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G-FRAMES AND THEIR DUALS FOR HILBERT C^*-MODULES

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Abstract. Certain facts about frames and generalized frames (g-frames) are extended for the g-frames for Hilbert C^*-modules. It is shown that g-frames for Hilbert C^*-modules share several useful properties with those for Hilbert spaces. We also characterize the operators which preserve the class of g-frames for Hilbert C^*-modules. Moreover, a necessary and sufficient condition is obtained for an operator T whose corresponding singleton set $\{T\}$ is a g-frame. Finally, some characterizations of dual g-frames for Hilbert spaces and Hilbert C^*-modules are given.

1. Introduction

Frames were first introduced in 1952 by Duffin and Schaeffer [5]. They abstracted the fundamental notion of Gabor [7] to study signal processing. It seems, however, that Duffin-Schaeffer ideas did not attract much interest outside the realm of nonharmonic Fourier series until the paper by Daubechies, et al. [4] was published in 1986.

The theory of frames was rapidly generalized and, until 2005, various generalizations consisting of vectors in Hilbert spaces or Hilbert C^*-modules were developed. In 2005, Sun [13] introduced the notion of g-frames as a generalization of frames for bounded operators on Hilbert

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spaces, and Nejati-Rahimi [11] developed methods to generate them. Casazza-Kutyniok [2] formulated a general method for piecing together local frames to get global ones. They also introduced the so-called fusion frames as a new type of g-frames. Frank-Larson [6] extended the theory for the elements of C^*-algebras and (finitely or countably generated) Hilbert C^*-modules. It is well known that the theory of Hilbert C^*-modules has applications in the study of locally compact quantum groups, complete maps between C^*-algebras, non-commutative geometry, and KK-theory. There are several differences between Hilbert C^*-modules and Hilbert spaces. For example, we know that the Riesz representation theorem for continuous linear functionals on Hilbert spaces dose not extend to Hilbert C^*-modules and there exist closed subspaces in Hilbert C^*-modules that have no orthogonal complement. Moreover, we know that every bounded operator on a Hilbert space has an adjoint, while there are bounded operators on Hilbert C^*-modules which do not have any. It is expected that problems about frames and g-frames for Hilbert C^*-modules to be more complicated than those for Hilbert spaces. This makes the study of the frames for Hilbert C^*-modules important and interesting. The properties of g-frames for Hilbert C^*-modules are further investigated in [8, 14]. The main purpose of the present work is to study the duals of g-frames for Hilbert C^*-modules.

The reminder of our work is organized as follows. We continue this introductory section with a review of the basic definitions and notations of Hilbert C^*-modules. Section 2 investigates some of the properties of g-frames for Hilbert C^*-modules and presents nontrivial examples of such g-frames. The main results of the paper are included in Section 3, where the duals of g-frames for Hilbert C^*-modules are studied.

Let us recall the definition and some of the basic properties of Hilbert C^*-modules and their frames. For more details, we refer the interested reader to the books by Lance [9] and Wegge-Olsen [12]. Let $\mathcal{A}$ be a C^*-algebra. A pre-Hilbert C^*-module over $\mathcal{A}$ or, simply, a pre-Hilbert $\mathcal{A}$-module, is a pair $(H, \langle\cdot,\cdot\rangle)$, where H is a complex linear space, which is an algebraic (left) $\mathcal{A}$-module, and $\langle\cdot,\cdot\rangle : H\times H \to \mathcal{A}$, called an $\mathcal{A}$-inner product, has the following properties:

(1) $\langle f, f\rangle \geq 0$, for any $f \in H$;
(2) $\langle f, f\rangle = 0$, if and only if $f = 0$;
(3) $\langle f, g\rangle = \langle g, f\rangle^*$, for any $f, g \in H$;
(4) $\langle \lambda f, h\rangle = \lambda\langle f, h\rangle$, whenever $\lambda \in \mathbb{C}$ and $f, h \in H$;
(5) $\langle af+bg, h\rangle = a\langle f, h\rangle+b\langle g, h\rangle$, whenever $a, b \in \mathcal{A}$ and $f, g, h \in H$.

As is clear from (4)-(5), the action of $\mathcal{A}$ on H is $\mathbb{C}$- and $\mathcal{A}$-linear and, in particular, $\lambda(af) = (\lambda a)f = a(\lambda f)$, whenever $\lambda \in \mathbb{C}$, $a \in \mathcal{A}$ and $f \in H$. The map $f \longmapsto \|f\| = \|\langle f, f\rangle\|_{\mathcal{A}}^{\frac{1}{2}}$ defines a norm on H. If $(H, \|\cdot\|)$ is a Banach space, then $(H, \langle\cdot,\cdot\rangle)$ is called a Hilbert C^*-module over $\mathcal{A}$ or, simply, a Hilbert $\mathcal{A}$-module. A net or sequence is usually denoted by $(u_j)_{j\in\Omega}$ or $\{u_j\}_{j\in\Omega}$, where the subscript $j \in \Omega$ may be dropped if understood from the context.

