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RING STRUCTURES OF MOD p EQUIVARIANT COHOMOLOGY RINGS AND RING HOMOMORPHISMS BETWEEN THEM

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Abstract. We consider a class of connected oriented (with respect to $\mathbb{Z}/p$) closed G-manifolds with a non-empty finite fixed point set, each of which is G-equivariantly formal, where $G = \mathbb{Z}/p$ and p is an odd prime. Using localization theorem and equivariant index, we give an explicit description of the mod p equivariant cohomology ring of such a G-manifold in terms of algebra. This makes it possible to determine the number of equivariant cohomology rings (up to isomorphism) of such 2-dimensional G-manifolds. Moreover, we obtain a description of the ring homomorphism between equivariant cohomology rings of such two G-manifolds induced by a G-equivariant map, and show a characterization of the ring homomorphism.

1. Introduction

Assume that $G = \mathbb{Z}/p$ and p is an odd prime unless stated otherwise. Let X be a G-space and EG be the universal free G-space. Then, the Borel construction $X_G := EG \times_G X$, the orbit space of the diagonal action on the product $EG \times X$, is the total space of the bundle $X \to X_G \to BG$ associated to the universal principal bundle

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$G \to EG \to BG$ ($BG := EG/G$, the classifying space of G). Applying cohomology with coefficients $\mathbb{Z}/p$ to X_G gives the equivariant cohomology ring $H_G^*(X;\mathbb{Z}/p) := H^*(X_G;\mathbb{Z}/p)$. It is well-known that the equivariant cohomology ring $H_G^*(X;\mathbb{Z}/p)$ is an $H^*(BG;\mathbb{Z}/p)$-module and $H_G^*(X^G;\mathbb{Z}/p)$ is a free $H^*(BG;\mathbb{Z}/p)$-module, where X^G denotes the fixed point set of the $G-$action.

Suppose that M is a connected oriented (with respect to $\mathbb{Z}/p$) closed manifold and admits a G-action with a non-empty finite fixed set M^G. For the fibration $M \to M_G \to BG$, if the restriction to a typical fiber $H_G^*(M;\mathbb{Z}/p) \longrightarrow H^*(M;\mathbb{Z}/p)$ is surjective, then M is called totally non-homologous to zero in M_G (cf. [3]). If M satisfies this condition, then M is also called G-equivariantly formal (cf. [5]). In 1998, Goresky, Kottwitz and MacPherson showed that the equivariant cohomology rings of a class of T^n-manifolds (i.e., GKM manifolds) can be explicitly expressed in terms of their associated graphs (cf. [5, 6]). Correspondingly, there is a mod 2 GKM theory (cf. [2]). Note that any odd dimensional oriented closed G-manifold must not have a non-empty finite fixed point set (cf. [4]). Then, we shall give explicit descriptions of the mod p equivariant cohomology rings of G-equivariantly formal manifolds at any even dimension in terms of algebra.

Let Λ_{2n} denote the set of all $2n$-dimensional connected oriented (with respect to $\mathbb{Z}/p$) closed G-manifolds with a non-empty finite fixed point set, each of which is G-equivariantly formal. Given an M in Λ_{2n}, we know from [1, Theorem 3.10.4] and [3, pp. 371-374] that M has the following properties:

(1) The order $|M^G|$ of M^G equals $\sum_{i=0}^{2n} b_i$, where b_i is the ith mod p Betti number of M;

(2) $H_G^*(M;\mathbb{Z}/p)$ is a free $H^*(BG;\mathbb{Z}/p)$-module;

(3) The inclusion $i : M^G \hookrightarrow M$ induces a monomorphism $i^* : H_G^*(M;\mathbb{Z}/p) \longrightarrow H_G^*(M^G;\mathbb{Z}/p)$.

Note that $b_i = b_{2n-i}$, by the Poincaré duality, and $b_0 = b_{2n} = 1$, since M is connected. For each $M \in \Lambda_{2n}$, by the property (1), we have that $|M^G| \geq 2$. Let $r \geq 2$ be a positive integer, and write $\Lambda_{2n}^r = \{M \in \Lambda_{2n} \mid |M^G| = r\}$. Then, $\Lambda_{2n} = \bigcup_{r\geq 2} \Lambda_{2n}^r$. From [1], we also have that $H^*(BG;\mathbb{Z}/p) = \Lambda(s) \bigotimes \mathbb{Z}/p[t] = \mathbb{Z}/p[s,t]/(s^2)$, with deg($s$)=1, deg($t$)=2, and $t = \beta(s)$, where β is the Bockstein homomorphism associated with the coefficient sequence $0 \to \mathbb{Z}/p \to \mathbb{Z}/p^2 \to \mathbb{Z}/p \to 0$. If $|M^G| = r$, since $H_G^*(M^G;\mathbb{Z}/p) = \bigoplus_{a\in M^G} H_G^*(\{a\};\mathbb{Z}/p)$ and the equivariant cohomology ring of a point is isomorphic to $H^*(BG;\mathbb{Z}/p) = \mathbb{Z}/p[s,t]/(s^2)$ where s and t are as above, we have that $H_G^*(M^G;\mathbb{Z}/p) \cong (\mathbb{Z}/p)^r[s,t]/(s^2)$ is a polynomial ring (or algebra). Thus, we obtain a monomorphism from $H_G^*(M;\mathbb{Z}/p)$ into $(\mathbb{Z}/p)^r[s,t]/(s^2)$, also denoted by i^*, and so $H_G^*(M;\mathbb{Z}/p)$ may be identified with a subring (or subalgebra) of $(\mathbb{Z}/p)^r[s,t]/(s^2)$.

