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INVARIANCE OF THE BARYCENTRIC SUBDIVISION OF A SIMPLICIAL COMPLEX

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Abstract. We prove that a simplicial complex is determined uniquely up to isomorphism by its barycentric subdivision as well as its comparability graph. We also put together several algebraic, combinatorial and topological invariants of simplicial complexes.

1. **Introduction and preliminaries**

Stanley-Reisner rings of simplicial complexes, which have had impressive applications in combinatorics [7], possess a rigidity property in the sense that they determine their underlying simplicial complexes uniquely up to isomorphism ([4] and [8]). Barycentric subdivision of a simplicial complex is another very important and applicable construction ([1, 2, 6, 7]), of which we aim to prove that it possesses the same rigidity property.

We first recall some basic definitions and facts about simplicial complexes and related topics which we will need later; see [3, 5] and [7], for details.

Let $[n] = \{1, 2, \ldots, n\}$. A (finite) simplicial complex Δ on n vertices is a collection of subsets of $[n]$ such that the following conditions hold:
(a) $\{i\} \in \Delta$, for all $i \in [n]$,
(b) if $E \in \Delta$ and $F \subseteq E$, then $F \in \Delta$.

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An element of Δ is called a face and a maximal face with respect to inclusion is called a facet. The set of facets of Δ is denoted by $\mathcal{F}(\Delta)$ and the set of minimal members of the collection of non-faces of Δ (subsets of $[n]$ not in Δ) is denoted by $\mathcal{N}(\Delta)$. The dimension of a face $F \in \Delta$ is defined to be $|F| - 1$ and the dimension of Δ is maximum of dimensions of its faces. A simplicial complex is called pure if all of its facets have the same dimension.

Let Δ be a simplicial complex of dimension $d-1$ on $[n]$. For each $0 \leq i \leq d-1$, the ith skeleton of Δ is the simplicial complex $\Delta^{(i)}$ on $[n]$, whose faces are those faces F of Δ with $|F| \leq i+1$. In particular the 1-skeleton $\Delta^{(1)}$ of Δ is the finite graph on $[n]$, whose edges are the 1-dimensional faces $\{i, j\}$ of Δ. We say that a simplicial complex Δ is connected if $\Delta^{(1)}$ is connected. A simplex is a simplicial complex with just one facet.

Let $S = K[x_1, \ldots, x_n]$ be the polynomial ring in n indeterminates with coefficients in a field K. Let I_Δ be the ideal of S generated by all monomials $x_{i_1} \cdots x_{i_s}$, provided that $\{i_1, \ldots, i_s\} \notin \Delta$. It is clear that the minimal generating set of I_Δ is all square-free monomials $x_{i_1} \cdots x_{i_s}$ such that $\{i_1, \ldots, i_s\} \in \mathcal{N}(\Delta)$. The quotient ring $K[\Delta] = S/I_\Delta$ is called the Stanley-Reisner ring of the simplicial complex Δ.

The facet ideal of Δ is the ideal $I(\Delta)$ of S, which is generated by square-free monomials $x_{i_1} \cdots x_{i_s}$, provided that $\{i_1, \ldots, i_s\}$ is a facet of Δ. The quotient ring $K_{\mathcal{F}}[\Delta] = S/I(\Delta)$ is called the facet ring of Δ.

For a given simplicial complex Δ on $[n]$, define $\Delta^\vee$ by

$$\Delta^\vee = \{[n] \setminus F : F \notin \Delta\}.$$

It is clear that $\Delta^\vee$ is a simplicial complex and $(\Delta^\vee)^\vee = \Delta$. The simplicial complex $\Delta^\vee$ is called the Alexander dual of Δ. Note that

$$\mathcal{F}(\Delta^\vee) = \{[n] \setminus F : F \in \mathcal{N}(\Delta)\}. \tag{1.1}$$

The complement simplicial complex Δ^c of Δ is defined to be the simplicial complex whose facets are complements of facets of Δ. One has

$$I_{\Delta^\vee} = I(\Delta^c). \tag{1.2}$$

A partially ordered set (poset) is a nonempty set P with an order $\leq$ such that for each x, y and z in P,
(a) $x \leq x$,
(b) if $x \leq y$ and $y \leq x$, then $x = y$,
(c) if $x \leq y$ and $y \leq z$, then $x \leq z$.