Let $\{(K_j, \langle\cdot,\cdot\rangle_j)\}_{j\in J}$ be a finite or countably infinite family of finitely or countably generated Hilbert $\mathcal{A}$-modules. Set

$$K_J = \{(f_j)_{j\in J} : f_j \in K_j, \|\sum_{j\in J}\langle f_j, f_j\rangle_j\| < \infty\}$$

and define

$$\langle (f_j)_{j\in J}, (g_j)_{j\in J}\rangle = \sum_{j\in J}\langle f_j, g_j\rangle_j.$$

It is known that $(K_J, \langle\cdot,\cdot\rangle_J)$ is a Hilbert $\mathcal{A}$-module. The mapping π_j, sending $(f_j)_{j\in J} \in K_J$ to f_j, is called the jth projection operator from K_J onto K_j.

Let $(H, \langle\cdot,\cdot\rangle_1)$ and $(K, \langle\cdot,\cdot\rangle_2)$ be Hilbert $\mathcal{A}$-modules. A (not necessarily linear or bounded) map $T : H \to K$ is said to be adjointable (with respect to the $\mathcal{A}$-inner products $(H, \langle\cdot,\cdot\rangle_1)$ and $(K, \langle\cdot,\cdot\rangle_2)$), if there exists a map $T^* : K \to H$ satisfying $\langle Tf, g\rangle_2 = \langle f, T^*g\rangle_1$, whenever $f \in H$, and $g \in K$. The map T^* is called the adjoint of T [12]. The class of all adjointable maps from H into K is denoted by $B_*(H, K)$ and the class of all bounded $\mathcal{A}$-module maps from H into K is denoted by $B_b(H, K)$. It is known that $B_*(H, K) \subseteq B_b(H, K)$. We write $B_*(H)$ and $B_b(H)$ for $B_*(H, H)$ and $B_b(H, H)$, respectively. We follow the usual notation $L(U, V)$ for ordinary continuous linear transformations from a normed linear space U into a normed linear space V (We avoid the classical notation $B(U, V)$ which has different usages by operator theorists and frame theorists).

Throughout the paper, we fix the notations $\mathcal{A}$ and J for a given unital C^*-algebra and a finite or countably infinite index set, respectively. Also, all Hilbert $\mathcal{A}$-modules are assumed to be finitely or countably generated.

2. g-frames for Hilbert C^*-modules

Sun [13] introduced the notion of a g-frame for a given separable Hilbert space H as a family of ordered pairs $\{(\Lambda_j, K_j) : j \in J\}$ consisting

of separable Hilbert spaces K_j and operators $\Lambda_j \in L(H, K_j)$ satisfying

$$A||f||^2 \leq \sum_{j\in J} ||\Lambda_j f||^2 \leq B||f||^2,$$

for all $f \in H$ and some positive constants A and B independent of f.

Before, the notion of frames for Hilbert spaces had been extended by Frank-Larson [6] to the notion of frames for Hilbert $\mathcal{A}$-modules as a family $\{f_j\}_{j\in J}$ in a Hilbert $\mathcal{A}$-module H satisfying

$$A\langle f, f\rangle \leq \sum_{j\in J} \langle f, f_j\rangle\langle f, f_j\rangle^* \leq B\langle f, f\rangle,$$

for all $f \in H$ and some positive constants A and B independent of f. Parallel to this, Khosravi-Khosravi [8] extended the concept of g-frames from Hilbert spaces to Hilbert C^*-modules as follows.

Definition 2.1. *By a g-frame for a given Hilbert $\mathcal{A}$-module H, we mean a family of ordered pairs $\{(\Lambda_j, K_j) : j \in J\}$ consisting of Hilbert $\mathcal{A}$-modules K_j and operators $\Lambda_j \in B_*(H, K_j)$ satisfying*

$$A\langle f, f\rangle \leq \sum_{j\in J} \langle \Lambda_j f, \Lambda_j f\rangle \leq B\langle f, f\rangle, \tag{2.1}$$

for all $f \in H$ and some positive constants A and B independent of f.

(Throughout the paper, series like (2.1) are assumed to be convergent in the norm sense.)

The numbers A and B are called the lower and the upper bounds of the g-frame, respectively. The frame bounds may be denoted by an ordered pair A and B. The optimal bounds are maximal for A and minimal for B. If $(A, B) = (\lambda, \lambda)$, then the g-frame is said to be λ-tight and in the special case $\lambda = 1$, it is called a Parseval g-frame or a normalized g-frame.

The family $\{(\Lambda_j, K_j) : j \in J\}$ is said to be a g-Bessel sequence for H if there exists a positive number B such that

$$\sum_{j\in J} \langle \Lambda_j f, \Lambda_j f\rangle \leq \ B\langle f, f\rangle, \ \forall f \in H. \tag{2.2}$$

The number B is called a g-Bessel bound.