Using the localization theorem and equivariant index, we give an explicit description of $H_G^*(M;\mathbb{Z}/p)$ in $(\mathbb{Z}/p)^r[s,t]/(s^2)$ (see Theorem 3.3). By using this result, we find that there is only one equivariant cohomology ring (up to isomorphism) of G-manifolds in Λ_2^r if Λ_2^r is non-empty (see Theorem 4.1). Furthermore, we give a description for the homomorphism between equivariant cohomology rings of two G-manifolds in Λ_{2n} induced by a G-equivariant map (see Theorem 5.1), obtaining a characterization of the homomorphism.

The reminder of our work is organized as follows. In Section 2, we review the localization theorem and reformulate the equivariant index from [1]. In Section 3, we study the equivariant cohomology structure of a G-manifold in Λ_{2n} and obtain an explicit description in terms of algebra. In Section 4, we determine the number of equivariant cohomology rings (up to isomorphism) of G-manifolds in Λ_2. In Section 5, we give a description of the homomorphism between equivariant cohomology rings of two G-manifolds in Λ_{2n} induced by a G-equivariant map, obtaining a characterization of the homomorphism.

2. **Preliminaries**

Let M be a $2n$-dimensional G-manifold with a non-empty finite set M^G. Let R denote the polynomial part of $H^*(BG;\mathbb{Z}/p)$, i.e., $R = \mathbb{Z}/p[t]$ and $S = R - (0)$. Then, we have the following well-known localization theorem (cf. [1, 7]).

Theorem 2.1. *$S^{-1}i^* : S^{-1}H_G^*(M;\mathbb{Z}/p) \longrightarrow S^{-1}H_G^*(M^G;\mathbb{Z}/p)$ is an isomorphism of $S^{-1}H^*(BG;\mathbb{Z}/p)$-algebras, where i is the inclusion of M^G into M.* $\square$

Take an isolated point $a \in M^G$. Let i_a be the inclusion of a into M. Then, we have the equivariant Gysin homomorphism

$$i_{a!} : H^*_G(\{a\}; \mathbb{Z}/p) \longrightarrow H^{*+2n}_G(M; \mathbb{Z}/p).$$

On the other hand, we also have a natural induced homomorphism

$$i^*_a : H^*_G(M; \mathbb{Z}/p) \longrightarrow H^*_G(\{a\}; \mathbb{Z}/p)$$

and we see that $i^* = \bigoplus_{a \in M^G} i^*_a$. Moreover, we have that the equivariant Euler class at a is

$$\chi_G(a) = i^*_a i_{a!}(1_a) \in H^{2n}_G(\{a\}; \mathbb{Z}/p) = H^{2n}(BG; \mathbb{Z}/p) = (\mathbb{Z}/p)t^n,$$

where $1_a \in H^*_G(\{a\}; \mathbb{Z}/p)$ is the identity and $(\mathbb{Z}/p)t^n = \{kt^n | k \in \mathbb{Z}/p\}$. So, we may write

$$\chi_G(a) = N_a t^n,$$

where $N_a \in \mathbb{Z}/p$. Let $\theta_a = i_{a!}(1_a)$. Then, $\theta_a \in H^{2n}_G(M; \mathbb{Z}/p)$ and $i^*_a(\theta_a) = \chi_G(a)$.

Lemma 2.2. *All elements $\theta_a, a \in M^G$ are linearly independent over $H^*(BG; \mathbb{Z}/p)$.*

Proof. Let $\sum_{a \in M^G} l_a \theta_a = 0$, where $l_a \in H^*(BG; \mathbb{Z}/p)$. By Lemma 5.3.14(2) of [1], we have that $i^*_b(\theta_a) = 0$, for $b \neq a$ in M^G, and so

$$i^*_b(\sum_{a \in M^G} l_a \theta_a) = \sum_{a \in M^G} l_a i^*_b(\theta_a) = l_b i^*_b(\theta_b) = l_b \chi_G(b) = 0.$$

Since $\chi_G(b)$ is a unit in $S^{-1}H^*_G(\{b\}; \mathbb{Z}/p) \cong S^{-1}H^*(BG; \mathbb{Z}/p)$, we have $l_b = 0$.

□

Lemma 2.3. *Let $\alpha \in S^{-1}H^*_G(M; \mathbb{Z}/p)$. Then,*

$$\alpha = \sum_{a \in M^G} \frac{f_a \theta_a}{\chi_G(a)},$$

where $f_a = S^{-1}i^*_a(\alpha) \in S^{-1}H^*(BG; \mathbb{Z}/p)$.

