

OPTIMAL INEQUALITIES FOR THE POWER, HARMONIC AND LOGARITHMIC MEANS

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ABSTRACT. For all $a, b > 0$, the following two optimal inequalities are presented: $H^\alpha(a, b)L^{1-\alpha}(a, b) \geq M_{\frac{1-4\alpha}{3}}(a, b)$ for $\alpha \in [\frac{1}{4}, 1)$, and $H^\alpha(a, b)L^{1-\alpha}(a, b) \leq M_{\frac{1-4\alpha}{3}}(a, b)$ for $\alpha \in (0, \frac{3\sqrt{5}-5}{40}]$. Here, $H(a, b)$, $L(a, b)$, and $M_p(a, b)$ denote the harmonic, logarithmic, and power means of order p of two positive numbers a and b , respectively.

1. Introduction

For $p \in \mathbb{R}$ the power mean $M_p(a, b)$ of order p and logarithmic mean $L(a, b)$ of two positive numbers a and b are defined by

$$(1.1) \quad M_p(a, b) = \begin{cases} \left(\frac{a^p + b^p}{2} \right)^{\frac{1}{p}}, & p \neq 0, \\ \sqrt{ab}, & p = 0 \end{cases}$$

and

$$(1.2) \quad L(a, b) = \begin{cases} \frac{b-a}{\log b - \log a}, & a \neq b, \\ a, & a = b, \end{cases}$$

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respectively.

It is well-known that $M_p(a, b)$ is continuous and strictly increasing with respect to $p \in \mathbb{R}$ for fixed $a, b > 0$ with $a \neq b$. Recently both mean values have been the subject of intensive research. In particular, many remarkable inequalities for $M_p(a, b)$ and $L(a, b)$ can be found in the literature [2, 3, 5, 6, 8, 9, 10, 11, 12, 13, 14, 16, 17, 18].

Let $I(a, b) = \frac{1}{e}(a^a/b^b)^{1/(a-b)}$, $A(a, b) = (a+b)/2$, $G(a, b) = \sqrt{ab}$, and $H(a, b) = 2ab/(a+b)$ be the identric, arithmetic, geometric, and harmonic means of two positive real numbers a and b with $a \neq b$, respectively. Then

$$(1.3) \quad \min\{a, b\} < H(a, b) = M_{-1}(a, b) < G(a, b) = M_0(a, b) < L(a, b) \\ < I(a, b) < A(a, b) = M_1(a, b) < \max\{a, b\}.$$

In [3], Alzer and Janous established the following sharp double inequality (see also [6, p.350]):

$$M_{\frac{\log 2}{\log 3}}(a, b) \leq \frac{2}{3}A(a, b) + \frac{1}{3}G(a, b) \leq M_{\frac{2}{3}}(a, b)$$

for all $a, b > 0$.

For any $\alpha \in (0, 1)$, Janous [11] found the greatest value p and the least value q such that

$$M_p(a, b) \leq \alpha A(a, b) + (1 - \alpha)G(a, b) \leq M_q(a, b)$$

for all $a, b > 0$.

In [15], Sándor proved

$$\frac{A(a, b)G(a, b)}{I(a, b)} < \sqrt{G(a, b)I(a, b)} < L(a, b)$$

and

$$L(a, b) < A(a, b) + G(a, b) - I(a, b) < \frac{1}{2}(G(a, b) + I(a, b))$$

for all $a, b > 0$ with $a \neq b$.

The following companion of (1.3) provides inequalities for the geometric and arithmetic means of L and I , the proof can be found in [1].

$$G(a, b)^{\frac{1}{2}}A(a, b)^{\frac{1}{2}} \leq L(a, b)^{\frac{1}{2}}I(a, b)^{\frac{1}{2}} \leq \frac{1}{2}L(a, b) + \frac{1}{2}I(a, b) \\ \leq \frac{1}{2}G(a, b) + \frac{1}{2}A(a, b)$$

for all $a, b > 0$.

Burk [7] established

$$L(a, b) < \left(\frac{a^{\frac{1}{3}} + b^{\frac{1}{3}}}{2} \right)^3$$

for all $a, b > 0$ with $a \neq b$.