Let $\{(\Lambda_j, K_j) : j \in J\}$ be a g-frame for the Hilbert $\mathcal{A}$-module H. It is said to be a g-Riesz basis if it satisfies

(1) $\Lambda_j \neq 0$, for any $j \in J$; and
(2) if $\{g_j\}_{j\in J} \in K_J$ and $\sum_{j\in J} \Lambda_j^* g_j = 0$, then $\Lambda_j^* g_j = 0$, for all $j \in J$.

Remark 2.2. *If $\{(\Lambda_j, K_j) : j \in J\}$ is a g-frame for the Hilbert $\mathcal{A}$-module H with an upper bound B, then $\{\Lambda_j\}_{j\in J}$ is uniformly bounded by $\sqrt{B}$. The proof is similar to the one given for ordinary frames [3] and can be obtained by using the properties of positive elements in C^*-algebras.*

To throw more light on the subject and appreciate the use of the concepts, we include some examples of nontrivial g-frames for Hilbert $\mathcal{A}$-modules.

Example 2.3. *Let H be the Hilbert $\mathbb{C}^2$-module $\mathbb{C}^2$ with the $\mathbb{C}^2$-inner product $\langle (x_1, x_2), (y_1, y_2)\rangle = (x_1\overline{y_1}, x_2\overline{y_2})$ and let $\mathcal{A}$ be the totality of all diagonal operators diag$\{a, b\}$ on $\mathbb{C}^2$, sending $(z_1, z_2)^t$ to $(az_1, bz_2)^t$. (Here, diag$\{a, b\}$ means a 2×2 matrix (a_{ij}) such that $a_{11} = a$, $a_{22} = b$ and $a_{12} = a_{21} = 0$.) Also, let $J = \mathbb{N}$ and fix nonzero sequences $\{a_j\}_{j\in J}$ and $\{b_j\}_{j\in J}$ in ℓ^2. Define*

$$\Lambda_j : \mathbb{C}^2 \to \mathbb{C}^2, \qquad (z_1, z_2) \mapsto (a_j z_1, b_j z_2).$$

Clearly, $\Lambda_j \in B_*(\mathbb{C}^2)$, for all $j \in J$. Now, for $z = (z_1, z_2) \in \mathbb{C}^2$,

$$\sum_{j\in J}\langle \Lambda_j z, \Lambda_j z\rangle = (\sum_{j\in J} |a_j|^2|z_1|^2 \ \ , \ \ \sum_{j\in J} |b_j|^2|z_2|^2).$$

The sequence $\{(\Lambda_j, \mathbb{C}^2) : j \in J\}$ is a g-frame with bounds

$$(\min\{\sum_{j\in J} |a_j|^2, \sum_{j\in J} |b_j|^2\} \ , \ \max\{\sum_{j\in J} |a_j|^2, \sum_{j\in J} |b_j|^2\}).$$

Example 2.4. *Let $\mathcal{A} = \ell^\infty$ and let $H = \mathcal{C}_0$, the Hilbert $\mathcal{A}$-module of the set of all null sequences equipped with the $\mathcal{A}$-inner product*

$$\langle (x_i)_{i\in\mathbb{N}}, (y_i)_{i\in\mathbb{N}}\rangle = (x_i\overline{y_i})_{i\in\mathbb{N}}.$$

The action of each sequence $(a_i) \in \mathcal{A}$ on a sequence $(x_i) \in H$ is implemented as $(a_i)(x_i) = (a_i x_i)$. Let $j \in J = \mathbb{N}$ and define $\Lambda_j \in B_*(H)$ by

$$\Lambda_j(a_i)_{i\in\mathbb{N}} = (\delta_{ij} a_j)_{i\in\mathbb{N}} \qquad \forall (a_i)_{i\in\mathbb{N}} \in H.$$

We observe that

$$\sum_{j\in\mathbb{N}}\langle \Lambda_j a, \Lambda_j a\rangle = \sum_{j\in\mathbb{N}} (a_j\overline{a_j})_{j\in\mathbb{N}} = \langle a, a\rangle, \qquad \forall a = (a_i)_{i\in\mathbb{N}} \in H.$$

Thus, $\{(\Lambda_j, H)\}_{j\in J}$ is a normalized g-frame for H. Now, we show that it is a g-Riesz basis. Obviously, $\Lambda_j \neq 0$, for all $j \in J$. Also, if $a(j) := (a_i(j))_i \in H$, for $j \in J$, and if $\sum_{j\in J} \Lambda_j^* a(j) = 0$, then $0 = \sum_{j\in J} \Lambda_j^* a(j) = \sum_{j\in J} \Lambda_j a(j) = \sum_{j\in J} (\delta_{ij} a_i(j))_i$, and hence $(\sum_{j\in J}$

$\delta_{ij}a_i(j))_i = 0$. *In particular,* $a_i(i) = 0$, *for all* $i \in \mathbb{N}$, *and thus* $\Lambda_j^* a(j) = \Lambda_j a(j) = (\delta_{ij}a_i(j))_i = 0$.