Proof. From Proposition 5.3.18(1) of [1], we know that

$$\alpha = \sum_{a \in M^G} S^{-1}i_{a!}(S^{-1}i^*_a(\alpha)/\chi_G(a)).$$

Since $f_a = S^{-1}i^*_a(\alpha) \in S^{-1}H^*(BG; \mathbb{Z}/p)$, we have that $\frac{f_a}{\chi_G(a)} \in S^{-1}H^*$ $(BG; \mathbb{Z}/p)$. Since $S^{-1}i_{a!}$ is a $S^{-1}H^*(BG; \mathbb{Z}/p)$-algebra homomorphism, we have

$$\begin{aligned} S^{-1}i_{a!}(S^{-1}i_a^*(\alpha)/\chi_G(a)) &= S^{-1}i_{a!}(\tfrac{f_a}{\chi_G(a)}) = \tfrac{f_a}{\chi_G(a)}S^{-1}i_{a!}(1_a) \\ &= \tfrac{f_a}{\chi_G(a)}i_{a!}(1_a) = \tfrac{f_a\theta_a}{\chi_G(a)}. \end{aligned}$$

Thus, $\alpha = \sum_{a\in M^G} \frac{f_a\theta_a}{\chi_G(a)}$. □

Remark 2.4. *From Lemma 2.2 and Lemma 2.3, we see that* $\{\frac{\theta_a}{\chi_G(a)} | a \in M^G\}$ *forms a basis of* $S^{-1}H_G^*(M;\mathbb{Z}/p)$ *as a* $S^{-1}H^*(BG;\mathbb{Z}/p)$*-algebra.*

The equivariant Gysin homomorphism of collapsing M to a point gives the G-index of M, i.e.,

$$\mathrm{Ind}_G : H_G^*(M;\mathbb{Z}/p) \longrightarrow H^{*-2n}(BG;\mathbb{Z}/p).$$

Theorem 2.5. *For any* $\alpha \in S^{-1}H_G^*(M;\mathbb{Z}/p)$,

$$S^{-1}\ \mathit{Ind}_G(\alpha) = \sum_{a\in M^G} \frac{f_a}{\chi_G(a)},$$

where $f_a = S^{-1}i_a^*(\alpha) \in S^{-1}H^*$
$(BG;Z/p)$. *In particular, if* $\alpha \in H_G^*(M;\mathbb{Z}/p)$, *then* $f_a = i_a^*(\alpha) \in H^*(BG;\mathbb{Z}/p)$ *and*

$$\mathit{Ind}_G(\alpha) = \sum_{a\in M^G} \frac{f_a}{\chi_G(a)} \in H^*(BG;\mathbb{Z}/p) = \mathbb{Z}/p[s,t]/(s^2).$$

Proof. From Lemma 5.3.19 of [1], we have that $\mathrm{Ind}_G(\theta_a) = 1_a$, and so by Lemma 2.3,

$$S^{-1}\mathrm{Ind}_G(\alpha) = \sum_{a\in M^G} \frac{f_a S^{-1}\mathrm{Ind}_G(\theta_a)}{\chi_G(a)} = \sum_{a\in M^G} \frac{f_a\cdot 1_a}{\chi_G(a)} = \sum_{a\in M^G} \frac{f_a}{\chi_G(a)}.$$

Since $H^*(BG;\mathbb{Z}/p) \longrightarrow S^{-1}H^*(BG;\mathbb{Z}/p)$ is injective, the last part of Theorem 2.5 follows immediately.

□

3. Equivariant cohomology structure

The purpose of this section is to study the structures of mod p equivariant cohomology rings of G-manifolds in Λ_{2n}.

Lemma 3.1. *Let* $M \in \Lambda_{2n}^r (r \geq 2)$. *Then,*

$$\dim_{\mathbb{Z}/p} H^i_G(M;\mathbb{Z}/p) = \begin{cases} \sum\limits_{j=0}^{i} b_j, & if\ i \le 2n-1, \\ \\ r, & if\ i \ge 2n, \end{cases}$$

where b_j is the jth mod p Betti number of M.

Proof. Let

$$P_z(M_G) = \sum_{i=0}^{\infty} \dim_{\mathbb{Z}/p} H^i_G(M;\mathbb{Z}/p) z^i$$

be the equivariant Poincaré polynomial of $H^*_G(M;\mathbb{Z}/p)$. Since $H^*_G(M;\mathbb{Z}/p)$ is a free $H^*(BG;\mathbb{Z}/p)$-module, we have that $H^*_G(M;\mathbb{Z}/p) = H^*(M;\mathbb{Z}/p) \bigotimes_{\mathbb{Z}/p} H^*(BG;\mathbb{Z}/p)$, and so

$$\begin{aligned} P_z(M_G) &= \sum_{i=0}^{\infty} \dim_{\mathbb{Z}/p} H^i_G(M;\mathbb{Z}/p) z^i = \frac{1}{1-z} \sum_{i=0}^{2n} \dim_{\mathbb{Z}/p} H^i(M;\mathbb{Z}/p) z^i \\ &= b_0 + (b_0+b_1)z + \cdots + (b_0+b_1+\cdots+b_{2n-1}) z^{2n-1} \\ &\quad + (b_0+b_1+\cdots+b_{2n})(z^{2n}+\cdots) \\ &= b_0 + (b_0+b_1)z + \cdots + (b_0+b_1+\cdots+b_{2n-1}) z^{2n-1} \\ &\quad + r(z^{2n}+\cdots). \end{aligned}$$