A simplicial complex can be considered as a poset ordered by inclusion.

Let G be a simple graph on the vertex set $V = \{v_l, ..., v_n\}$. Let $S = K[x_1, \ldots, x_n]$. The edge ideal $I(G)$ is defined to be the ideal of S, generated by all square-free monomials $x_i x_j$, provided that v_i is adjacent to v_j in G. The quotient ring $R(G) = S/I(G)$ is called the edge ring of G. We say that a set $F \subseteq V$ is an independent set in G if no two of its vertices are adjacent. Define the independence complex of G, the simplicial complex Δ_G, by

$$\Delta_G = \{F \subseteq V : F \text{ is an independent set in } G\}.$$

For a vertex v in a graph G, define the degree of v to be the number of vertices in G adjacent to v. A path in the graph G is a sequence of vertices $v_1, \ldots, v_r$ such that v_i is adjacent to v_{i+1}, for each i, $1 \leq i < r$. A cycle is a path with $v_1 = v_r$.

Let Δ be a simplicial complex on the vertex set $[n]$. The barycentric subdivision of Δ, denoted by $\Delta^\flat$, is a simplicial complex with vertex set consisting of all nonempty faces of Δ. A face in $\Delta^\flat$ consists of comparable vertices, that is, two vertices lie in a face in $\Delta^\flat$ if one is a subset of the other. In other words, the facets of $\Delta^\flat$ are the maximal chains of faces of Δ considered as a poset.

It is easy to see that the minimal non-faces of $\Delta^\flat$ are subsets of Δ with exactly two non-comparable elements. Therefore, $\Delta^\flat$ is a clique complex and the ideal $I_{\Delta^\flat}$ is generated by square-free quadrics. It is known that the dimensions (and depths, respectively) of a simplicial complex and its barycentric subdivision are equal ([2] and [6]).

The 1-skeleton of $\Delta^\flat$ is called the comparability graph of Δ and is denoted by $G(\Delta)$. The complement $\overline{G(\Delta)}$ of $G(\Delta)$ is called the non-comparability graph of Δ. The ideal $I_{\Delta^\flat}$ is the edge ideal of the graph $\overline{G(\Delta)}$ and the simplicial complex $\Delta^\flat$ is the independence complex of this graph.

It is not true that any graph is the comparability graph of some simplicial complex. For example, there is no simplicial complex with comparability graph equal to a cycle of length 3, 4 or 5. A necessary condition for a graph to be comparability graph of some simplicial complex is to be transitively orientable. That is, there is an orientation on the graph such that if (x, y) and (y, z) are oriented edges, then it contains the oriented edge (x, z).

A (convex) polytope is the convex hull of a finite set of points in the n-dimensional Euclidean space for some n. A proper face of a polytope is the intersection of the polytope with a supporting hyperplane. The empty set and the polytope itself are called improper faces. A polyhedral complex is the union of a finite set of polytopes such that intersection of any two is a face of each.

It is known that the geometric realizations of Δ and $\Delta^\flat$ are homeomorphic as topological spaces and therefore, they share topological properties such as Cohen-Macauleyness's (see [7], p. 101]).

2. **The main result**

It is natural to ask whether a given graph is the comparability graph of some simplicial complex and, how many non-isomorphic simplicial complexes are there with the same comparability graphs. Here, we prove that there is only one simplicial complex with a given comparability graph (up to isomorphism). An isomorphism of simplicial complexes Δ_1 and Δ_2 is a bijection between their vertex sets which preserves faces and facets. It is enough to check that the image and the inverse image of any facet is again a facet. The face lattice of a polyhedral complex is a generalization of the notion of simplicial complex (see [1] for definitions). In the case of polyhedral complexes with at least two maximal faces, Bayer has proved the following result.