Example 2.5. *Let* $\{(\Lambda_j, H)\}_{j\in J}$ *be as in Example* 2.4 *and define* $\Gamma_1 = \Gamma_2 = \Lambda_1 + \Lambda_2$, *and* $\Gamma_j = \Lambda_j$, *for* $j \geq 3$. *It is clear that* $\{(\Gamma_j, H)\}_{j\in J}$ *is a g-frame with the lower and upper bounds* $(1, 2)$.

Example 2.6. *For separable Hilbert spaces* U *and* V, *let* $\mathcal{A} = L(V)$ *and let* H *be the Hilbert* $L(V)$*-module* $L(U, V)$ *with the* $L(V)$*-valued inner product* $\langle T_1, T_2\rangle = T_1T_2^*$ *and the action* $\theta T = \theta oT$, *for* $\theta \in L(V)$ *and* $T, T_1, T_2 \in L(U, V)$. *Also, let* $J = \mathbb{N}$ *and fix* $(\Lambda_j)_j \in \ell^2$. *Define*

$$\Lambda_j(T) = \Lambda_j T, \ \forall T \in L(U, V), \forall j \in \mathbb{N}.$$

Clearly, $\Lambda_j \in B_*(L(U, V))$, *for all* $j \in \mathbb{N}$ *and that* $\{(\Lambda_j, B_*(L(U, V))\}_{j\in J}$ *is a g-frame in* $L(U, V)$. *To see this, observe that*

$$\sum_{j\in J}\langle \Lambda_j T, \Lambda_j T\rangle = \sum_{j\in J} \Lambda_j T\overline{\Lambda_j}T^* = \sum_{j\in J} |\Lambda_j|^2\langle T, T\rangle, \ \forall T \in L(U, V).$$

Furthermore, the exact equalities reveal that the g-frame is $\sum_{j\in J} |\Lambda_j|^2$*-tight.*

The following lemma is known for the case T is bijective (See [1]); we modify the proof to show that the proposition remains valid under slightly weaker conditions.

Lemma 2.7. *Let* $T \in B_*(H, K)$. *Then, the following assertions are true.*

(1) *If* T *is injective and has a closed range, then* T^*T *is an invertible selfadjoint operator satisfying* $\|(T^*T)^{-1}\|^{-1} \leq T^*T \leq \|T\|^2$.
(2) *If* T *is surjective, then* TT^* *is an invertible selfadjoint map satisfying* $\|(TT^*)^{-1}\|^{-1} \leq TT^* \leq \|T\|^2$.

Proof. (1) Since the adjointable map T^* is surjective, it follows that for any $f \in H$, there exists $g \in K$ such that $T^*g = f$. Since $K = ker T^* \oplus Im T$, it follows from Theorem 15.3.8 of [12] that $g = g_1 + Th$, for some $g_1 \in ker T^*$ and some $h \in H$. Thus, $f = T^*(g_1 + Th) = T^*Th$, and hence T^*T is surjective. If $T^*Tf = 0$, then $Tf \in ker T^* \cap Im T = \{0\}$, which implies that $f = 0$; therefore, T^*T is an injective positive map. Hence, T^*T is an invertible element of the C^*-algebra $B_*(H)$, $0 \leq (T^*T)^{-1} \leq \|(T^*T)^{-1}\|$ and $0 \leq (T^*T) \leq \|(T^*T)\|$. Therefore, $\|(T^*T)^{-1}\|^{-1} \leq T^*T \leq \|T\|^2$.

(2) Let T be surjective. Then, T^* is injective and T^* has a closed range. By substituting T^* for T in (1), we have TT^* is invertible and $\|(TT^*)^{-1}\|^{-1} \leq TT^* \leq \|T\|^2$. □

Remark 2.8. *Let $T \in B_*(H,K)$. The following examples involve g-frames consisting of a single term. Similar to this work was done by Sun [13] in the case of g-frames for Hilbert spaces.*

(1) *Assume T is an injective map with a closed range. Then, the family $\{(T,K)\}$ is a g-frame in H. In fact, by Lemma 2.7,*

$$\begin{aligned} \|(T^*T)^{-1}\|^{-1}\langle f,f\rangle \leq \langle Tf,Tf\rangle &= \langle T^*Tf,f\rangle \\ &\leq |T\|^2\langle f,f\rangle \forall f \in H. \end{aligned}$$

Then, $\{(T,K)\}$ is a g-frame in H with bounds $(\|(T^*T)^{-1}\|^{-1}, \|T\|^2)$.