So, the result follows. □

Let $x = (x_1, \cdots, x_r)^T$ and $y = (y_1, \cdots, y_r)^T$ be two vectors in $(\mathbb{Z}/p)^r$. Define $x \circ y$ by

$$x \circ y = (x_1 y_1, \cdots, x_r y_r)^T.$$

Then, $(\mathbb{Z}/p)^r$ forms a commutative ring with respect to two operations $+$ and $\circ$. Let $a_1, \cdots, a_r$ be all fixed points in M^G and

$$\mathcal{V}_r^{(M)} = \{x = (x_1, \cdots, x_r)^T \in (\mathbb{Z}/p)^r \mid |x| = \sum_{i=1}^{r} \tfrac{x_i}{N_{a_i}} = 0\},$$

where N_{a_i} is as above. Then, it is easy to see that $\mathcal{V}_r^{(M)}$ is an $(r-1)$-dimensional subspace of $(\mathbb{Z}/p)^r$. Generally speaking, the operation $\circ$ in $\mathcal{V}_r^{(M)}$ is not closed.

If $M \in \Lambda^r_{2n}$, then we have that the inclusion $i : M^G \hookrightarrow M$ induces a monomorphism

$$i^* : H^*_G(M;\mathbb{Z}/p) \longrightarrow (\mathbb{Z}/p)^r[s,t]/(s^2).$$

By Lemma 3.1, there are subspaces V_i^M with $\dim V_i^M = \sum_{j=0}^{i} b_j$ $(i = 0, \cdots, 2n-1)$ of $(\mathbb{Z}/p)^r$ such that

$$H_G^i(M;\mathbb{Z}/p) \cong i^*(H_G^i(M;\mathbb{Z}/p)) = \begin{cases} V_i^M t^{\frac{i}{2}}, & if\ i \leq 2n-2\ and\ i\ is\ even, \\ V_i^M st^{\frac{i-1}{2}}, & if\ i \leq 2n-1\ and\ i\ is\ odd, \\ (\mathbb{Z}/p)^r t^{\frac{i}{2}}, & if\ i \geq 2n\ and\ i\ is\ even, \\ (\mathbb{Z}/p)^r st^{\frac{i-1}{2}}, & if\ i \geq 2n+1\ and\ i\ is\ odd, \end{cases}$$

where $V_i^M t^{\frac{i}{2}} = \{vt^{\frac{i}{2}} | v \in V_i^M\}$ and $V_i^M st^{\frac{i-1}{2}} = \{vst^{\frac{i-1}{2}} | v \in V_i^M\}$.

Lemma 3.2. *There are the following properties:*

(1) $V_i^M \subset V_{2n-1}^M = \mathcal{V}_r^{(M)}$, for $i < 2n-1$, $V_0^M \cong \mathbb{Z}/p$ is generated by $(1, \cdots, 1)^\top \in (\mathbb{Z}/p)^r$, $V_i^M \subset V_{i+1}^M$, for $i < 2n-1$ and i even, and $V_i^M \subset V_{i+2}^M$, for $i+1 < 2n-1$.

(2) For $d = \sum_{i \leq 2n-2 \text{ and } i \text{ even}} i \cdot d_i < 2n$, with each $d_i \geq 0$, $v_{\omega_{d_0}} \circ v_{\omega_{d_2}} \circ \cdots \circ v_{\omega_{d_{2n-2}}} \in V_d^M$, where $v_{\omega_{d_i}} = v_1^{(i)} \circ \cdots \circ v_{d_i}^{(i)}$, with each $v_j^{(i)} \in V_i^M$.

(3) For $d = \sum_{i \leq 2n-2 \text{ and } i \text{ even}} i \cdot d_i + k < 2n$, with k odd, $1 \leq k \leq 2n-1$ and each $d_i \geq 0$, $v_{\omega_{d_0}} \circ v_{\omega_{d_2}} \circ \cdots \circ v_{\omega_{d_{2n-2}}} \circ v_1^{(k)} \in V_d^M$, where $v_{\omega_{d_i}} = v_1^{(i)} \circ \cdots \circ v_{d_i}^{(i)}$, with each $v_j^{(i)} \in V_i^M$ and $v_1^{(k)} \in V_k^M$.

Proof. (1) For an element $\alpha \in H_G^*(M;\mathbb{Z}/p)$ of even degree d, we have that $i^*(\alpha) = vt^{\frac{d}{2}}$, where $v \in (\mathbb{Z}/p)^r$. Since $i^* = \bigoplus_{a \in M^G} i_a^*$, by Theorem 2.5 we have that

$$\begin{aligned} \mathrm{Ind}_G(\alpha) &= \sum_{a \in M^G} \frac{i_a^*(\alpha)}{\chi_G(a)} = \sum_{a \in M^G} \frac{i_a^*(\alpha)}{N_a t^n} = \frac{1}{t^n} \sum_{a \in M^G} \frac{i_a^*(\alpha)}{N_a} \\ &= |v| t^{\frac{d}{2}-n} \in \mathbb{Z}/p[s,t]/(s^2). \end{aligned}$$

Thus, if $d < 2n$ and d is even, then $|v|$ must be zero. For an element $\alpha' \in H_G^*(M;\mathbb{Z}/p)$ of odd degree d, we have that $i^*(\alpha') = v'st^{\frac{d-1}{2}}$, where $v' \in (\mathbb{Z}/p)^r$. Similarly, we have that if $d < 2n$ and d is odd, $|v'|$ also must be zero. Thus, if $i < 2n$, V_i^M is a subspace of $\mathcal{V}_r^{(M)}$, and $V_{2n-1}^M = \mathcal{V}_r^{(M)}$, by reason of dimension.

