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THE EXISTENCE RESULTS FOR A COUPLED SYSTEM OF NONLINEAR FRACTIONAL DIFFERENTIAL EQUATIONS WITH MULTI-POINT BOUNDARY CONDITIONS

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ABSTRACT. In this paper, we study a coupled system of nonlinear fractional differential equations with multi-point boundary conditions. The differential operator is taken in the Riemann-Liouville sense. Applying the Schauder fixed-point theorem and the contraction mapping principle, two existence results are obtained for the following system

$$\begin{cases} D_{0+}^{\alpha}x(t) = f\left(t, y(t), D_{0+}^{p}y(t)\right), & t \in (0,1), \\ D_{0+}^{\beta}y(t) = g\left(t, x(t), D_{0+}^{q}x(t)\right), & t \in (0,1), \\ x(0) = x'(0) = x''(0) = \cdots = x^{(m-2)}(0) = 0, & x(1) = \lambda x(\xi), \\ y(0) = y'(0) = y''(0) = \cdots = y^{(m-2)}(0) = 0, & y(1) = \lambda y(\xi), \end{cases}$$

where $0 < \xi < 1$, $m \in \mathbb{N}$, $m \geq 2$, α, $\beta \in (m-1, m)$ and α, β, p, q, λ satisfy certain conditions.

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1. **Introduction**

In this paper, we consider the existence of the solution of the following multi-point boundary value problem

$$
(1.1)\quad \begin{cases} D_{0+}^{\alpha}x(t) = f\left(t, y(t), D_{0+}^{p}y(t)\right), & t \in (0,1), \\ D_{0+}^{\beta}y(t) = g\left(t, x(t), D_{0+}^{q}x(t)\right), & t \in (0,1), \\ x(0) = x'(0) = \cdots = x^{(m-2)}(0) = 0,\ x(1) = \lambda x(\xi), \\ y(0) = y'(0) = \cdots = y^{(m-2)}(0) = 0,\ y(1) = \lambda y(\xi), \end{cases}
$$

where $0 < \xi < 1,\ m \in \mathbb{N},\ m \geq 2,\ \alpha,\ \beta \in (m-1, m),\ \alpha - q \geq 1,\ \beta - p \geq 1,\ p,\ q \geq 0,\ \lambda \geq 0,\ 1 - \lambda\xi^{\alpha-1} > 0,\ 1 - \lambda\xi^{\beta-1} > 0,\ f, g : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are given continuous functions and D_{0+} is the standard Riemann-Liouville fractional derivative.

Fractional differential equations have been shown to be very useful in the study of models of many phenomena in various fields of science and engineering, such as viscoelasticity, electrochemistry, control, porous media, electromagnetic, aerodynamics, polymer rheology, etc (see[7, 10, 11]). Recently, there are a large number of papers dealing with the fractional differential equations (see[1–6, 8, 12–16]). Among them, initial and boundary value problems for nonlinear differential equations of fractional order have been a subject of intensive studies for quite a long time and continue to play an important role in the theory of differential equations (see[2–4, 12–14]). From [2, 3, 6, 8, 13], we know that the coupled system of differential equations of fractional order is also important and several authors have done a lot of work in this topic. In [13], Xinwei Su considered the coupled system with two-point boundary conditions, while Bashir Ahmad and Juan J. Nieto investigated the three-point boundary value problem of the coupled system in [2]. Both papers discussed the coupled system by means of the fixed-point theorem where $\alpha,\ \beta \in (1, 2]$.

In this paper, we study the system under the multi-point boundary conditions, and give the existence results of the solution of the system (1.1). Besides, we prove the existence and uniqueness of the solution applying the contraction mapping principle.

2. **Preliminaries**

We present the necessary definitions and fundamental facts on the fractional calculus theory. They can be found in [7, 10, 11].

Definition 2.1 (*[7, 10, 11]*). The Riemann-Liouville fractional integral of order $\nu > 0$ of a function $h : (0, \infty) \to \mathbb{R}$ is given by

$$I_{0+}^{\nu} h(t) = \frac{1}{\Gamma(\nu)} \int_0^t (t-s)^{\nu-1} h(s) ds \tag{2.1}$$

or

$$D_{0+}^{-\nu} h(t) = \frac{1}{\Gamma(\nu)} \int_0^t (t-s)^{\nu-1} h(s) ds, \tag{2.2}$$

provided that the right-hand side is pointwise defined on $(0, \infty)$.

