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ON A SUBCLASS OF MULTIVALENT ANALYTIC FUNCTIONS ASSOCIATED WITH AN EXTENDED FRACTIONAL DIFFERINTEGRAL OPERATOR

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ABSTRACT. Making use of an extended fractional differintegral operator (introduced recently by Patel and Mishra), we introduce a new subclass of multivalent analytic functions and investigate certain interesting properties of the subclass.

1. Introduction and preliminaries

Let $A(p)$ denote the class of functions of the form

$$f(z) = z^p + \sum_{n=1}^{\infty} a_n z^{n+p} \quad (p \in N = \{1, 2, 3, \cdots\}), \tag{1.1}$$

which are analytic in the open unit disk $U = \{z : z \in C \text{ and } |z| < 1\}$.

Suppose that $f(z)$ and $g(z)$ are analytic in U. We say that the function $f(z)$ is subordinate to $g(z)$ in U, and we write $f(z) \prec g(z)$ $(z \in U)$, if there exists an analytic function $w(z)$ in U with $w(0) = 0$ and $|w(z)| < 1$ for all $z \in U$, such that $f(z) = g(w(z)) \quad (z \in U)$. If $g(z)$ is univalent in U, then the following equivalence relationship holds:

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \text{ and } f(U) \subset g(U).$$

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For functions $f_j(z) \in A(p) \quad (j = 1, 2)$, given by

$$f_j(z) = z^p + \sum_{n=1}^{\infty} a_{n,j} z^{n+p} \quad (j = 1, 2),$$

we define the Hadamard product (or convolution) of $f_1(z)$ and $f_2(z)$ by

$$(f_1 * f_2)(z) = z^p + \sum_{n=1}^{\infty} a_{n,1} a_{n,2} z^{n+p} = (f_2 * f_1)(z).$$

In [10] (see also [11] and [16]), Owa introduced the following definitions of fractional calculus (that is, fractional integrals and fractional derivatives of an arbitrary order).

Definition 1.1. *The fractional integral of order λ ($\lambda > 0$) is defined, for a function $f(z)$, analytic in a simply-connected region of the complex plane containing the origin, by*

$$D_z^{-\lambda} f(z) = \frac{1}{\Gamma(\lambda)} \int_0^z \frac{f(\xi)}{(z-\xi)^{1-\lambda}} d\xi,$$

where the multipicity of $(z-\xi)^{\lambda-1}$ is removed by requiring $log(z-\xi)$ to be real when $z - \xi > 0$.

Definition 1.2. *Under the hypothesis of Definition 1.1 , the fractional derivative of $f(z)$ of order λ ($\lambda \geq 0$) is defined by*

$$D_z^{\lambda} f(z) = \begin{cases} \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int_0^z \frac{f(\xi)}{(z-\xi)^{\lambda}} d\xi & (0 \leq \lambda < 1), \\ \frac{d^n}{dz^n} D_z^{\lambda-n} f(z) & (n \leq \lambda < n+1; n \in N \cup \{0\}), \end{cases}$$

where the multiplicity of $(z-\xi)^{-\lambda}$ is removed as in Definition 1.1.

Very recently, Patel and Mishra [12] defined the extended fractional differintegral operator $\Omega_z^{(\lambda,p)} : A(p) \to A(p)$ for a function $f(z) \in A(p)$ and for a real number λ ($-\infty < \lambda < p+1$) by

$$\Omega_z^{(\lambda,p)} f(z) = \frac{\Gamma(p+1-\lambda)}{\Gamma(p+1)} z^{\lambda} D_z^{\lambda} f(z), \tag{1.2}$$

where $D_z^{\lambda} f$ is, respectively the fractional integral of f of order $-\lambda$, when $-\infty < \lambda < 0$, and the fractional derivative of f of order λ, when $0 \leq \lambda < p+1$.

It is easily seen from (1.2) that for a function $f(z)$ of the form (1.1), we have

$$\Omega_z^{(\lambda,p)} f(z) = z^p + \sum_{n=1}^{\infty} \frac{\Gamma(n+p+1)\Gamma(p+1-\lambda)}{\Gamma(p+1)\Gamma(n+p+1-\lambda)} a_n z^{n+p} \quad (z \in U) \tag{1.3}$$

and

$$z(\Omega_z^{(\lambda,p)} f(z))' = (p-\lambda)\Omega_z^{(\lambda+1,p)} f(z) + \lambda \Omega_z^{(\lambda,p)} f(z) \quad (-\infty < \lambda < p; z \in U). \tag{1.4}$$

We also note from (1.3) and (1.4) that

$$\Omega_z^{(-1,p)} f(z) = \frac{p+1}{z} \int_0^z f(t)dt, \quad \Omega_z^{(0,p)} f(z) = f(z),$$

$$\Omega_z^{(1,p)} f(z) = \frac{zf'(z)}{p},$$

and, in general,

$$\Omega_z^{(m,p)} f(z) = \frac{(p-m)! z^m f^{(m)}(z)}{p\,!} \quad (m \in N; m < p+1).$$