In particular, when $\alpha = 1$ (the identity of $H_G^*(M;\mathbb{Z}/p)$), $i^*(\alpha) = \bigoplus_{a \in M^G} i_a^*(1) = (1, \cdots, 1)^\top \in (\mathbb{Z}/p)^r$. So, $V_0^M \cong \mathbb{Z}/p$

is generated by $(1,\cdots,1)^\top \in (\mathbb{Z}/p)^r$, since $\dim V_0^M = b_0 = 1$. Since $H_G^*(M;\mathbb{Z}/p) = H^*(M;\mathbb{Z}/p)\bigotimes_{\mathbb{Z}/p} H^*(BG;\mathbb{Z}/p)$, we have that $(1,\cdots,1)^\top s \in i^*(H_G^1(M;$
$\mathbb{Z}/p))$ and $(1,\cdots,1)^\top t \in i^*(H_G^2(M;\mathbb{Z}/p))$. Thus, for any $v \in V_i^M$ with $i < 2n-1$ and i even, $[(1,\cdots,1)^\top s]\circ(vt^{\frac{i}{2}}) = vst^{\frac{i}{2}} \in V_{i+1}^M st^{\frac{i}{2}}$, and so we have $v \in V_{i+1}^M$. For any $v \in V_i^M$ with $i+1 < 2n-1$ and i odd, $[(1,\cdots,1)^\top t]\circ(vst^{\frac{i-1}{2}}) = vst^{\frac{i+1}{2}} \in V_{i+2}^M st^{\frac{i+1}{2}}$, and so $v \in V_{i+2}^M$. Similarly, we have that $V_i^M \subset V_{i+2}^M$ for $i+1 < 2n-1$ and i even. This completes the proof of (1).

(2) If i is even and $v_j^{(i)} \in V_i^M$, since $i^*: H_G^*(M;\mathbb{Z}/p) \longrightarrow (\mathbb{Z}/p)^r[s,t]/(s^2)$ is injective, there is a class $\alpha_j^{(i)}$ of degree i in $H_G^*(M;\mathbb{Z}/p)$ such that $i^*(\alpha_j^{(i)}) = v_j^{(i)}t^{\frac{i}{2}}$. If k is odd and $v_1^{(k)} \in V_k^M$, there is also a class $\beta_1^{(k)}$ of degree k in $H_G^*(M;\mathbb{Z}/p)$ such that $i^*(\beta_1^{(k)}) = v_1^{(k)}st^{\frac{k-1}{2}}$.

For $d = \sum_{i\le 2n-2 \textit{ and } i \textit{ even}} i\cdot d_i < 2n$ with each $d_i \ge 0$, since $i^* = \bigoplus\limits_{a\in M^G} i_a^*$ is a ring homomorphism, we have

$$\begin{aligned} i^*\Big(\prod_{i\le 2n-2 \textit{ and } i \textit{ even}} \prod_{j=1}^{d_i} \alpha_j^{(i)}\Big) &= \bigoplus_{a\in M^G} i_a^*\Big(\prod_{i\le 2n-2 \textit{ and } i \textit{ even}} \prod_{j=1}^{d_i} \alpha_j^{(i)}\Big) \\ &= \bigoplus_{a\in M^G} \prod_{i\le 2n-2 \textit{ and } i \textit{ even}} \prod_{j=1}^{d_i} i_a^*(\alpha_j^{(i)}) \\ &= v_{\omega_{d_0}}\circ v_{\omega_{d_2}}\circ\cdots\circ v_{\omega_{d_{2n-2}}} t^{\frac{d}{2}}, \end{aligned}$$

and so $v_{\omega_{d_0}}\circ v_{\omega_{d_2}}\circ\cdots\circ v_{\omega_{d_{2n-2}}} \in V_d^M$. The proof of (2) is now complete.

(3) Similarly, for $d = \sum_{i\le 2n-2 \textit{ and } i \textit{ even}} i\cdot d_i + k < 2n$ with k odd, $1 \le k \le 2n-1$ and each $d_i \ge 0$, we have that

$$\begin{aligned} & i^*\Big(\Big[\prod_{i\le 2n-2 \textit{ and } i \textit{ even}} \prod_{j=1}^{d_i} \alpha_j^{(i)}\Big]\cdot \beta_1^{(k)}\Big) \\ &= \bigoplus_{a\in M^G} i_a^*\Big(\Big[\prod_{i\le 2n-2 \textit{ and } i \textit{ even}} \prod_{j=1}^{d_i} \alpha_j^{(i)}\Big]\cdot \beta_1^{(k)}\Big) \\ &= \bigoplus_{a\in M^G} \Big(\Big[\prod_{i\le 2n-2 \textit{ and } i \textit{ even}} \prod_{j=1}^{d_i} i_a^*(\alpha_j^{(i)})\Big]\cdot i_a^*(\beta_1^{(k)})\Big) \end{aligned}$$

$$= v_{\omega_{d_0}} \circ v_{\omega_{d_2}} \circ \cdots \circ v_{\omega_{d_{2n-2}}} \circ v_1^{(k)} st^{\frac{d-1}{2}},$$

and so $v_{\omega_{d_0}} \circ v_{\omega_{d_2}} \circ \cdots \circ v_{\omega_{d_{2n-2}}} \circ v_1^{(k)} \in V_d^M$, which completes the proof.