Theorem 2.1. *(Bayer* [1]*) Let P be the face lattice of a connected polyhedral complex with at least two maximal faces, and let P^* be its dual poset. If Q is a poset with $Q^\flat = P^\flat$, then either $Q = P$ or $Q = P^*$.*

A simplicial complex is the face lattice of the polyhedral complex of its geometric realization and hence, by the above theorem, for a given connected simplicial complex with at least two facets, there are at most one more simplicial complex with the same barycentric subdivision. Let Δ be a simplicial complex. The barycentric subdivision $\Delta^\flat$ is the clique complex of the comparability graph $G(\Delta)$ and $G(\Delta)$ is the 1-skeleton of $\Delta^\flat$. Therefore, knowing one of them is enough to construct the other. Accordingly, two barycentric subdivisions are isomorphic as simplicial complexes if and only if their 1-skeletons are isomorphic as graphs. If Δ_1 and Δ_2 are two simplicial complexes, then they are isomorphic if and only if there is a rearrangement of their connected components such that the corresponding components are isomorphic separately. Therefore, in connection with the isomorphism problem of simplicial complexes, it is enough to consider connected complexes, which is equivalent to considering connected barycentric subdivision.

In the proof of the above theorem, Bayer has shown that for a given poset P with the mentioned conditions, there are exactly two transitive orientations on the 1-skeleton graph of $P^\flat$, which are reverses of each other. We will show that in the case of simplicial complexes, there is no need for these conditions and, either only one of the orientations on the graph corresponds to a simplicial complex or, the simplicial complexes corresponding to these two orientations are isomorphic.

Let Δ be a simplicial complex of dimension d and $G = G(\Delta)$ be its comparability graph. Let $\overrightarrow{G}$ be the orientation of G corresponding to the inclusion order on Δ_1, that is, (x, y) is a directed edge in $\overrightarrow{G}$ if and only if $x \subset y$ in Δ. We give a grade to each vertex x of $\overrightarrow{G}$ equal to cardinality of the set x in Δ. Note that there is no edge between two vertices with the same grade. The graph $\overrightarrow{G}$ is $(d+1)$-partite, each part consisting of all vertices of the same grade. There is a part in $\overrightarrow{G}$ consisting of all vertices with all arrows directed out, which we call initial points. This is the set of vertices (faces of dimension 0) of the underlying simplicial complex. The vertices of $\overrightarrow{G}$ for which all the connecting arrows are directed in, which we call terminal points, are indeed facets of Δ. The number of vertices of the longest directed path with end point x in $\overrightarrow{G}$ is equal to grade of x. Hence, the underlying simplicial complex Δ can be uniquely determined by $\overrightarrow{G}$. We will denote a vertex of $\overrightarrow{G}$ and the corresponding face in Δ with the same letters.

Lemma 2.2. *Let Δ be a connected simplicial complex and $\overrightarrow{G}$ be its directed comparability graph. Let E, F and H be vertices in $\overrightarrow{G}$.*

- (i) *If (E, F) and (F, H) are directed edges, then the directed edge (E, H) belongs to the graph.*
- (ii) *Let E and F be vertices with grades r and s, respectively, such that $r < s$. If E and F are adjacent, then there is a path $E = H_0, H_1, \ldots, H_l = F$ such that $l = s - r$, and for each $i = 0, \ldots, l$, grade of E_i is $r + i$.*
- (iii) *Let E and F be vertices with the same grade r. Then, there is a path connecting E and F such that for each vertex H in the path, grade$(H) \leq 2r$.*

