \frac{1 - s_k xy^{-1}t + z_k \left(xy^{-1}t\right)^2}{\left(A_{p(1,k,c)} - z_k A_{p(0,k,c)} xy^{-1}t\right)(1 - xy^{-1}t)},$$

and hence the Riordan representation of $H[x,y]^{-1}$ is

$$H[x,y]^{-1} = \left(\frac{1 - s_k xy^{-1}t + z_k \left(xy^{-1}t\right)^2}{\left(A_{p(1,k,c)} - z_k A_{p(0,k,c)} xy^{-1}t\right)}, y^{-2}t\right).$$

Secondly, since $f_C(t) = \frac{t}{1 - xy^{-1}t}$, for the matrix $C[x,y]$ we get $\overline{f}_C(t) = t\left(1 + xy^{-1}t\right)^{-1}$ and

$$\frac{1}{g_C\left(\overline{f}_C(t)\right)} = \frac{A_{p(1,k,c)} + \left(A_{p(1,k,c)} - z_k A_{p(0,k,c)}\right) xy^{-1}t}{1 + (2 - s_k) xy^{-1}t + (1 - s_k + z_k) (xy^{-1}t)^2}.$$

Thus, the Riordan representation of $C[x,y]^{-1}$ is

$$C[x,y]^{-1} = \left(\frac{A_{p(1,k,c)} + \left(A_{p(1,k,c)} - z_k A_{p(0,k,c)}\right) xy^{-1}t}{1 + (2 - s_k) xy^{-1}t + (1 - s_k + z_k) (xy^{-1}t)^2}, \frac{t}{1 + xy^{-1}t}\right).$$

Thirdly, since $f_{C'}(t) = \frac{y^2 t}{1-xyt}$, for the matrix $C'[x,y]$ we get $\overline{f}_{C'}(t) = t\left(y^2+xyt\right)^{-1}$ and

$$\frac{1}{g_{C'}\left(\overline{f}_{C'}(t)\right)} = \frac{A_{p(1,k,c)} - z_k A_{p(0,k,c)} xy^{-1}t}{\left(1 - s_k xy^{-1}t + z_k\left(xy^{-1}t\right)^2\right)\left(1+xy^{-1}t\right)}.$$

Thus the Riordan representation of $C'[x,y]^{-1}$ is

$$C'[x,y]^{-1} = \left(\frac{A_{p(1,k,c)} - z_k A_{p(0,k,c)} xy^{-1}t}{\left(1 - s_k xy^{-1}t + z_k\left(xy^{-1}t\right)^2\right)\left(1+xy^{-1}t\right)}, \frac{t}{y^2+xyt}\right).$$

Finally, since $f_{H'}(t) = t$, for the matrix $H'[x,y]$ we get $\overline{f}_{H'}(t) = t$ and

$$\frac{1}{g_{H'}\left(\overline{f}_{H'}(t)\right)} = \frac{1 - s_k xy^{-1}t + z_k\left(xy^{-1}t\right)^2}{\left(A_{p(1,k,c)} - z_k A_{p(0,k,c)} xy^{-1}t\right)}.$$

Thus, the Riordan representation of $C'[x,y]^{-1}$ is

$$H'[x,y]^{-1} = \left(\frac{1 - s_k xy^{-1}t + z_k\left(xy^{-1}t\right)^2}{\left(A_{p(1,k,c)} - z_k A_{p(0,k,c)} xy^{-1}t\right)}, t\right).$$

□

When $k=1$, $p=1$, $q=-1$ and $c=0$, the inverses of $H[x,y]$ and $C[x,y]$ are the inverses of the infinite generalized Fibonacci matrix $\mathcal{F}[x,y]$ and the matrix $L[x,y]$, respectively. Moreover, when $k=1$, $p=2$, $q=-1$ and $c=0$, the inverses of $H[x,y]$ and $C[x,y]$ are the inverses of the generalized Pell matrix $S[x,y]$ and the matrix $M[x,y]$, respectively.

Corollary 2.8. *For the generalized Pascal matrix* $\Phi[x,y]$, *we have*

$$\Phi[x,y]^{-1} = C[x,y]^{-1} * H[x,y]^{-1}$$

and

$$\Phi[x,y]^{-1} = H'[x,y]^{-1} * C'[x,y]^{-1}.$$

Proof. From [9], we have the inverse of $\Phi[x,y]$ as

$$\Phi[x,y]^{-1} = \left(\frac{1}{1+xy^{-1}t}, \frac{t}{y^2+xyt}\right).$$

From theorems 2.1 and 2.4, we know that $\Phi[x,y] = H[x,y] * C[x,y]$, $\Phi[x,y] = C'[x,y] * H'[x,y]$, respectively. Thus, the proof is complete.

□

Corollary 2.9. *For $n \geq 1$, we have*

$$(i)\, H\left[x,y\right]^{n} = \left(\prod_{m=1}^{n} \frac{A_{p(1,k,c)} - z_k A_{p(0,k,c)} x y^{2m-1} t}{1 - s_k x y^{2m-1} t + z_k \left(x y^{2m-1} t\right)^2}, y^{2n} t\right),$$

$$(ii)\, H\left[x,y\right]^{-n} = \left(\prod_{m=1}^{n} \frac{1 - s_k x y^{-2m+1} t + z_k \left(x y^{-2m+1} t\right)^2}{A_{p(1,k,c)} - z_k A_{p(0,k,c)} x y^{-2m+1} t}, y^{-2n} t\right).$$

Proof. The desired result follow from induction and the use of Riordan representations of $H\left[x,y\right]$ and $H\left[x,y\right]^{-1}$. $\square$

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