□

Remark 3.3. *An observation shows that Lemma 3.2 gives a subring structure of*

$$\mathcal{R}_M = V_0^M + V_1^M s + \cdots + V_{2n-2}^M t^{n-1} + V_{2n-1}^M st^{n-1} + (\mathbb{Z}/p)^r (t^n + st^n + \cdots)$$

in $(\mathbb{Z}/p)^r[s,t]/(s^2)$.

Combining lemmas 3.1, 3.2, and Remark 3.3, we have the following.

Theorem 3.4. *Let* $M \in \Lambda_{2n}^r (r \geq 2)$. *Then, there are subspaces* V_i^M *with* $\dim V_i^M = \sum_{j=0}^{i} b_j (i = 0, \cdots, 2n-1)$ *of* $\mathcal{V}_r^{(M)}$ *such that* $H_G^*(M; \mathbb{Z}/p)$ *is isomorphic to the graded ring*

$\mathcal{R}_M = V_0^M + V_1^M s + \cdots + V_{2n-2}^M t^{n-1} + V_{2n-1}^M st^{n-1} + (\mathbb{Z}/p)^r (t^n + st^n + \cdots)$, *where the ring structure of* $\mathcal{R}_M$ *is determined by*

(1) $V_i^M \subset V_{2n-1}^M = \mathcal{V}_r^{(M)}$, *for* $i < 2n-1$, $V_0^M \cong \mathbb{Z}/p$ *is generated by* $(1, \cdots, 1)^\top \in (\mathbb{Z}/p)^r$, $V_i^M \subset V_{i+1}^M$, *for* $i < 2n-1$ *and* i *even, and* $V_i^M \subset V_{i+2}^M$, *for* $i + 1 < 2n - 1$.

(2) *For* $d = \sum_{i \leq 2n-2 \text{ and } i \text{ even}} i \cdot d_i < 2n$ *with each* $d_i \geq 0$, $v_{\omega_{d_0}} \circ v_{\omega_{d_2}} \circ \cdots \circ v_{\omega_{d_{2n-2}}} \in V_d^M$, *where* $v_{\omega_{d_i}} = v_1^{(i)} \circ \cdots \circ v_{d_i}^{(i)}$ *with each* $v_j^{(i)} \in V_i^M$.

(3) *For* $d = \sum_{i \leq 2n-2 \text{ and } i \text{ even}} i \cdot d_i + k < 2n$ *with* k *odd,* $1 \leq k \leq 2n-1$ *and each* $d_i \geq 0$, $v_{\omega_{d_0}} \circ v_{\omega_{d_2}} \circ \cdots \circ v_{\omega_{d_{2n-2}}} \circ v_1^{(k)} \in V_d^M$, *where* $v_{\omega_{d_i}} = v_1^{(i)} \circ \cdots \circ v_{d_i}^{(i)}$ *with each* $v_j^{(i)} \in V_i^M$ *and* $v_1^{(k)} \in V_k^M$.

Remark 3.5. *Since* $H_G^*(M; \mathbb{Z}/p)$ *is a free* $H^*(BG; \mathbb{Z}/p)$*-module, we have that* $\mathcal{R}_M$ *is also a free* $\mathbb{Z}/p[s,t]/(s^2)$*-module.*

4. 2-dimensional case

Let $M \in \Lambda_2^r (r \geq 2)$. From Theorem 3.4, we have that

$$H_G^*(M;\mathbb{Z}/p) \cong V_0^M + \mathcal{V}_r^{(M)}s + (\mathbb{Z}/p)^r(t + st + \cdots).$$

Theorem 4.1. *Let $M_1, M_2 \in \Lambda_2$. Then, $H_G^*(M_1;\mathbb{Z}/p)$ and $H_G^*(M_2;\mathbb{Z}/p)$ are isomorphic as graded rings if and only if $|M_1^G| = |M_2^G|$.*

Proof. If $|M_1^G| = |M_2^G| = r$, then let $a_1, \cdots, a_r$ be all fixed points in M_1^G and $b_1, \cdots, b_r$ be all fixed points in M_2^G. Thus,

$$\mathcal{V}_r^{(M_1)} = \{x = (x_1, \cdots, x_r)^\top \in (\mathbb{Z}/p)^r |\ |x| = \sum_{i=1}^{r} \tfrac{x_i}{N_{a_i}} = 0\}$$

and

$$\mathcal{V}_r^{(M_2)} = \{y = (y_1, \cdots, y_r)^T \in (\mathbb{Z}/p)^r |\ |y| = \sum_{i=1}^{r} \tfrac{y_i}{N_{b_i}} = 0\}.$$

By Theorem 3.4, we have that

$$H_G^*(M_1;\mathbb{Z}/p) \cong \mathcal{R}_{M_1} = V_0^{M_1} + \mathcal{V}_r^{(M_1)}s + (\mathbb{Z}/p)^r(t + st + \cdots)$$

and

$$H_G^*(M_2;\mathbb{Z}/p) \cong \mathcal{R}_{M_2} = V_0^{M_2} + \mathcal{V}_r^{(M_2)}s + (\mathbb{Z}/p)^r(t + st + \cdots).$$