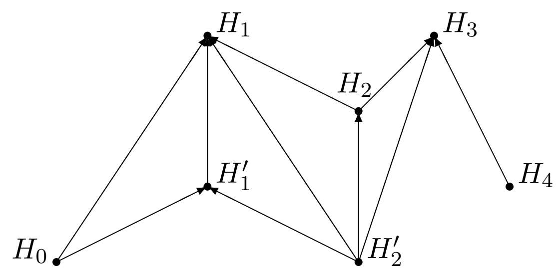

FIGURE 1. A path with lower grade

Proof. Parts (i) and (ii) are clear. To prove (iii), note that $\overrightarrow{G}$ is connected and there is a path connecting E and F. By (i) and (ii), it is enough to prove the statement for the case that $\text{grade}(H) \geq r$, for each H in the path. Let $E = H_0, H_1, \ldots, H_l = F$ be a path. We may assume that $\text{grade}(E) < \text{grade}(H_1) > \text{grade}(H_2) < \cdots < \text{grade}(H_{l-1}) > \text{grade}(F)$. Let H_2' be a subset of H_2 and $|H_2'| = r$. Then, $H_2' \in \Delta$ and the edges (H_2', H_1), and (H_2', H_3) are in the graph. On the other hand, H_1 contains $H_0 \cup H_2' = H_1'$, and we may put H_1' instead of H_1 and H_2' instead of H_2 in the path and get another path from F to E. In the new path, we have $\text{grade}(H_2') = r$ and $\text{grade}(H_1') = \text{grade}(H_0 \cup H_2') \leq 2r$ (Figure 1). Continuing this process, we will get a path with the required property.

□

Theorem 2.3. *Let Δ_1 and Δ_2 be two simplicial complexes. Then, Δ_1 and Δ_2 are isomorphic if and only if the graphs $G(\Delta_1)$ and $G(\Delta_2)$ are isomorphic.*

Proof. If Δ_1 and Δ_2 are isomorphic, then it is clear that $G(\Delta_1)$ and $G(\Delta_2)$ are isomorphic. For the converse, as mentioned above, it is enough to consider connected simplicial complexes. Therefore, let Δ_1 and Δ_2 be connected. Without loss of generality, we may assume that $G(\Delta_1) = G(\Delta_2) = G$. If Δ_1 is of dimension 0, then it is a single point and the theorem holds. Suppose that dimension of $\Delta_1 = d \geq 1$. If Δ_1 has only one facet, then it is a simplex and in the graph G there is a unique vertex corresponding to the facet, which is adjacent to all other vertices. The only possibility for Δ_2 with a non-empty face comparable with all others is to be a simplex. Therefore, Δ_2 is a simplex of dimension d and any two simplexes of the same dimension are isomorphic.

Now, we consider the case that Δ_1 has at least two facets. In this case, by Theorem 2.1, there are at most two simplicial complexes with comparability graph G. One of them is Δ_1 with directed comparability graph $\overrightarrow{G}$. Let $\overleftarrow{G}$ be the graph G with directions being the reversal of those in $\overrightarrow{G}$. Assume that $\overleftarrow{G}$ is the directed comparability graph of some simplicial complex Δ_2. We want to prove that Δ_1 and Δ_2 are isomorphic. We denote a face in Δ_1 or the corresponding vertex in $\overrightarrow{G}$ by a capital letter as F and the same vertex in $\overleftarrow{G}$ or in $\overleftarrow{G}$ by F^*.

First, we prove that Δ_1 and Δ_2 are pure, or equivalently, in $\overrightarrow{G}$ and $\overleftarrow{G}$, all maximal chains have the same length. To the contrary, assume that this is not the case and there are two facets F_1 and F_2 in Δ_1 with $\dim(F_1) < \dim(F_2)$. The vertices F_1 and F_2 are terminal points in $\overrightarrow{G}$, and therefore, F_1^* and F_2^* are initial points in $\overleftarrow{G}$ and they are not adjacent. The vertices F_1^* and F_2^* have grade 1 and by Lemma 2.2 (iii), there is a path P^* in $\overleftarrow{G}$ from F_1^* to F_2^* with vertices alternatively of grades 1 and 2. Therefore, the path P in $\overrightarrow{G}$ consists of terminal and subterminal vertices alternatively. But, by Lemma 2.2 (ii), this is impossible if $\text{grade}(F_1) \neq \text{grade}(F_2)$, because F_1 is not adjacent to any vertex with a higher grade. Therefore, Δ_1 and Δ_2 are pure.