Here, the converse is also true; that is, if $\{(T,K)\}$ is a g-frame with bounds (A,B), then

$$A\langle f,f\rangle \leq \langle Tf,Tf\rangle \leq B\langle f,f\rangle, \forall f \in H. \tag{2.3}$$

It follows that $(T^*T)^{-1}$ is bounded, and hence T is injective and has a closed range.

(2) *Assume $T \in B_*(H,K)$ is a surjective map. Then, T^* is injective and its range is closed [9, Theorem 3.2]. Therefore, $\{(T^*,H)\}$ is a g-frame in K.*

(3) *Let $T \in B_*(H)$ be a normal adjointable map. Then, it follows directly from the definition that $\{(T,H)\}$ is a g-frame in H if and only if $\{(T^*,H)\}$ is a g-frame in H. For a general $T \in B_*(H,K)$, its bijectivity is equivalent with the bijectivity of its adjoint T^*; hence, $T \in B_*(H,K)$ is bijective if and only if both $\{(T,K)\}$ and $\{(T^*,H)\}$ are g-frames.*

Frame operator is an important notion in the reconstruction formula in the theory of ordinary frames. Given a g-frame $\{(\Lambda_j,K_j)\}_{j\in J}$ in the Hilbert C^*-module H with bounds (A,B), its corresponding *g-frame operator* is defined in [8] as an operator $S \in B_*(H)$ satisfying $Sf = \sum_{j\in J}\Lambda_j^*\Lambda_j f$, for all $f \in H$. The operator S is well defined, positive, invertible and adjointable; moreover, it satisfies $\|S\| \leq B$, $A \leq S \leq B$ and $B^{-1} \leq S^{-1} \leq A^{-1}$. In the same paper [8], one can also find the following result as Theorem 3.2.

Theorem 2.9. *Let $\{(\Lambda_j, K)\}_{j\in J}$ be a g-frame in the Hilbert C^*-module H with bounds (A, B), and let S be the corresponding g-frame operator. Let $T \in B_*(H, K)$ be invertible. Then, $\{(\Lambda_j T^*, K)\}_{j\in J}$ is a g-frame in K with bounds $(A\|(T^*T)^{-1}\|^{-1}, B\|T\|^2)$ and the g-frame operator TST^*.*

The following modification of the proof of Theorem 3.5 of [8] reveals that the bijectivity condition on T can be relaxed to the surjectivity of T and that the converse of the theorem remains true.

Proof. Assume $\{(\Lambda_j T^*, K)\}_{j\in J}$ is a g-frame in K with the g-frame operator S_1. Then,

$$S_1 g = \sum_{j\in J} (\Lambda_j T^*)^* \Lambda_j T^* g = T(\sum_{j\in J} \Lambda_j^* \Lambda_j) T^* g = TST^* g, \quad \forall g \in K,$$

and hence $S_1 = TST^*$. To show T is surjective, observe that $g = S_1 S_1^{-1} g = T(ST^* S_1^{-1} g)$, for all $g \in K$.

Conversely, if T^* is surjective, then set $f = T^* g$ in (2.1) to deduce that

$$A\langle T^*g, T^*g\rangle \;\leq\; \sum_{j\in J} \langle \Lambda_j T^* g, \Lambda_j T^* g\rangle \;\leq\; B\langle T^*g, T^*g\rangle, \quad \forall g \in K,$$

and conclude from Lemma 2.7 that

$$A\|(TT^*)^{-1}\|^{-1}\langle g, g\rangle \;\leq\; \sum_{j\in J} \langle \Lambda_j T^* g, \Lambda_j T^* g\rangle \;\leq\; B\|T\|^2 \langle g, g\rangle.$$

□

The next example shows that surjectivity is a necessary condition.

Example 2.10. *Let $\{(\Lambda_j, \mathbb{C}^2)\}_{j\in J}$ be as in Example 2.3 and let $T : \mathbb{C}^2 \to \mathbb{C}^2$ be the mapping $(z_1, z_2)^t \mapsto (z_1, 0)^t$. One can easily check that T is a non-surjective, self-adjoint $\mathbb{C}^2$-linear transformation and $\Lambda_j T^*(z_1, z_2)^t = (a_j z_1, 0)^t$, for all $(z_1, z_2)^t \in \mathbb{C}^2$ and $j \in J$. Now, if $g_j = (0, 1)^t$, for all $j \in J$, then $\sum_j \Lambda_j T^* g_j = 0$, defying the essential property (2.1) of a g-frame.*

Also, Theorem 3.5 of [8] can be extended as follows. Recall that if S is a selfadjoint element of a C^*-algebra $\mathcal{A}$, then the functional calculus can be applied to define $f(A)$ for any continuous function defined on the spectrum of S; in particular, S^r is well-defined if S is a positive element [10]. For the proof of the following corollary, we apply the functional

calculus for the selfadjoint element S of the C^*-algebra $B_*(H)$ to write $S = S^{(t-1)/2}S^{(3-t)/2}$, and then apply Theorem 2.9 to $T = S^{(t-1)}$.