Alzer and Qiu [4] proved

$$M_c(a, b) \leq \frac{1}{2}L(a, b) + \frac{1}{2}I(a, b)$$

for all $a, b > 0$ with the best possible parameter $c = \frac{\log 2}{1 + \log 2}$.

$$[G(a, b)]^{A(a, b)} < [L(a, b)]^{I(a, b)} < [A(a, b)]^{G(a, b)}$$

for $a, b \geq e$, and

$$[A(a, b)]^{G(a, b)} < [I(a, b)]^{L(a, b)} < [G(a, b)]^{A(a, b)}$$

for $0 < a, b < e$.

The main purpose of this paper is to present the optimal bounds for $H^\alpha(a, b)L^{1-\alpha}(a, b)$ in terms of the power mean $M_p(a, b)$ for some $\alpha \in (0, 1)$.

2. Set up

In order to establish our main results we need a lemma, which we present in this section.

Lemma 2.1. *Let $r \in (0, 1)$, $p = \frac{1-4r}{3}$ and $g(t) = [rt^{p+2} - t^{p+1} + (r-1)t^p - (1-r)t^2 - t + r] \log t + (1-r)(t^{p+2} - t^p + t^2 - 1)$. Then the following statements are true:*

- (1) *If $r \in (\frac{1}{4}, 1)$, then $g(t) < 0$ for $t \in (1, \infty)$;*
- (2) *If $r \in (0, \frac{3\sqrt{5}-5}{40}]$, then $g(t) > 0$ for $t \in (1, \infty)$.*

Proof. Let $g_1(t) = t^{1-p}g'(t)$, $g_2(t) = t^p g'_1(t)$, $g_3(t) = t^{1-p}g'_2(t)$, $g_4(t) = t^p g'_3(t)$, $g_5(t) = t^{1-p}g'_4(t)$, $g_6(t) = t^{3+p}g'_5(t)$, $g_7(t) = t^{1-p}g'_6(t)$, $g_8(t) =$

$t^{p-1}g_7'(t)$, $g_9(t) = t^2g_8'(t)$, and $g_{10}(t) = t^{1-p}g_9'(t)$. Then simple computations lead to

$$(2.1) \quad g(1) = 0,$$

$$g_1(t) = [r(p+2)t^2 - (p+1)t - 2(1-r)t^{2-p} - t^{1-p} \\ - p(1-r)] \log t + (p-pr+2-r)t^2 - t \\ + (1-r)t^{2-p} + rt^{-p} - t^{1-p} - (p+1)(1-r),$$

$$(2.2) \quad g_1(1) = 0,$$

$$g_2(t) = [2r(p+2)t^{1+p} - (p+1)t^p + 2(p-2)(1-r)t \\ + p-1] \log t + (2p+4-pr)t^{1+p} - (p+2)t^p \\ - p(1-r)t^{p-1} - p(1-r)t - prt^{-1} + p-2,$$

$$(2.3) \quad g_2(1) = 0,$$

$$g_3(t) = [2r(p+1)(p+2)t + 2(p-2)(1-r)t^{1-p} \\ - p(p+1)] \log t + (2p^2 - p^2r + pr + 6p + 4r + 4)t \\ - p(p-1)(1-r)t^{-1} + prt^{-1-p} + (p-1)t^{-p} \\ + (1-r)(p-4)t^{1-p} - (p^2 + 3p + 1),$$

$$(2.4) \quad g_3(1) = 6p + 8r - 2 = 0,$$

$$g_4(t) = 2[r(p+1)(p+2)t^p - (p-1)(p-2)(1-r)] \log t \\ + p(p-1)(1-r)t^{p-2} - p(p+1)t^{p-1} - p(p-1)t^{-1} \\ - p(p+1)rt^{-2} + (2p^2 + p^2r + 7pr + 6p + 8r + 4)t^p \\ - (p^2 - 7p + 8)(1-r),$$