The fractional differential operator $\Omega_z^{(\lambda,p)}$ with $0 \leq \lambda < 1$ was investigated by Srivastava and Aouf [17] and studied by Srivastava and Mishra [18]. Patel and Mishra [12] also obtained several interesting properties and characteristics for certain subclasses of multivalent analytic functions involving the differintegral operator $\Omega_z^{(\lambda,p)}$, when $-\infty < \lambda < p+1$. We further observe that $\Omega_z^{(\lambda,1)} = \Omega_z^{\lambda}$ is the operator introduced and studied by Owa and Srivastava [11]. In the present sequel to these earlier works, we shall derive certain interesting properties of the extended fractional differintegral operator $\Omega_z^{(\lambda,p)}$.

Let P be the class of functions $h(z)$ with $h(0) = 1$, which are analytic and convex univalent in U.

Recently, many authors have introduced and studied some new subclasses of analytic functions defined by various other linear operators (see, e.g., Dziok and Srivastava [1, 2], Srivastava et al. [17–20], Yang et al. [23], Patel et al. [12,13]], Liu and Srivastava [7], Liu [5,6], and Wang et al. [21]). Now, we introduce the following subclass of $A(p)$ associated with the operator $\Omega_z^{(\lambda,p)}$.

Definition 1.3. *A function $f(z) \in A(p)$ is said to be in the class $H_p(\lambda, \alpha; h)$, if it satisfies the subordination condition,*

$$(1.5) \qquad (1-\alpha)z^{-p}\Omega_z^{(\lambda,p)}f(z) + \frac{\alpha}{p}z^{-p+1}(\Omega_z^{(\lambda,p)}f(z))' \prec h(z),$$

where α is a complex number and $h(z) \in P$.

Remark 1.4.

(1) *For $p = 1, \lambda = 1, \alpha = 1$ and $h(z) = \frac{1+z}{1-z}$, $H_1(1, 1; \frac{1+z}{1-z})$ coincides with R, as investigated by Singh and Singh [15].*

(2) *For $p = 1, \lambda = 1$ and $h(z) = \frac{1+az}{1+bz}$ $(-1 \le b < 1, a > b)$, $H_1(1, \alpha; \frac{1+az}{1+bz})$ reduces to $H(\alpha, a, b)$, as studied by Yang [22].*

(3) *For $p = 1, \lambda = 1$ and $h(z) = 1+Mz (M > 0)$, $H_1(1, \alpha; 1+Mz) = S(\alpha, M)$, as introduced and studied by Zhou and Owa [24] and Liu [4] respectively.*

A function $f(z) \in A(1)$ is said to be in the class $S^*(\rho)$, if

$$(1.6) \qquad \mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \rho \quad (z \in U),$$

for some $\rho (\rho < 1)$. When $0 \le \rho < 1$, $S^*(\rho)$ is the class of starlike functions of order ρ in U. A function $f(z) \in A(1)$ is said to be prestarlike of order ρ in U, if

$$(1.7) \qquad \frac{z}{(1-z)^{2(1-\rho)}} * f(z) \in S^*(\rho) \quad (\rho < 1).$$

We denote this class by $R(\rho)$ (see [14]). It is clear from (1.6) and (1.7) that a function $f(z) \in A(1)$ is in the class $R(0)$ if and only if $f(z)$ is convex univalent in U and

$$R\left(\frac{1}{2}\right) = S^*\left(\frac{1}{2}\right).$$

We need the following lemmas in order to derive our main results for the class $H_p(\lambda, \alpha; h)$.

Lemma 1.5. *Let $g(z)$ be analytic in U and $h(z)$ be analytic and convex univalent in U with $h(0) = g(0)$. If*

$$(1.8) \qquad g(z) + \frac{1}{\mu}zg'(z) \prec h(z),$$

where $Re\mu \ge 0$ and $\mu \ne 0$, then

$$g(z) \prec \widetilde{h}(z) = \mu z^{-\mu}\int_0^z t^{\,\mu-1}h(t)dt \prec h(z),$$

and $\widetilde{h}(z)$ *is the best dominant of* (1.8).

Lemma 1.6. *Let* $\rho < 1$, $f(z) \in S^*(\rho)$ *and* $g(z) \in R(\rho)$. *Then, for any analytic function* $F(z)$ *in* U,

$$\frac{g * (fF)}{g * f}(U) \subset \overline{co}(F(U)),$$

where $\overline{co}(F(U))$ *denotes the closed convex hull of* $F(U)$.

Lemma 1.5 is due to Miller and Mocanu [9] (see also [3]) and Lemma 1.6 can be found in Ruscheweyh [14].