Corollary 2.11. *Let $\{\Lambda_j \in B_*(H, K_j) : j \in J\}$ be a g-frame for H with respect to $\{K_j\}_{j\in J}$ with frame bounds (A, B). If S is the g-frame operator, then the sequence $\{\Lambda_j S^{\frac{t-1}{2}} \in B_*(H, K_j) : j \in J\}$ is a g-frame with g-frame operator S^t and g-frame bounds $(A\|S^{1-t}\|^{-1}\ ,\ B\|S^{\frac{t-1}{2}}\|^2)$, for all $t \in \mathbb{R}$. Moreover, $A^t \leq S^t \leq B^t$, for $t \in (0, 1)$.*

3. The duals of g-frames for Hilbert C^*-modules

In this section, we will characterize the dual g-frames for Hilbert C^*-modules in terms of g-Bessel sequences. Similar results can be proved for g-frames for Hilbert spaces.

For a sequence $\{\Lambda_j \in B_*(H, K_j) : j \in J\}$, the pre-frame operator θ from H into K_J is defined by $\theta f = (\Lambda_j f)_{j\in J}$ [8]. The relation between the g-Bessel sequences and their pre-frame operators plays an important role in characterizing the dual g-frames.

If $\{(\Lambda_j, K_j)_{j\in J}\}$ is a g-Bessel sequence in the C^*-module H with bound B, then the pre-frame operator θ is adjointable and $\|\theta\| \leq \sqrt{B}$. Moreover, $\theta^*(k_j)_{j\in J} = \sum_{j\in J} \Lambda_j^* k_j$, for all $(k_j)_{j\in J} \in K_J$. The adjointable map θ^* is called the synthesis operator of $\{\Lambda_j\}_{j\in J}$. In addition, if $\{(\Lambda_j, K_j)_{j\in J}\}$ is a g-frame in the C^*-module H with the g-frame operator S, then $S = \theta^*\theta$.

The following proposition shows that every adjointable map θ from H into K_J induces a g-Bessel sequence in H, whose pre-frame operator is θ.

Proposition 3.1. *Let θ be an adjointable map from H into K_J. Define $\Lambda_j f = \pi_j \theta f$, for $f \in H$ and $j \in J$. Then, Λ_j is an adjointable map for all $j \in J$ and $\{(\Lambda_j, K_j)_{j\in J}\}$ is a g-Bessel sequence with the upper bound $\|\theta\|^2$.*

Proof. Clearly, every Λ_j is adjointable. Then,

$$\sum_{j\in J} \langle \Lambda_j f, \Lambda_j f\rangle = \sum_{j\in J} \langle \pi_j \theta f, \pi_j \theta f\rangle = \langle \theta f, \theta f\rangle = \langle \theta^*\theta f, f\rangle$$
$$\leq \|\theta^*\theta\|\langle f, f\rangle = \|\theta\|^2\langle f, f\rangle, \quad \forall f \in H.$$

So, $\{\Lambda_j\}_{j\in J}$ is a g-Bessel sequence with the upper bound $\|\theta\|^2$. □

Duals and canonical duals of g-frames for Hilbert spaces are defined by Sun [13]; here, we extend the definitions to Hilbert C^*-modules.

Definition 3.2. *Let $\{(\Lambda_j, K_j)\}_{j\in J}$ be a g-frame in H with g-frame operator S and pre-frame operator θ. A g-frame $\{(\Omega_j, K_j)\}_{j\in J}$ is a dual g-frame of $\{(\Lambda_j, K_j)\}_{j\in J}$ if $\sum_{j\in J}\Lambda_j^*\Omega_j = I$. In particular, the g-frame $\{(\widetilde{\Lambda}_j, K_j)\}_{j\in J} := \{(\Lambda_j S^{-1}, K_j)\}_{j\in J}$ is called the canonical dual g-frame of $\{(\Lambda_j, K_j)\}_{j\in J}$.*

We now give a characterization of dual g-frames in terms of right inverses of the synthesis operators. The following example reveals that although it may be easy to find the canonical dual g-frame, the characterization of all dual g-frames does not seem to be an easy task.

Example 3.3. *Let H be the Hilbert C^*-module $\mathbb{C}^2$ as in Example 2.3 and define*

$$\Lambda_j : \mathbb{C}^2 \to \mathbb{C}^2,\ (z_1, z_2)^t \mapsto (z_1/j, z_2/j)^t,\ j \in J.$$

Then,

$$\sum_{j\in J}\langle \Lambda_j z, \Lambda_j z\rangle = \sum_{j\in J} j^{-2}\langle z, z\rangle,\ \forall z \in \mathbb{C}^2.$$

For $A = \sum_{j\in J} j^{-2}$, the sequence $\{(\Lambda_j, \mathbb{C}^2\}_{j\in J}$ is an A-tight g-frame. Set $\Gamma_j = A^{-1}\Lambda_j$, for all $j \in J$. Clearly, $\{\Gamma_j\}_{j\in J}$ is the canonical dual g-frame of $\{(\Lambda_j, \mathbb{C}^2)\}_{j\in J}$.