$$(2.5) \quad g_4(1) = 12p + 16r - 4 = 0,$$

$$g_5(t) = 2pr(p+1)(p+2) \log t - 2(p-1)(p-2)t^{-p} \\ + p(p-1)t^{-1-p} - p(p+1)(p-1)t^{-1} \\ + p(p-1)(p-2)(1-r)t^{-2} + 2p^3 + p^3r + 9p^2r \\ + 6p^2 + 14pr + 4p + 4r,$$

$$(2.6) \quad g_5(1) = 2p^3 + 16rp^2 + 2p^2 + 8pr + 12p + 8r - 4$$

$$= \frac{8(r-1)}{27}(80r^2 + 20r - 1),$$

$$g_6(t) = p[2(p-1)(p-2)(1-r)t^2 - (p-1)(p+1)t \\ - 2r(p+1)(p+2) + 2r(p+1)(p+2)t^{2+p} \\ + (p-1)(p+1)t^{1+p} - 2(p-1)(p-2)(1-r)t^p],$$

$$(2.7) \quad g_6(1) = 0,$$

$$\begin{aligned}
g_7(t) &= p[2r(p+1)(p+2)^2t^2 + (p-1)(p+1)^2t \\
&\quad - (p-1)(p+1)t^{1-p} + 4(p-1)(p-2)(1-r)t^{2-p} \\
&\quad - 2p(p-1)(p-2)(1-r)], \\
(2.8) \quad g_7(1) &= p[4rp^3 - p^3 + 10p^2 + 32pr - 17p - 8] \\
&= \frac{4(1-4r)}{81}(-64r^4 + 64r^3 - 192r^2 + 169r + 23), \\
g_8(t) &= p[4r(p+1)(p+2)^2t^p + (p-1)(p+1)^2t^{p-1} \\
&\quad + (p-1)^2(p+1)t^{-1} - 4(p-1)(p-2)^2(1-r)], \\
(2.9) \quad g_8(1) &= 2p[4rp^3 - p^3 + 10p^2 + 32pr - 17p - 8] \\
&= \frac{8(1-4r)}{81}(-64r^4 + 64r^3 - 192r^2 + 169r + 23), \\
g_9(t) &= p[4rp(p+1)(p+2)^2t^{p+1} + (p-1)^2(p+1)^2t^p \\
&\quad - (p+1)(p-1)^2], \\
(2.10) \quad g_9(1) &= p^2(p+1)[4r(p+2)^2 + (p-1)^2] > 0, \\
(2.11) \quad g_{10}(t) &= p^2(p+1)^2[4r(p+2)^2t + (p-1)^2], \\
(2.12) \quad g_{10}(1) &= p^2(p+1)^2[4r(p+2)^2 + (p-1)^2] > 0.
\end{aligned}$$

(1) If $r \in (\frac{1}{4}, 1)$, then from (2.6), (2.8) and (2.9) we have

$$\begin{aligned}
(2.13) \quad &g_8(1) < 0, \\
(2.14) \quad &g_7(1) < 0, \\
(2.15) \quad &g_5(1) < 0, \\
(2.16) \quad &\lim_{t \rightarrow +\infty} g_8(t) < 0, \\
(2.17) \quad &\lim_{t \rightarrow +\infty} g_7(t) = -\infty.
\end{aligned}$$

Equation (2.11) implies that $g_{10}(t)$ is strictly increasing in $(1, \infty)$, then from (2.10), (2.12) and the monotonicity of $g_{10}(t)$ we clearly see that $g_9(t) > 0$ for all $t \in (1, \infty)$. Hence we get that $g_8(t)$ is strictly increasing in $(1, \infty)$.

From (2.13) and (2.16) together with the monotonicity of $g_8(t)$ we see that $g_8(t) < 0$ in $(1, \infty)$. Hence we obtain that $g_7(t)$ is strictly decreasing in $(1, \infty)$. From (2.14) and (2.17) together with the monotonicity of $g_7(t)$

we clearly see that $g_7(t) < 0$ in $(1, \infty)$. So we get that $g_6(t)$ is strictly decreasing in $(1, \infty)$.

Therefore, Lemma 2.1(1) follows from the monotonicity of $g_6(t)$, (2.7), (2.15) and (2.1)-(2.5).