Let $f_0 : V_0^{M_1} \longrightarrow V_0^{M_2}$ be the identity map,

$f_1 : \mathcal{V}_r^{(M_1)}s \longrightarrow \mathcal{V}_r^{(M_2)}s$ be $f_1((x_1, \cdots, x_r)^\top s) = (\frac{x_1}{N_{a_1}}N_{b_1}, \cdots, \frac{x_r}{N_{a_r}}N_{b_r})^T s$,

$f_i : (\mathbb{Z}/p)^r t^{\frac{i}{2}} \longrightarrow (\mathbb{Z}/p)^r t^{\frac{i}{2}}$ be the identity map for $i \geq 2$ and i even,

$f_i : (\mathbb{Z}/p)^r st^{\frac{i-1}{2}} \longrightarrow (\mathbb{Z}/p)^r st^{\frac{i-1}{2}}$ be $f_i((x_1, \cdots, x_r)^\top st^{\frac{i-1}{2}}) = (\frac{x_1}{N_{a_1}}N_{b_1}, \cdots, \frac{x_r}{N_{a_r}}N_{b_r})^\top st^{\frac{i-1}{2}}$ for $i \geq 3$ and i odd.

Then, it is easy to check that $f = \sum_{i=0}^{\infty} f_i$ is an isomorphism between $\mathcal{R}_{M_1}$ and $\mathcal{R}_{M_2}$. Thus, $H_G^*(M_1;\mathbb{Z}/p)$ and $H_G^*(M_2;\mathbb{Z}/p)$ are isomorphic as graded rings.

If $|M_1^G| \neq |M_2^G|$, then obviously $H_G^*(M_1;\mathbb{Z}/p)$ and $H_G^*(M_2;\mathbb{Z}/p)$ are not isomorphic by their ring structures. $\square$

An observation shows that S^2 admits a G-action such that $|(S^2)^G| = 2$. Thus, Λ_2 is non-empty.

As a consequence of Theorem 4.1, we have the following.

Corollary 4.2. *Let $r \geq 2$ be a positive integer. All G-manifolds in Λ_2^r determine a unique equivariant cohomology up to isomorphism if Λ_2^r is non-empty.*

Remark 4.3. *For each $M \in \Lambda_2^r (r \geq 2)$, its equivariant cohomology ring $H_G^*(M; \mathbb{Z}/p)$ can be expressed in a simpler way. We have that*

$$H_G^*(M; \mathbb{Z}/p) \cong$$

$$\{\alpha = (\alpha_1, \cdots, \alpha_r) \in (\mathbb{Z}/p)^r[s,t]/(s^2) | \begin{cases} \alpha_1 = \cdots = \alpha_r, \, if\, deg\alpha = 0 \\ \sum\limits_{i=1}^{r} \frac{\alpha_i}{N_{a_i}} = 0, \quad if\, deg\alpha = 1 \end{cases} \},$$

where $a_1, \cdots, a_r$ are all fixed points in M^G and $\chi_G(a_i) = N_{a_i} t$.

5. Ring homomorphisms induced by G-equivariant maps

In this section, the task is to give a description for the homomorphism between equivariant cohomology rings of two G-manifolds in Λ_{2n} induced by a G-equivariant map and to show a characterization of the homomorphism.

Theorem 5.1. *Let $f : M_1 \to M_2$ be a G-equivariant map for $M_1 \in \Lambda_{2n}^{r_1}$, $M_2 \in \Lambda_{2n}^{r_2} (r_1, r_2 \geq 2)$ and f^* be the induced homomorphism between graded rings*

$$\begin{aligned} \mathcal{R}_{M_2} &= V_0^{M_2} + V_1^{M_2} s + \cdots + V_{2n-2}^{M_2} t^{n-1} + V_{2n-1}^{M_2} st^{n-1} + \\ &\quad (\mathbb{Z}/p)^{r_2}(t^n + st^n + \cdots) \end{aligned}$$

and

$$\begin{aligned} \mathcal{R}_{M_1} &= V_0^{M_1} + V_1^{M_1} s + \cdots + V_{2n-2}^{M_1} t^{n-1} + V_{2n-1}^{M_1} st^{n-1} + \\ &\quad (\mathbb{Z}/p)^{r_1}(t^n + st^n + \cdots). \end{aligned}$$

Then, there is a linear map σ from $(\mathbb{Z}/p)^{r_2}$ to $(\mathbb{Z}/p)^{r_1}$ such that $f^* = \sum\limits_{i\ even} \sigma t^{\frac{i}{2}} + \sum\limits_{j\ odd} \sigma s t^{\frac{j-1}{2}}$, where $f^*(\beta) = \sum\limits_{i\ even} \sigma(v_i) t^{\frac{i}{2}} + \sum\limits_{j\ odd} \sigma(v_j) s t^{\frac{j-1}{2}}$, for $\beta = \sum\limits_{i\ even} v_i t^{\frac{i}{2}} + \sum\limits_{j\ odd} v_j s t^{\frac{j-1}{2}} \in \mathcal{R}_{M_2}$. In particular, if $r_1 = r_2 = r$ and f^* is an isomorphism, then $\sigma \in \mathrm{GL}(r, \mathbb{Z}/p)$.