Let Δ_1 and Δ_2 be pure complexes of dimension $d-1$. Let $\mathbf{F} = \{F_1, F_2, \ldots, F_r\}$ be the set of vertices of grade d and $\mathbf{E} = \{E_1, E_2, \ldots, E_t\}$ be the set of vertices of grade $d-1$ in $\overrightarrow{G}$. Then, $\{F_1^*, F_2^*, \ldots, F_r^*\}$ and $\{E_1^*, E_2^*, \ldots, E_t^*\}$ are the sets of vertices of grade 1 and 2, respectively. Let $\overleftarrow{G}_{0,1}$ be the induced subgraph of $\overleftarrow{G}$ on these vertices. Define similarly the induced subgraph $\overrightarrow{G}_{d-1,d}$ of $\overrightarrow{G}$ on the set of vertices $\mathbf{F} \cup \mathbf{E}$. In the graph $\overleftarrow{G}_{0,1}$, all vertices in $\mathbf{E}$ have degree 2 (they are corresponding to faces of Δ_2 with two elements), and vertices in $\mathbf{F}$ have degree d (they are corresponding to facets of Δ_1). By Lemma 2.2 (iii), the graph $\overleftarrow{G}_{0,1}$ is connected. If $d = 2$, then the graph $\overleftarrow{G}_{0,1}$ is in fact $\overleftarrow{G}$ and it is a connected bipartite graph such that all vertices have degree 2. Therefore, it is a cycle with even number of vertices and both parts have the same cardinality (Figure 2). Therefore, the simplicial complexes Δ_1 and Δ_2 are isomorphic.

Now, let $d > 2$. Then, $\overleftarrow{G}_{0,1}$, and equivalently, $\overrightarrow{G}_{d-1,d}$ are connected. Let P be a path in $\overrightarrow{G}_{d-1,d}$ of the form F_1, E_1, F_2, E_2, F_3 such that $\{F_1, F_2, F_3\} \subseteq \mathbf{F}$ and $\{E_1, E_2\} \subseteq \mathbf{E}$. Then, $|F_1 \cap F_2| = |E_1| = d-1$, and

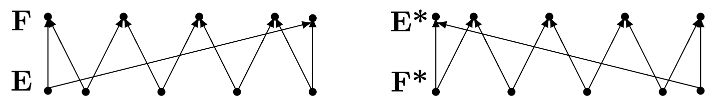

FIGURE 2. Simplicial complexes Δ_1 and Δ_2 with dimension 1

$|F_2 \cap F_3| = |E_2| = d-1$. Thus, $F_1 \cap F_3 = B \neq \varnothing$ and $B \in \Delta_1$. Therefore, $B^* = F_1^* \cup F_3^* \in \Delta_2$. But, $|B^*| = 2$ and therefore, $\text{grade}(B) = d-1$. This means that $B \in \mathbf{E}$ and it is adjacent to both F_1 and F_3. By connectedness of $\overrightarrow{G}_{d-1,d}$, we conclude that for any two vertices in $\mathbf{F}$ there is a vertex in $\mathbf{E}$ adjacent to both of them. Let F_i and F_j be two vertices in $\mathbf{F}$ and E_k be a vertex in $\mathbf{E}$ adjacent to both of them. In this case, $F_i^* \cup F_j^* = E_k^*$ and therefore, F_i and F_j determine E_k uniquely. This is equivalent to saying that for each two vertices in $\mathbf{F}$, there is exactly one vertex in $\mathbf{E}$ adjacent to both of them.