(2) If $r \in (0, \frac{3\sqrt{5}-5}{40}]$, then from (2.6), (2.8) and (2.9) we have

$$(2.18) \quad g_8(1) > 0,$$

$$(2.19) \quad g_7(1) > 0,$$

$$(2.20) \quad g_5(1) \geq 0.$$

Equation (2.11) implies that $g_{10}(t)$ is strictly increasing in $(1, \infty)$. Therefore, Lemma 2.1(2) follows from (2.12), (2.10), (2.18), (2.19), (2.7), (2.20) and (2.1)-(2.5) together with the monotonicity of $g_{10}(t)$. $\square$

3. The main result

Theorem 3.1. *For all $a, b > 0$ we*

$$M_{\frac{1-4\alpha}{3}}(a, b) \leq H^\alpha(a, b)L^{1-\alpha}(a, b)$$

for $\alpha \in (\frac{1}{4}, 1)$, and

$$H^\alpha(a, b)L^{1-\alpha}(a, b) \leq M_{\frac{1-4\alpha}{3}}(a, b)$$

for $\alpha \in (0, \frac{3\sqrt{5}-5}{40}]$. Each inequality becomes equality if and only if $a = b$, and the parameter $\frac{1-4\alpha}{3}$ in each inequality is the best possible.

Proof. If $a = b$, then from (1.1) we clearly see that

$$a = M_0(a, b) = H^\alpha(a, b)L^{1-\alpha}(a, b) = M_{\frac{1-4\alpha}{3}}(a, b) = b.$$

Without loss of generality, we assume that $a > b$. Let $t = \frac{a}{b} > 1$. Then (1.1) and (1.2) lead to

$$(3.1) \quad \begin{aligned} & M_p(a, b) - H^\alpha(a, b)L^{1-\alpha}(a, b) \\ &= b \left[\left(\frac{t^p + 1}{2} \right)^{\frac{1}{p}} - \left(\frac{2t}{t+1} \right)^\alpha \left(\frac{t-1}{\log t} \right)^{1-\alpha} \right]. \end{aligned}$$

Let

$$(3.2) \quad \begin{aligned} f(t) &= \frac{1}{p} \log \frac{1+t^p}{2} - \alpha \log \left(\frac{2t}{t+1} \right) - (1-\alpha)[\log(t-1) \\ &\quad + \log(\log t)]. \end{aligned}$$

Then simple computations yield

$$(3.3) \quad \lim_{t \rightarrow 1^+} f(t) = 0,$$

$$(3.4) \quad f'(t) = \frac{g(t)}{t(t-1)(t+1)(t^p+1)\log t},$$

where

$$g(t) = [\alpha t^{p+2} - t^{p+1} + (\alpha - 1)t^p - (1 - \alpha)t^2 - t + \alpha] \log t + (1 - \alpha)(t^{p+2} - t^p + t^2 - 1).$$

(1) If $\alpha \in (\frac{1}{4}, 1)$, then from Lemma 2.1(1) and (3.4) we get

$$(3.5) \quad f'(t) < 0$$

for all $t \in (1, \infty)$.

Therefore, from (3.1)-(3.3) and (3.5) we see that the inequality

$$H^\alpha(a, b)L^{1-\alpha}(a, b) > M_{\frac{1-4\alpha}{3}}(a, b)$$

holds for all $a, b > 0$ with $a \neq b$.

(2) If $\alpha \in (0, \frac{3\sqrt{5}-5}{40}]$, from Lemma 2.1(2) and (3.4) we have

$$(3.6) \quad f'(t) > 0$$

for all $t \in (1, \infty)$.

Therefore, from (3.1)-(3.3) and (3.6) we obtain that inequality

$$H^\alpha(a, b)L^{1-\alpha}(a, b) < M_{\frac{1-4\alpha}{3}}(a, b)$$

for all $a, b > 0$ with $a \neq b$.

Next, we prove that the parameter $\frac{1-4\alpha}{3}$ in each inequality is optimal.