Proof. Since f^* is a ring homomorphism, $f^*((1\underbrace{,\cdots,}_{r_2}1)^T) = (1\underbrace{,\cdots,}_{r_1}1)^T$ and $f^*((k\underbrace{,\cdots,}_{r_2}k)^T) = (k\underbrace{,\cdots,}_{r_1}k)^T$, for $k \in \mathbb{Z}/p$. It is easy to check that f^* is linear. Since the restriction $f^*|_{(\mathbb{Z}/p)^{r_2}t^n} : (\mathbb{Z}/p)^{r_2}t^n \longrightarrow (\mathbb{Z}/p)^{r_1}t^n$ is linear, there exists a linear map σ from $(\mathbb{Z}/p)^{r_2}$ to $(\mathbb{Z}/p)^{r_1}$ such that $f^*|_{(\mathbb{Z}/p)^{r_2}t^n} = \sigma t^n$.

By Remark 3.5 we know that $\mathcal{R}_{M_i}, i = 1, 2$, are free $\mathbb{Z}/p[s,t]/(s^2)$-modules, and that f^* is a module homomorphism between $\mathcal{R}_{M_2}$ and $\mathcal{R}_{M_1}$. If $i > 2n$ and i is even, since $f^*|_{(\mathbb{Z}/p)^{r_2}t^n} = \sigma t^n$, we have that for $x \in (\mathbb{Z}/p)^{r_2}$,

$$f^*(xt^{\frac{i}{2}}) = f^*(xt^n)t^{\frac{i}{2}-n} = \sigma(x)t^{\frac{i}{2}}.$$

Thus, $f^*|_{(\mathbb{Z}/p)^{r_2}t^{\frac{i}{2}}} = \sigma t^{\frac{i}{2}}$, for $i > 2n$ and i even. Similarly, we have that $f^*|_{(\mathbb{Z}/p)^{r_2}st^{\frac{i-1}{2}}} = \sigma st^{\frac{i-1}{2}}$, for $i > 2n$ and i odd.

Finally we consider the case of the dimension being less than $2n$. For $0 \leq i < 2n$ and i even, let $v \in V_i^{M_2} \subset (\mathbb{Z}/p)^{r_2}$. Since $f^*|_{(\mathbb{Z}/p)^{r_2}t^n} = \sigma t^n$, we have that

$$f^*(vt^{\frac{i}{2}})t^{n-\frac{i}{2}} = f^*(vt^n) = \sigma(v)t^n,$$

and so $f^*(vt^{\frac{i}{2}}) = \sigma(v)t^{\frac{i}{2}}$. Thus, $f^*|_{V_i^{M_2}t^{\frac{i}{2}}} = \sigma t^{\frac{i}{2}}$, for $0 \leq i < 2n$ and i even.

In a similar way, using $f^*|_{(\mathbb{Z}/p)^{r_2}st^n} = \sigma st^n$, we have that $f^*|_{V_i^{M_2}st^{\frac{i-1}{2}}} = \sigma st^{\frac{i-1}{2}}$, for $0 \leq i < 2n$ and i odd.

If $r_1 = r_2 = r$ and f^* is an isomorphism, it is easy to see that $\sigma \in \mathrm{GL}(r, \mathbb{Z}/p)$. This completes the proof. $\square$

Remark 5.2. *σ in Theorem 5.1 has the following property:*
$\sigma(x \circ y) = \sigma(x) \circ \sigma(y)$, for any $x, y \in (\mathbb{Z}/p)^{r_2}$.
In fact, since $f^*|_{(\mathbb{Z}/p)^{r_2}t^n} = \sigma t^n$ and $f^*|_{(\mathbb{Z}/p)^{r_2}t^{2n}} = \sigma t^{2n}$, for any $x, y \in (\mathbb{Z}/p)^{r_2}$, we have

$$\sigma(x \circ y)t^{2n} = f^*(x \circ yt^{2n}) = f^*(xt^n)f^*(yt^n) = \sigma(x) \circ \sigma(y)t^{2n},$$

and so $\sigma(x \circ y) = \sigma(x) \circ \sigma(y)$.

Combining Theorem 5.1 and Remark 5.2, we have the following.

Proposition 5.3. *With the notation of Theorem 5.1, let $f : M_1 \to M_2$ be a G-equivariant map for $M_1 \in \Lambda_{2n}^{r_1}$ and $M_2 \in \Lambda_{2n}^{r_2}$. Then, there is a linear map σ from $(\mathbb{Z}/p)^{r_2}$ to $(\mathbb{Z}/p)^{r_1}$ satisfying the property that $\sigma(x \circ y) = \sigma(x) \circ \sigma(y)$, for any $x, y \in (\mathbb{Z}/p)^{r_2}$ and that σ linearly maps $V_i^{M_2}$ to $V_i^{M_1}$, for $i < 2n$.*

Remark 5.4. *Let $M_1, M_2 \in \Lambda_{2n}$. We see that if $|M_1^G| \neq |M_2^G|$, then $H_G^*(M_1; \mathbb{Z}/p)$ and $H_G^*(M_2; \mathbb{Z}/p)$ must not be isomorphic as graded rings.*

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