Consider $|\mathbf{F}| = r$. Then, $|\mathbf{E}| = rd/2$ and according to the above argument, $|\mathbf{E}| = \binom{r}{2} = r(r-1)/2$. It means that $r = d+1$. Let $A = F_1 \cup \cdots \cup F_r$. Take an arbitrary element $x \in A$. Then, there is $1 \leq i \leq r$ such that F_i contains x. For convenience, suppose $i = 1$ and $E_1, \ldots, E_d$ are all subsets of F_1 with $d-1$ elements. Then, x belongs to all these subsets except one of them, for example E_d. For each i, $1 \leq i \leq d-1$, the vertex E_i is adjacent to F_1 and F_j for a j, $2 \leq j \leq d+1$. In other hand, for j, k, $1 \leq j < k \leq d-1$, the vertices other than F_1 to which E_j and E_k are adjacent, are different. Then, there are d vertices in $\mathbf{F}$ adjacent to some vertices in $\{E_1, \ldots, E_{d-1}\}$. Therefore, x belongs to all sets in $\mathbf{F}$, except one of them. If $A \setminus F_1$ has more than one element, for example x, y, then $\{x, y\} \subseteq F_i$, for each $2 \leq i \leq d+1$. Then, $\{x, y\} \subseteq F_2 \setminus F_1$, which is impossible, since $|F_1 \cap F_2| = d-1$. Therefore, A is a set with cardinality $d+1$ and $F_1, \ldots, F_{d+1}$ are all maximal proper subsets of A. Then, Δ_1 is $d-1$-skeleton of a simplex of dimension d. This is true for Δ_2 and therefore, Δ_1 and Δ_2 are isomorphic. $\square$

By Theorem 2.3, the barycentric subdivision and the comparability graph are invariants of a simplicial complex, and conversely, the underlying simpicial complex is an invariant of its barycentric subdivision and comparability graph. In the following theorem, we summarize more invariants of simplicial complexes.

Theorem 2.4. *Let Δ_1 and Δ_2 be two simplicial complexes. The following conditions are equivalent.*

(i) Δ_1 *and* Δ_2 *are isomorphic as simplicial complexes.*
(ii) $\Delta_1^\vee$ *and* $\Delta_2^\vee$ *are isomorphic as simplicial complexes.*
(iii) Δ_1^c *and* Δ_2^c *are isomorphic as simplicial complexes.*
(iv) $\Delta_1^\flat$ *and* $\Delta_2^\flat$ *are isomorphic as simplicial complexes.*
(v) $\Delta_1^{\flat n}$ *and* $\Delta_2^{\flat n}$ *are isomorphic as simplicial complexes, for some positive integer* n.
(vi) $K[\Delta_1]$ *and* $K[\Delta_2]$ *are isomorphic as* K*-algebras.*
(vii) $K_{\mathcal{F}}[\Delta_1]$ *and* $K_{\mathcal{F}}[\Delta_2]$ *are isomorphic as* K*-algebras.*
(viii) $G(\Delta_1)$ *and* $G(\Delta_2)$ *are isomorphic as graphs.*

Proof. The equivalences of (i), (ii) and (iii) are clear from equations (1.1) and (1.2). The equivalence of (i) and (iv) has been proved in Theorem 2.3. Items (iv) and (v) are equivalent by the equivalence of (i) and (iv). The equivalences of (i), (vi) and (vii) are proved in [4] and [8]. By the argument just after Theorem 2.1, the equivalence of (iv) and (viii) is also clear. □

Remark 2.5. *Let A be a square-free monomial algebra, i.e., an algebra of the form $K[x_1, \dots, x_n]/I$, where I is an ideal generated by some square-free monomials. Then, there is a simplicial complex Δ and a graph G such that*

$$\mathrm{Aut}(A) \cong \mathrm{Aut}(\Delta) \cong \mathrm{Aut}(G),$$

where $\mathrm{Aut}(A)$ *is the group of K-algebra automorphisms of A,* $\mathrm{Aut}(\Delta)$ *is the group of simplicial complex automorphisms of Δ and* $\mathrm{Aut}(G)$ *is the group of graph automorphisms of G. Also, for a given graph G there are square-free monomial algebra A and a simplicial complex Δ satisfying the above isomorphisms. For a given simplicial complex Δ, the similar statement is also true.*

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