Definition 2.2 (*[7, 10, 11]*). The Riemann-Liouville fractional derivative of order $\nu > 0$ of a continuous function $h : (0, \infty) \to \mathbb{R}$ is given by

$$D_{0+}^{\nu} h(t) = \frac{1}{\Gamma(n-\nu)} \left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-\nu-1} h(s) ds, \tag{2.3}$$

where $n = [\nu] + 1$, provided that the right-hand side is pointwise defined on $(0, \infty)$.

Lemma 2.1. ([4]) *Assume that $h \in C(0,1) \bigcap L(0,1)$ with a fractional derivative of order $\nu > 0$ that belongs to $C(0,1) \bigcap L(0,1)$. Then*

$$I_{0+}^{\nu} D_{0+}^{\nu} h(t) = h(t) + C_1 t^{\nu-1} + C_2 t^{\nu-2} + \cdots + C_N t^{\nu-N}, \forall t \in (0,1) \tag{2.4}$$

for some $C_i \in \mathbb{R},\ i = 1,\ 2, \ldots, N$, where N is the smallest integer greater than or equal to ν.

Lemma 2.2. ([7, 10, 11]) *If $\nu_1, \nu_2, \nu > 0,\ t \in [0,1]$ and $h \in L[0,1]$, then we have*

$$I_{0+}^{\nu_1} I_{0+}^{\nu_2} h(t) = I_{0+}^{\nu_1+\nu_2} h(t), \qquad D_{0+}^{\nu} I_{0+}^{\nu} h(t) = h(t). \tag{2.5}$$

Lemma 2.3. ([9, 11]) *If $h \in C[0,1]$ and $\nu > 0$, then for $s, t \in [0,1]$ we have*

$$\left[I_{0+}^{\nu} h(t)\right]_{t=0} = 0, \quad \text{or} \quad \lim_{t \to 0} \frac{1}{\Gamma(\nu)} \int_0^t (t-s)^{\nu-1} h(s) ds = 0, \tag{2.6}$$

Lemma 2.4. ([9, 11]) *If* $h \in C^n[0,1]$ *and* $n-1<\nu<n \in \mathbb{N}$, *then*

$$(2.7)\quad I^{\nu}_{0+}D^{\nu}_{0+}h(t) = h(t) - \sum_{k=1}^{n}\left[D^{\nu-k}_{0+}h(t)\right]_{t=0}\frac{t^{\nu-k}}{\Gamma(\nu-k+1)}, \forall t \in [0,1]\ .$$

Lemma 2.5. ([11]) *If* $h \in C^n[0,1]$ *and* $n-1<\nu<n \in \mathbb{N}$, *then the conditions*

$$\left[D^{\nu-k}_{0+}h(t)\right]_{t=0} = 0, \qquad k = 1,2,\cdots,n,$$

are equivalent to

$$h^{(k)}(0) = 0, \qquad k = 0,1,\cdots,n-1.$$

Define the space

$$\mathcal{C} = \left\{x:\ x \in C^{m-2}[0,1],\ x(0) = x'(0) = \cdots = x^{(m-2)}(0) = 0\right\}$$

and

$$X = \left\{x:\ x \in \mathcal{C} \text{ and } D^{q}_{0+}x(t) \in C[0,1]\right\}$$

endowed with the norm

$$\|x\|_X = \max_{k\le m-2}\max_{t\in[0,1]}\left|x^{(k)}(t)\right| + \max_{t\in[0,1]}\left|D^q_{0+}x(t)\right|,$$

where $k \in \mathbb{N}$.

Lemma 2.6. *If* $x \in X$, *then we have*

$$I^q_{0+}D^q_{0+}x(t) = x(t), \forall t \in [0,1]$$

Proof. By Lemma 2.4, we have

$$I^q_{0+}D^q_{0+}x(t) = x(t) - \sum_{k=1}^{[q]+1}\left[D^{q-k}_{0+}x(t)\right]_{t=0}\frac{t^{q-k}}{\Gamma(q-k+1)}\ .$$

Since $\alpha - q \ge 1$, $q>0$ and $\alpha \in (m-1,m)$, then $0<q\le\alpha-1<m-1$ and $[q]<m-2$. By the definition of the space X, we can get

$$x(0) = x'(0) = x''(0) = \cdots = x^{([q])}(0) = 0.$$

Then Lemma 2.5 implies that

$$\left[D_{0+}^{q-k}x(t)\right]_{t=0}=0, \qquad k=1,2,\cdots,[q]+1.$$

In consequence,

$$I_{0+}^{q}D_{0+}^{q}x(t)=x(t).$$

This ends the proof. □

Lemma 2.7. $(X,\|\cdot\|_X)$ *is a Banach space.*

Remark 2.1. Analogous to the proof of Lemma 3.2 in [13], we can prove Lemma 2.7. One thing we should note is that in the proof of Lemma 3.2 in [13] $I_{0+}^{q}D_{0+}^{q}x(t)=x(t)$ holds in X' only when $0<q<1$(we denote the space X in [13] as X'), and in this paper we prove that $I_{0+}^{q}D_{0+}^{q}x(t)=x(t)$ in X, in which q can be equal or greater 1.