Theorem 3.4. *Let $\{(\Lambda_j, K_j)\}_{j\in J}$ be a g-frame in H with the pre-frame operator θ, the g-frame operator S and the canonical dual g-frame $\{(\widetilde{\Lambda}_j, K_j)\}_{j\in J}$. Let $\{(\Omega_j, K_j)\}_{j\in J}$ be an arbitrary dual g-frame of $\{(\Lambda_j, K_j)\}_{j\in J}$ with the pre-frame operator η. Then, the following assertions are true.*

(1) $\theta^*\eta = I$.
(2) $\Omega_j = \pi_j\eta$, *for all* $j \in J$.
(3) *If $\eta' : H \to K_J$ is any adjointable right inverse of θ^*, then $\{(\pi_j\eta', K_j)\}_{j\in J}$ is a dual g-frame of $\{(\Lambda_j, K_j)\}_{j\in J}$ with the pre-frame operator η'.*
(4) *The g-frame operator S_Ω of $\{(\Omega_j, K_j)\}_{j\in J}$ is equal to $S^{-1}+\eta^*(I-\theta S^{-1}\theta^*)\eta$.*
(5) *Every adjointable right inverse η' of θ^* is of the form*

$$\eta' = \theta S^{-1} + (I - \theta S^{-1}\theta^*)\xi, \tag{3.1}$$

for some adjointable map $\xi : H \to K_J$, and vice versa.

(6) *There exists a g-Bessel sequence* $\{(\Delta_j, K_j)\}_{j\in J}$ *in* H *whose pre-frame operator is* η *and yields*

$$\Omega_j = \widetilde{\Lambda}_j + \Delta_j - \sum_{k\in J} \widetilde{\Lambda}_j \Lambda_k^* \Delta_k, \text{ for all } j \in J.$$

Proof. (1) For $f, g \in H$,

$$\begin{aligned} \langle \theta^*\eta f, g\rangle &= \langle \eta f, \theta g\rangle = \langle (\Omega_j f), (\Lambda_j g)\rangle \\ &= \sum_j \langle \Omega_j f, \Lambda_j g\rangle = \sum_j \langle \Lambda_j^* \Omega_j f, g\rangle \\ &= \langle \sum_j \Lambda_j^* \Omega_j f, g\rangle = \langle f, g\rangle. \end{aligned}$$

(2) The proof is clear from the definition.

(3) Since η' is adjointable, it follows from Proposition 3.1 that $\{(\pi_j \eta')\}_{j\in J}$ is a g-Bessel sequence in H. Also, since $\eta'^*\theta = I$, η'^* is surjective, by Lemma 2.7(1), for $f \in H$,

$$\|(\eta'^*\eta)^{-1}\|^{-1}\langle f, f\rangle \leq \langle \eta' f, \eta' f\rangle = \sum_{j\in J} \langle \pi_j f, \pi_j f\rangle.$$

Clearly, η' is the pre-frame of $\{(\pi_j\eta', K_j)\}_{j\in J}$.

(4) $S_\Omega = \eta^*\eta = \eta^*\theta S^{-1} + \eta^*(I - \theta S^{-1}\theta^*)\eta = S^{-1} + \eta^*(I - \theta S^{-1}\theta^*)\eta$.

(5) If η' is such a right inverse, then

$$\theta S^{-1} + (I - \theta S^{-1}\theta^*)\eta' = \theta S^{-1} + \eta' - \theta S^{-1}\theta^*\eta' = \eta'.$$

The converse is straightforward.

(6) Let $\{(\Delta_j, K_j)\}_{j\in J}$ be a g-Bessel sequence in H with the pre-frame operator η. For $j \in J$, let $\Omega_j = \widetilde{\Lambda}_j + \Delta_j - \sum_{k\in J} \widetilde{\Lambda}_j \Lambda_k^* \Delta_k$. Let S and θ be the g-frame operator and the pre-frame operator of $\{(\Lambda_j, K_j)\}_{j\in J}$, respectively. Define the linear operator $\xi : H \to K_J$ by $\xi f = (\Omega_j f)_{j\in J}$. Clearly, ξ is adjointable. For every $j \in J$, we have

$$\begin{aligned} \pi_j \xi = \Omega_j &= \Lambda_j S^{-1} + \Delta_j - \Lambda_j S^{-1} \sum_{k\in J} \Lambda_k^* \Delta_k \\ &= \pi_j(\theta S^{-1} + \eta - \theta S^{-1}\theta^*\eta). \end{aligned}$$