Lemma 1.7. *(see [8]). Let* $g(z) = 1 + \sum_{n=k}^{\infty} b_n z^n \quad (k \in N)$ *be analytic in* U. *If* $Re\{g(z)\} > 0 \quad (z \in U)$, *then*

$$Re\{g(z)\} \geq \frac{1 - |z|^k}{1 + |z|^k} \quad (z \in U).$$

2. **Main results**

Theorem 2.1. *Let* $0 \leq \alpha_1 < \alpha_2$. *Then*

$$H_p(\lambda, \alpha_2; h) \subset H_p(\lambda, \alpha_1; h).$$

Proof. Let $0 \leq \alpha_1 < \alpha_2$ and suppose that

$$g(z) = z^{-p} \Omega_z^{(\lambda,p)} f(z), \tag{2.1}$$

for $f(z) \in H_p(\lambda, \alpha_2; h)$. Then, the function $g(z)$ is analytic in U with $g(0) = 1$. Differentiating both sides of (2.1) with respect to z and using (1.5), we have

$$\begin{aligned} &(1-\alpha_2) z^{-p} \Omega_z^{(\lambda,p)} f(z) + \frac{\alpha_2}{p} z^{-p+1} (\Omega_z^{(\lambda,p)} f(z))' \\ &= g(z) + \frac{\alpha_2}{p} z g'(z) \prec h(z). \end{aligned} \tag{2.2}$$

Hence, an application of Lemma 1.5 yields

$$g(z) \prec h(z). \tag{2.3}$$

Noting that $0 \leq \frac{\alpha_1}{\alpha_2} < 1$ and that $h(z)$ is convex univalent in U, it follows from (2.1), (2.2) and (2.3) that

$$\begin{aligned}
&(1-\alpha_1)z^{-p}\Omega_z^{(\lambda,p)}f(z) + \frac{\alpha_1}{p}z^{-p+1}(\Omega_z^{(\lambda,p)}f(z))' \\
&= \frac{\alpha_1}{\alpha_2}\left((1-\alpha_2)z^{-p}\Omega_z^{(\lambda,p)}f(z) + \frac{\alpha_2}{p}z^{-p+1}(\Omega_z^{(\lambda,p)}f(z))'\right) \\
&\quad + \left(1-\frac{\alpha_1}{\alpha_2}\right)g(z) \\
&\prec h(z).
\end{aligned}$$

Thus, $f(z) \in H_p(\lambda, \alpha_1; h)$, and the proof of Theorem 2.1 is complete. $\square$

Remark 2.2. *Theorem 2.1 generalizes a result by Yang [22].*

Theorem 2.3. *Let $\alpha > 0, \gamma > 0$ and $f(z) \in H_p(\lambda, \alpha; \gamma h + 1 - \gamma)$. If $\gamma \leq \gamma_0$, where*

$$\gamma_0 = \frac{1}{2}\left(1 - \frac{p}{\alpha}\int_0^1 \frac{u^{\frac{p}{\alpha}-1}}{1+u}du\right)^{-1}, \tag{2.4}$$

then $f(z) \in H_p(\lambda, 0; h)$. The bound γ_0 is sharp when $h(z) = \frac{1}{1-z}$.

Proof. Define

$$g(z) = z^{-p}\Omega_z^{(\lambda,p)}f(z), \tag{2.5}$$

for $f(z) \in H_p(\lambda, \alpha; \gamma h + 1 - \gamma)$, with $\alpha > 0$ and $\gamma > 0$. Then, we have

$$\begin{aligned}
g(z) + \frac{\alpha}{p}zg'(z) &= (1-\alpha)z^{-p}\Omega_z^{(\lambda,p)}f(z) + \frac{\alpha}{p}z^{-p+1}(\Omega_z^{(\lambda,p)}f(z))' \\
&\prec \gamma h(z) + 1 - \gamma.
\end{aligned}$$

Hence, an application of Lemma 1.5 yields

$$g(z) \prec \frac{\gamma p}{\alpha}z^{-\frac{p}{\alpha}}\int_0^z t^{\frac{p}{\alpha}}h(t)dt + 1 - \gamma = (h * \psi)(z), \tag{2.6}$$

where

$$\psi(z) = \frac{\gamma p}{\alpha}z^{-\frac{p}{\alpha}}\int_0^z \frac{t^{\frac{p}{\alpha}-1}}{1-t}dt + 1 - \gamma. \tag{2.7}$$

If $0<\gamma\leq\gamma_0$, where $\gamma_0>1$ is given by (2.4), then it follows from (2.7) that

$$\begin{aligned}\mathrm{Re}\psi(z) &= \frac{\gamma p}{\alpha}\int_0^1 u^{\frac{p}{\alpha}-1}\mathrm{Re}\left(\frac{1}{1-uz}\right)du+1-\gamma\\ &> \frac{\gamma p}{\alpha}\int_0^1 \frac{u^{\frac{p}{\alpha}-1}}{1+u}du+1-\gamma\\ &\geq \frac{1}{2}\quad (z\in U).\end{aligned}$$