Case A. If $\alpha \in [\frac{1}{4}, 1)$, then for any $0 < \varepsilon < (4\alpha - 1)/3$ and $x > 0$ one has

$$\begin{aligned} (3.7) \quad & M_{\frac{1-4\alpha}{3}+\varepsilon}(x+1, 1) - H^\alpha(x+1, 1)L^{1-\alpha}(x+1, 1) \\ &= \left[\frac{(x+1)^{\frac{1-4\alpha}{3}+\varepsilon} + 1}{2} \right]^{\frac{1}{\frac{1-4\alpha}{3}+\varepsilon}} - \left(\frac{x+1}{1+\frac{x}{2}} \right)^\alpha \cdot \left[\frac{x}{\log(x+1)} \right]^{1-\alpha} \\ &= \frac{f_1(x)}{\left(1+\frac{x}{2}\right)^\alpha [\log(1+x)]^{1-\alpha}}, \end{aligned}$$

where

$$f_1(x) = \left[\frac{(x+1)^{\frac{1-4\alpha}{3}+\varepsilon} + 1}{2} \right]^{\frac{1-4\alpha}{3}+\varepsilon} [\log(1+x)]^{1-\alpha} \left(1 + \frac{x}{2}\right)^\alpha - (1+x)^\alpha x^{1-\alpha}.$$

Letting $x \rightarrow 0$ and making use of Taylor expansion we get

$$(3.8) \quad f_1(x) = \left[\frac{\varepsilon}{8} x^2 + o(x^2) \right] x^{1-\alpha}.$$

Equations (3.7) and (3.8) imply that for any $\alpha \in [\frac{1}{4}, 1)$ and $0 < \varepsilon < \frac{4\alpha-1}{3}$ there exists $\delta_1(\varepsilon, \alpha) > 0$, such that

$$H^\alpha(x+1, 1)L^{1-\alpha}(x+1, 1) < M_{\frac{1-4\alpha}{3}+\varepsilon}(x+1, 1)$$

for $x \in (0, \delta_1)$.

Case B. If $\alpha \in (0, \frac{3\sqrt{5}-5}{40}]$, then for any $0 < \varepsilon < \frac{1-4\alpha}{3}$ and $x > 0$ one has

$$(3.9) \quad \begin{aligned} & H^\alpha(x+1, 1)L^{1-\alpha}(x+1, 1) - M_{\frac{1-4\alpha}{3}-\varepsilon}(x+1, 1) \\ &= \left(\frac{x+1}{1+\frac{x}{2}} \right)^\alpha \left[\frac{x}{\log(x+1)} \right]^{1-\alpha} - \left[\frac{(x+1)^{\frac{1-4\alpha}{3}-\varepsilon} + 1}{2} \right]^{\frac{1}{\frac{1-4\alpha}{3}-\varepsilon}} \\ &= \frac{f_2(x)}{\left(1 + \frac{x}{2}\right)^\alpha [\log(1+x)]^{1-\alpha}}, \end{aligned}$$

where

$$\begin{aligned} f_2(x) &= (1+x)^\alpha \cdot x^{1-\alpha} - \left[\frac{(x+1)^{\frac{1-4\alpha}{3}-\varepsilon} + 1}{2} \right]^{\frac{1-4\alpha}{3}-\varepsilon} [\log(1+x)]^{1-\alpha} \\ &\quad \times \left(1 + \frac{x}{2}\right)^\alpha. \end{aligned}$$

Letting $x \rightarrow 0$ and making use of Taylor expansion we have

$$(3.10) \quad f_2(x) = \left[\frac{\varepsilon}{8} x^2 + o(x^2) \right] x^{1-\alpha}.$$

Equations (3.9) and (3.10) imply that for any $\alpha \in (0, \frac{3\sqrt{5}-5}{40}]$ and $0 < \varepsilon < \frac{1-4\alpha}{3}$ there exists $\delta_2(\varepsilon, \alpha) > 0$, such that

$$H^\alpha(1+x, 1)L^{1-\alpha}(1+x, 1) > M_{\frac{1-4\alpha}{3}-\varepsilon}(1+x, 1)$$

for $x \in (0, \delta_2)$. □

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