Then, $\xi = \theta S^{-1} + \eta - \theta S^{-1}\theta^*\eta$. By parts (3) and (5) of the theorem, $\{(\Omega_j, H)\}_{j\in J}$ becomes a dual g-frame of $\{(\Lambda_j, H)\}_{j\in J}$. □

Example 3.5. *Let* $\mathcal{A}$ *be a Hilbert* $\mathcal{A}$*-module over itself (See [14]). Let* $\{f_j\}_{j\in J} \subset \mathcal{A}$*. For* $j \in J$*, define the adjointable* $\mathcal{A}$*-module map* Λ_{fj} :

$\mathcal{A} \to \mathcal{A}$ *by* $\Lambda_{fj} f = \langle f, f_j \rangle$. *Clearly,* $\{f_j\}_{j\in J}$ *is a frame in* $\mathcal{A}$ *if and only if* $\{(\Lambda_{fj}, \mathcal{A})\}_{j\in J}$ *is a g-frame in* $\mathcal{A}$. *In the following, we study the duals of such g-frames.*

(*a*) *Let* $\{g_j\}_{j\in J} \subset \mathcal{A}$. *For all* $f \in \mathcal{A}$,

$$\sum_{j\in J} \Lambda^*_{gj}\Lambda_{fj} f = \sum_{j\in J} \langle f, f_j\rangle g_j = \sum_{j\in J} \langle f, g_j\rangle f_j = \sum_{j\in J} \Lambda^*_{fj}\Lambda_{gj} f.$$

Therefore, $\{g_j\}_{j\in J}$ *is a dual frame of* $\{f_j\}_{j\in J}$ *if and only if* $\{(\Lambda_{gj}, \mathcal{A})\}_{j\in J}$ *is a dual g-frame of* $\{(\Lambda_{fj}, \mathcal{A})\}_{j\in J}$.

(*b*) *Let* S *and* S_Λ *be the frame operators of* $\{f_j\}_{j\in J}$ *and* $\{(\Lambda_{fj}, \mathcal{A})\}_{j\in J}$, *respectively. Now, for all* $f \in \mathcal{A}$,

$$\sum_{j\in J} \langle f, f_j\rangle f_j = \sum_{j\in J} f f_j^* f_j = \sum_{j\in J} \langle \langle f, f_j\rangle, f_j^*\rangle = \sum_{j\in J} \Lambda^*_{fj}\Lambda_{fj} f.$$

It follows that $S = S_\Lambda$.

(*c*) *It is easy to see that a sequence* $\{h_j\}_{j\in J} \subset \mathcal{A}$ *is a Bessel sequence if and only if* $\{(\Lambda_{hj}, \mathcal{A})\}_{j\in J}$ *is a g-Bessel sequence.*

(*d*) *For a Bessel sequence* $\{h_j\}_{j\in J}$, *define*

$$g_j = S^{-1} f_j + h_j - \sum_{k\in J} \langle S^{-1} f_j, f_k\rangle h_k, \ (j \in J).$$

Then, the sequence $\{g_j\}_{j\in j}$ *is a dual frame of* $\{f_j\}_{j\in J}$ [3]. *By Theorem 3.4, the sequence* $\{(\Gamma_j, \mathcal{A})\}_{j\in J}$ *is a dual g-frame of* $\{\Lambda_{fj}\}_{j\in J}$, *where* $\Gamma_j = \widetilde{\Lambda}_{fj} + \Lambda_{hj} - \sum_{k\in J} \widetilde{\Lambda}_{fj}\Lambda^*_{fk}\Lambda_{hk}$, *for all* $j \in J$. *Now, we claim that* $\Gamma_j = \Lambda_{gj}$. *In fact, for all* $f \in \mathcal{A}$,

$$\begin{aligned}
\Gamma_j f &= \Lambda_{fj} S^{-1} f + \Lambda_{hj} f - \sum_{k\in J} \Lambda_{fj} S^{-1} \Lambda^*_{fk} \langle f, h_k\rangle \\
&= \langle S^{-1} f, f_j\rangle + \langle f, h_j\rangle - \sum_{k\in J} \langle S^{-1} f h_k^* f_k, f_j\rangle \\
&= \langle f, S^{-1} f_j\rangle + \langle f, h_j\rangle - \sum_{k\in J} \langle f h_k^* f_k, S^{-1} f_j\rangle \\
&= \langle f, S^{-1} f_j + h_j - \sum_{k\in J} \langle S^{-1} f_j, f_k\rangle h_k\rangle \\
&= \langle f, g_j\rangle = \Lambda_{gj} f.
\end{aligned}$$

Therefore, every dual g-frame of $\{(\Lambda_{fj}\}_{j\in J}$ *has the form* $\widetilde{\Lambda}_{fj} + \Lambda_{hj} - \sum_{k\in J} \widetilde{\Lambda}_{fj}\Lambda^*_{fk}\Lambda_{hk}$, *where* $\{h_j\}_{j\in J}$ *is a Bessel sequence in* $\mathcal{A}$.

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