Now,by using the Herglotz representation for $\psi(z)$, from (2.5) and (2.6) we arrive at

$$z^{-p}\Omega_z^{(\lambda,p)}f(z)\prec(h*\psi)(z)\prec h(z),$$

because $h(z)$ is convex univalent in U. This shows that $f(z)\in H_p(\lambda,0;h)$. For $h(z)=\frac{1}{1-z}$ and $f(z)\in A(p)$ defined by

$$z^{-p}\Omega_z^{(\lambda,p)}f(z)=\frac{\gamma p}{\alpha}z^{-\frac{p}{\alpha}}\int_0^z\frac{t^{\frac{p}{\alpha}-1}}{1-t}dt+1-\gamma,$$

it is easy to verify that

$$(1-\alpha)z^{-p}\Omega_z^{(\lambda,p)}f(z)+\frac{\alpha}{p}z^{-p+1}(\Omega_z^{(\lambda,p)}f(z))'=\gamma h(z)+1-\gamma.$$

Thus, $f(z)\in H_p(\lambda,\alpha,;\gamma h+1-\gamma)$. Also, for $\gamma>\gamma_0$, we have

$$\mathrm{Re}\{z^{-p}\Omega_z^{(\lambda,p)}f(z)\}\to\frac{\gamma p}{\alpha}\int_0^1\frac{u^{\frac{p}{\alpha}-1}}{1+u}du+1-\gamma<\frac{1}{2}\quad(z\to-1),$$

which implies that $f(z)\notin H_p(\lambda,0;h)$. Hence, the bound γ_0 cannot be increased when $h(z)=\frac{1}{1-z}$. □

Theorem 2.4. *Let $f(z)\in H_p(\lambda,\alpha;h)$,*

$$g(z)\in A(p)\ \text{and}\ Re\{z^{-p}g(z)\}>\frac{1}{2}\quad(z\in U).\tag{2.8}$$

Then

$$(f*g)(z)\in H_p(\lambda,\alpha;h).$$

Proof. For $f(z) \in H_p(\lambda, \alpha; h)$ and $g(z) \in A(p)$, we have

$$
\begin{aligned}
&(1-\alpha)z^{-p}\Omega_z^{(\lambda,p)}(f*g)(z) + \frac{\alpha}{p}z^{-p+1}(\Omega_z^{(\lambda,p)}(f*g)(z))' \\
&= (1-\alpha)(z^{-p}g(z)) * (z^{-p}\Omega_z^{(\lambda,p)}f(z)) \\
&\quad + \frac{\alpha}{p}(z^{-p}g(z)) * (z^{-p+1}(\Omega_z^{(\lambda,p)}f(z))') \\
&= (z^{-p}g(z)) * \psi(z), \qquad (2.9)
\end{aligned}
$$

where

$$
\begin{aligned}
\psi(z) &= (1-\alpha)z^{-p}\Omega_z^{(\lambda,p)}f(z) + \frac{\alpha}{p}z^{-p+1}(\Omega_z^{(\lambda,p)}f(z))' \\
&\prec h(z). \qquad (2.10)
\end{aligned}
$$

In view of (2.8), the function $z^{-p}g(z)$ has the Herglotz representation,

$$z^{-p}g(z) = \int_{|x|=1} \frac{d\mu(x)}{1-xz} \quad (z \in U), \tag{2.11}$$

where $\mu(x)$ is a probability measure defined on the unit circle $|x| = 1$ and

$$\int_{|x|=1} d\mu(x) = 1.$$

Since $h(z)$ is convex univalent in U, it follows from (2.9), (2.10) and (2.11) that

$$
\begin{aligned}
&(1-\alpha)z^{-p}\Omega_z^{(\lambda,p)}(f*g)(z) + \frac{\alpha}{p}z^{-p+1}(\Omega_z^{(\lambda,p)}(f*g)(z))' \\
&= \int_{|x|=1} \psi(xz)d\mu(x) \prec h(z).
\end{aligned}
$$

This shows that $(f*g)(z) \in H_p(\lambda, \alpha; h)$ and the theorem is proved. $\square$

Remark 2.5. *For $p = 1, \lambda = 1$ and $h(z) = \frac{1+az}{1+bz}(-1 \le b < 1, a > b)$, we get a result by Yang [22](Throrem 4).*

Corollary 2.6. *Let $f(z) \in H_p(\lambda, \alpha; h)$ be given by (1.1) and let*

$$s_m(z) = z^p + \sum_{n=1}^{m-1} a_n z^{n+p} \quad (m \in N \setminus \{1\}).$$

Then, the function

$$\sigma_m(z) = \int_0^1 t^{-p} s_m(tz)dt$$

is also in the class $H_p(\lambda, \alpha; h)$.

Proof. We have

$$\begin{aligned}\sigma_m(z) &= z^p + \sum_{n=1}^{m-1} \frac{a_n}{n+1} z^{n+p} \\ &= (f * g_m)(z) \quad (m \in N \setminus \{1\}),\end{aligned} \tag{2.12}$$

where

$$f(z) = z^p + \sum_{n=1}^{\infty} a_n z^{n+p} \in H_p(\lambda, \alpha; h)$$

and

$$g_m(z) = z^p + \sum_{n=1}^{m-1} \frac{z^{n+p}}{n+1} \in A(p).$$

Also, for $m \in N \setminus \{1\}$, it is known from [8] that

$$\mathrm{Re}\{z^{-p} g_m(z)\} = \mathrm{Re}\left\{1 + \sum_{n=1}^{m-1} \frac{z^n}{n+1}\right\} > \frac{1}{2} \quad (z \in U). \tag{2.13}$$

In view of (2.12) and (2.13), an application of Theorem 2.4 leads to $\sigma_m(z) \in H_p(\lambda, \alpha; h)$. □

Theorem 2.7. *Let* $f(z) \in H_p(\lambda, \alpha; h)$,

$$g(z) \in A(p) \text{ and } z^{-p+1} g(z) \in R(\rho) \quad (\rho < 1).$$

Then

$$(f * g)(z) \in H_p(\lambda, \alpha; h).$$

Proof. For $f(z) \in H_p(\lambda, \alpha; h)$ and $g(z) \in A(p)$, from (2.9) (used in the proof of Theorem 2.4), we can write

$$\begin{aligned}&(1-\alpha) z^{-p} \Omega_z^{(\lambda,p)} (f * g)(z) + \frac{\alpha}{p} z^{-p+1} (\Omega_z^{(\lambda,p)} (f * g)(z))' \\ &= \frac{(z^{-p+1} g(z)) * (z\psi(z))}{(z^{-p+1} g(z)) * z} \quad (z \in U),\end{aligned} \tag{2.14}$$

where $\psi(z)$ is defined as in (2.10).
Since $h(z)$ is convex univalent in U,

$$\psi(z) \prec h(z), z^{-p+1} g(z) \in R(\rho) \text{ and } z \in S^*(\rho) \quad (\rho < 1),$$

the desired result follows from (2.14) and Lemma 1.6. □

Taking $\rho = 0$ and $\rho = \frac{1}{2}$, Theorem 2.7 reduces to the following.

Corollary 2.8. *Let $f(z) \in H_p(\lambda, \alpha; h)$ and let $g(z) \in A(p)$ satisfy any one of the following conditions:*
(i) $z^{-p+1}g(z)$ is convex univalent in U
or
(ii) $z^{-p+1}g(z) \in S^*(\frac{1}{2})$.
Then

$$(f * g)(z) \in H_p(\lambda, \alpha; h).$$

Theorem 2.9. *Let $f(z) \in H_p(\lambda, \alpha; h)$. Then, the function $F(z)$, defined by*

$$F(z) = \frac{\mu + p}{z^{\mu}} \int_0^z t^{\mu-1} f(t)dt \quad (Re\mu > -p) \tag{2.15}$$

is in the class $H_p(\lambda, \alpha; \widetilde{h})$, where

$$\widetilde{h}(z) = (\mu + p)z^{-(\mu+p)} \int_0^z t^{\mu+p-1} h(t)dt \prec h(z).$$

Proof. For $f(z) \in A(p)$ and $\mathrm{Re}\mu > -p$, we find from (2.15) that $F(z) \in A(p)$ and

$$(\mu + p)f(z) = \mu F(z) + zF'(z). \tag{2.16}$$

Define $G(z)$ by

$$z^p G(z) = (1 - \alpha)\Omega_z^{(\lambda,p)} F(z) + \frac{\alpha}{p} z(\Omega_z^{(\lambda,p)} F(z))'. \tag{2.17}$$

Differentiating both sides of (2.17) with respect to z, we get

$$\begin{aligned} zG'(z) + pG(z) &= (1 - \alpha)z^{-p}\Omega_z^{(\lambda,p)}(zF'(z)) \\ &\quad + \frac{\alpha}{p} z^{-p+1}(\Omega_z^{(\lambda,p)}(zF'(z)))'. \end{aligned} \tag{2.18}$$

Furthermore, it follows from (2.16), (2.17) and (2.18) that

$$\begin{aligned}
&(1-\alpha)z^{-p}\Omega_z^{(\lambda,p)}f(z)+\frac{\alpha}{p}z^{-p+1}(\Omega_z^{(\lambda,p)}f(z))' \\
&=(1-\alpha)z^{-p}\Omega_z^{(\lambda,p)}\left(\frac{\mu F(z)+zF'(z)}{\mu+p}\right) \\
&\quad+\frac{\alpha}{p}z^{-p+1}\left(\Omega_z^{(\lambda,p)}\left(\frac{\mu F(z)+zF'(z)}{\mu+p}\right)\right)' \\
&=\frac{\mu}{\mu+p}G(z)+\frac{1}{\mu+p}(zG'(z)+pG(z)) \\
&=G(z)+\frac{zG'(z)}{\mu+p}. \qquad (2.19)
\end{aligned}$$

Let $f(z)\in H_p(\lambda,\alpha;h)$. Then, by (2.19),

$$G(z)+\frac{zG'(z)}{\mu+p}\prec h(z)\quad (\mathrm{Re}\mu>-p),$$

and so it follows from Lemma 1.5 that

$$G(z)\prec\widetilde{h}(z)=(\mu+p)z^{-(\mu+p)}\int_0^z t^{\mu+p-1}h(t)dt\prec h(z).$$

Therefore, we conclude that

$$F(z)\in H_p(\lambda,\alpha;\widetilde{h})\subset H_p(\lambda,\alpha;h).$$

□

Theorem 2.10. *Let $f(z)\in A(p)$ and $F(z)$ be defined as in Theorem 2.9. If*

$$(1-\gamma)z^{-p}\Omega_z^{(\lambda,p)}F(z)+\gamma z^{-p}\Omega_z^{(\lambda,p)}f(z)\prec h(z)\quad(\gamma>0), \qquad (2.20)$$

then $F(z)\in H_p(\lambda,0;\widetilde{h})$, where $Re\mu>-p$ and

$$\widetilde{h}(z)=\frac{\mu+p}{\gamma}z^{-\frac{\mu+p}{\gamma}}\int_0^z t^{\frac{\mu+p}{\gamma}-1}h(t)dt\prec h(z).$$

Proof. Define

$$G(z)=z^{-p}\Omega_z^{(\lambda,p)}F(z). \qquad (2.21)$$

Then, $G(z)$ is analytic in U, with $G(0)=1$, and

$$zG'(z)=-pG(z)+z^{-p+1}(\Omega_z^{(\lambda,p)}F(z))'. \qquad (2.22)$$

Making use of (2.16), (2.20), (2.21) and (2.22), we deduce that

$$\begin{aligned}&(1-\gamma)z^{-p}\Omega_z^{(\lambda,p)}F(z)+\gamma z^{-p}\Omega_z^{(\lambda,p)}f(z)\\&=(1-\gamma)z^{-p}\Omega_z^{(\lambda,p)}F(z)+\frac{\gamma}{\mu+p}(\mu z^{-p}\Omega_z^{(\lambda,p)}F(z)\\&\quad+z^{-p+1}(\Omega_z^{(\lambda,p)}F(z))')\\&=G(z)+\frac{\gamma}{\mu+p}zG'(z)\prec h(z),\end{aligned}$$

for $\mathrm{Re}\mu>-p$ and $\gamma>0$. Therefore, an application of Lemma 1.5 yields the assertion of Theorem 2.10. □

Theorem 2.11. *Let $F(z)\in H_p(\lambda,\alpha;h)$. If the function $f(z)$ is defined by*

$$F(z)=\frac{\mu+p}{z^{\mu}}\int_0^z t^{\mu-1}f(t)dt\quad(\mu>-p),\tag{2.23}$$

then

$$\sigma^{-p}f(\sigma z)\in H_p(\lambda,\alpha;h),$$

where

$$\sigma=\sigma_p(\mu)=\frac{\sqrt{1+(\mu+p)^2}-1}{\mu+p}\in(0,1).\tag{2.24}$$

The bound σ is sharp, when

$$h(z)=\beta+(1-\beta)\frac{1+z}{1-z}\quad(\beta\neq 1).\tag{2.25}$$

Proof. For $F(z)\in A(p)$, it is easy to verify that

$$F(z)=F(z)*\frac{z^p}{1-z}\text{ and }zF'(z)=F(z)*\left(\frac{z^p}{(1-z)^2}+(p-1)\frac{z^p}{1-z}\right).$$

Hence, by (2.23), we have

$$f(z)=\frac{\mu F(z)+zF'(z)}{\mu+p}=(F*g)(z)\quad(z\in U;\mu>-p),\tag{2.26}$$

where

$$g(z)=\frac{1}{\mu+p}\left((\mu+p-1)\frac{z^p}{1-z}+\frac{z^p}{(1-z)^2}\right)\in A(p).\tag{2.27}$$

Next, we show that

$$\mathrm{Re}\{z^{-p}g(z)\}>\frac{1}{2}\quad(|z|<\sigma),\tag{2.28}$$

where $\sigma = \sigma_p(\mu)$ is given by (2.24). Setting

$$\frac{1}{1-z} = Re^{i\theta} \quad (R > 0) \quad \text{and} \quad |z| = r < 1,$$

we see that

$$\cos\theta = \frac{1 + R^2(1-r^2)}{2R} \quad \text{and} \quad R \geq \frac{1}{1+r}. \tag{2.29}$$

For $\mu > -p$, it follows from (2.27) and (2.29) that

$$\begin{aligned}
2\mathrm{Re}\{z^{-p}g(z)\} &= \frac{2}{\mu+p}[(\mu+p-1)R\cos\theta + R^2(2\cos^2\theta - 1)] \\
&= \frac{1}{\mu+p}[(\mu+p-1)(1+R^2(1-r^2)) \\
&\quad +(1+R^2(1-r^2))^2 - 2R^2] \\
&= \frac{R^2}{\mu+p}[R^2(1-r^2)^2 + (\mu+p+1)(1-r^2) - 2] + 1 \\
&\geq \frac{R^2}{\mu+p}[(1-r)^2 + (\mu+p+1)(1-r^2) - 2] + 1 \\
&= \frac{R^2}{\mu+p}(\mu+p-2r-(\mu+p)r^2) + 1.
\end{aligned}$$

This evidently gives (2.28), which is equivalent to

$$\mathrm{Re}\{z^{-p}\sigma^{-p}g(\sigma z)\} > \frac{1}{2} \quad (z \in U). \tag{2.30}$$

Let $F(z) \in H_p(\lambda, \alpha; h)$. Then, by using (2.26) and (2.30), an application of Theorem 2.4 yields

$$\sigma^{-p}f(\sigma z) = F(z) * (\sigma^{-p}g(\sigma z)) \in H_p(\lambda, \alpha; h).$$

For $h(z)$ given by (2.25), we consider the function $F(z) \in A(p)$ defined by

$$\begin{aligned}
&(1-\alpha)z^{-p}\Omega_z^{(\lambda,p)}F(z) + \frac{\alpha}{p}z^{-p+1}(\Omega_z^{(\lambda,p)}F(z))' \\
&= \beta + (1-\beta)\frac{1+z}{1-z} \quad (\beta \neq 1).
\end{aligned} \tag{2.31}$$

Then, by (2.31), (2.17) and (2.19) (used in the proof of Theorem 2.9), we find that

$$\begin{aligned}&(1-\alpha)z^{-p}\Omega_z^{(\lambda,p)}f(z)+\frac{\alpha}{p}z^{-p+1}(\Omega_z^{(\lambda,p)}f(z))'\\&=\beta+(1-\beta)\frac{1+z}{1-z}+\frac{z}{\mu+p}\left(\beta+(1-\beta)\frac{1+z}{1-z}\right)'\\&=\beta+\frac{(1-\beta)(\mu+p+2z-(\mu+p)z^2)}{(\mu+p)(1-z)^2}\\&=\beta\quad(z=-\sigma).\end{aligned}$$

Therefore, we conclude that the bound $\sigma=\sigma_p(\mu)$ cannot be increased for each $\mu(\mu>-p)$. □

Theorem 2.12. *Let* $\alpha\geq 0$ *and*

$$f_j(z)=z^p+\sum_{n=1}^{\infty}a_{n,j}z^{n+p}\in H_p(\lambda,\alpha;h_j)\quad(j=1,2),\tag{2.32}$$

where

$$h_j(z)=\beta_j+(1-\beta_j)\frac{1+z}{1-z}\qquad \textit{and}\qquad \beta_j<1.\tag{2.33}$$

If $f(z)\in A(p)$ *is defined by*

$$\Omega_z^{(\lambda,p)}f(z)=\Omega_z^{(\lambda,p)}f_1(z)*\Omega_z^{(\lambda,p)}f_2(z),\tag{2.34}$$

then $f(z)\in H_p(\lambda,\alpha;h)$*, where*

$$h(z)=\beta+(1-\beta)\frac{1+z}{1-z}\tag{2.35}$$

and the parameter β *is given by*

$$\beta=\begin{cases}1-4(1-\beta_1)(1-\beta_2)(1-\frac{p}{\alpha}\int_0^1\frac{u^{\frac{p}{\alpha}-1}}{1+u}du) & (\alpha>0),\\ 1-2(1-\beta_1)(1-\beta_2) & (\alpha=0).\end{cases}\tag{2.36}$$

The bound β *is the best possible.*

Proof. We consider the case when $\alpha>0$. By setting

$$F_j(z)=(1-\alpha)z^{-p}\Omega_z^{(\lambda,p)}f_j(z)+\frac{\alpha}{p}z^{-p+1}(\Omega_z^{(\lambda,p)}f_j(z))'\quad(j=1,2),$$

for $f_j(z)\quad(j=1,2)$, given by (2.32), we find that

$$F_j(z)=1+\sum_{n=1}^{\infty}b_{n,j}z^n\prec h_j(z)\quad(j=1,2),\tag{2.37}$$

where $h_j(z)\quad (j=1,2)$, given by (2.33), and

$$\Omega_z^{(\lambda,p)}f_j(z)=\frac{p}{\alpha}z^{-\frac{p(1-\alpha)}{\alpha}}\int_0^z t^{\frac{p}{\alpha}-1}F_j(t)dt\quad (j=1,2). \tag{2.38}$$

Now, if $f(z)\in A(p)$ is defined by (2.34), then we find from (2.38) that

$$\begin{aligned}\Omega_z^{(\lambda,p)}f(z) &= \Omega_z^{(\lambda,p)}f_1(z)*\Omega_z^{(\lambda,p)}f_2(z)\\ &= \left(\frac{p}{\alpha}z^p\int_0^1 u^{\frac{p}{\alpha}-1}F_1(uz)du\right)*\left(\frac{p}{\alpha}z^p\int_0^1 u^{\frac{p}{\alpha}-1}F_2(uz)du\right)\\ &= \frac{p}{\alpha}z^p\int_0^1 u^{\frac{p}{\alpha}-1}F(uz)du, \end{aligned}\tag{2.39}$$

where

$$F(z)=\frac{p}{\alpha}\int_0^1 u^{\frac{p}{\alpha}-1}(F_1*F_2)(uz)du. \tag{2.40}$$

Also, by using (2.37) and the Herglotz theorem, we see that

$$\mathrm{R}e\left\{\left(\frac{F_1(z)-\beta_1}{1-\beta_1}\right)*\left(\frac{1}{2}+\frac{F_2(z)-\beta_2}{2(1-\beta_2)}\right)\right\}>0\quad (z\in U),$$

which leads to

$$\mathrm{R}e\left\{(F_1*F_2)(z)\right\}>\beta_0=1-2(1-\beta_1)(1-\beta_2)\quad (z\in U).$$

According to Lemma 1.7, we have

$$\mathrm{R}e\left\{(F_1*F_2)(z)\right\}\geq\beta_0+(1-\beta_0)\frac{1-|z|}{1+|z|}\quad (z\in U). \tag{2.41}$$

Now, it follows from (2.39),(2.40) and (2.41) that

$$\begin{aligned}&\mathrm{R}e\left\{(1-\alpha)z^{-p}\Omega_z^{(\lambda,p)}f(z)+\frac{\alpha}{p}z^{-p+1}(\Omega_z^{(\lambda,p)}f(z))'\right\}=\mathrm{R}e\{F(z)\}\\ &=\frac{p}{\alpha}\int_0^1 u^{\frac{p}{\alpha}-1}\mathrm{R}e\left\{(F_1*F_2)(uz)\right\}du\\ &\geq\frac{p}{\alpha}\int_0^1 u^{\frac{p}{\alpha}-1}\left(\beta_0+(1-\beta_0)\frac{1-u|z|}{1+u|z|}\right)du\\ &>\beta_0+\frac{p(1-\beta_0)}{\alpha}\int_0^1 u^{\frac{p}{\alpha}-1}\frac{1-u}{1+u}du\\ &=1-4(1-\beta_1)(1-\beta_2)\left(1-\frac{p}{\alpha}\int_0^1\frac{u^{\frac{p}{\alpha}-1}}{1+u}du\right)\\ &=\beta\quad (z\in U),\end{aligned}$$

which proves that $f(z) \in H_p(\lambda, \alpha; h)$ for the function $h(z)$ given by (2.35) and β given by (2.36).
In order to show that the bound β is sharp, we take the functions $f_j(z) \in A(p) \quad (j = 1, 2)$, defined by

$$\Omega_z^{(\lambda,p)} f_j(z) \quad = \quad \frac{p}{\alpha} z^{-\frac{p(1-\alpha)}{\alpha}} \int_0^z t^{\frac{p}{\alpha}-1} \left(\beta_j + (1-\beta_j) \frac{1+t}{1-t} \right) dt \quad (j = 1, 2),$$

for which we have

$$\begin{aligned} F_j(z) \quad &= \quad (1-\alpha) z^{-p} \Omega_z^{(\lambda,p)} f_j(z) + \frac{\alpha}{p} z^{-p+1} (\Omega_z^{(\lambda,p)} f_j(z))' \\ &= \quad \beta_j + (1-\beta_j) \frac{1+z}{1-z} \quad (j = 1, 2) \end{aligned}$$

and

$$(F_1 * F_2)(z) = 1 + 4(1-\beta_1)(1-\beta_2) \frac{z}{1-z}.$$

Hence, for $f(z) \in A(p)$ given by (2.34), we obtain

$$\begin{aligned} &(1-\alpha) z^{-p} \Omega_z^{(\lambda,p)} f(z) + \frac{\alpha}{p} z^{-p+1} (\Omega_z^{(\lambda,p)} f(z))' \\ &= \frac{p}{\alpha} \int_0^1 u^{\frac{p}{\alpha}-1} \left(1 + 4(1-\beta_1)(1-\beta_2) \frac{uz}{1-uz} \right) du \\ &\to \beta \quad (\text{ as } z \to -1). \end{aligned}$$

Finally, for the case when $\alpha = 0$, the proof of Theorem 2.12 is simple, and so we omit the details. □

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