Similarly, we can define the Banach space

$$Y=\left\{y(t):\ y(t)\in\mathcal{C}\ \mathit{and}\ D_{0+}^{p}y(t)\in C[0,1]\right\}$$

endowed with the norm

$$\|y\|_Y=\max_{k\le m-2}\max_{t\in[0,1]}\left|y^{(k)}(t)\right|+\max_{t\in[0,1]}\left|D_{0+}^{p}y(t)\right|,$$

where $k\in\mathbb{N}$.

For $(x,y)\in X\times Y$, let

$$\|(x,y)\|_{X\times Y}=\max\left\{\|x\|_X,\ \|y\|_Y\right\}.$$

Then clearly $(X\times Y,\|\cdot\|_{X\times Y})$ is a Banach space.

3. **The main results**

In this section, we will reduce the problem (1.1) to the equivalent system of integral equations, and then obtain the existence result of the solution of problem (1.1) under certain conditions.

First, we present the Green's function of system (1.1) by the following lemma.

Lemma 3.1. *Given a function $h \in C[0,1]$, $m \in \mathbb{N}$, $m \geq 2$ and $\alpha \in (m-1, m)$, then the unique solution of the boundary value problem,*

$$D_{0+}^{\alpha} x(t) = h(t), \qquad t \in (0,1), \tag{3.1}$$

$$\begin{gathered} x(0) = x'(0) = \cdots = x^{(m-2)}(0) = 0, \; x(1) = \lambda x(\xi), \\ 0 < \xi < 1, \; 1 - \lambda \xi^{\alpha-1} > 0, \end{gathered} \tag{3.2}$$

is given by

$$\begin{aligned} x(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s) ds \\ &- \frac{t^{\alpha-1}}{\Gamma(\alpha)(1-\lambda\xi^{\alpha-1})} \left[\int_0^1 (1-s)^{\alpha-1} h(s) ds - \lambda \int_0^{\xi} (\xi - s)^{\alpha-1} h(s) ds \right]. \end{aligned} \tag{3.3}$$

Proof. Applying Lemma 2.1, we can reduce (3.1) to an equivalent integral equation

$$x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s) ds - C_1 t^{\alpha-1} - C_2 t^{\alpha-2} - \cdots - C_m t^{\alpha-m}, \tag{3.4}$$

where C_1, C_2, $\cdots$, $C_m \in \mathbb{R}$ are arbitrary constants. By $x(0) = x'(0) = \cdots = x^{(m-2)}(0) = 0$, we can obtain $C_2 = C_3 = \cdots = C_m = 0$. Then we can write (3.4) as

$$x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s) ds - C_1 t^{\alpha-1}.$$

Using $x(1) = \lambda x(\xi)$, we get

$$C_1 = \frac{1}{\Gamma(\alpha)(1-\lambda\xi^{\alpha-1})} \left[\int_0^1 (1-s)^{\alpha-1} h(s) ds - \lambda \int_0^{\xi} (\xi - s)^{\alpha-1} h(s) ds \right].$$

Substituting the values of C_1, C_2, $\cdots$, C_m in (3.4), we obtain (3.3). This completes the proof. □

Eq. (3.3) can be written as

$$x(t) = \int_0^1 G_1(t,s) h(s) ds$$

where $G_1(t,s)$ is the Green's function given by

$$G_1(t,s)=\frac{1}{\Gamma(\alpha)(1-\lambda\xi^{\alpha-1})}\begin{cases}G_{11}(t,s), & 0\le t\le\xi,\\ G_{12}(t,s), & \xi\le t\le 1,\end{cases}$$

$$G_{11}(t,s)=\begin{cases}P(t,s,\alpha)-Q(t,s,\alpha), & 0\le s\le t,\\ -Q(t,s,\alpha), & t<s\le\xi,\\ -t^{\alpha-1}(1-s)^{\alpha-1}, & \xi<s\le 1,\end{cases}\tag{3.5}$$

$$G_{12}(t,s)=\begin{cases}P(t,s,\alpha)-Q(t,s,\alpha), & 0\le s\le\xi,\\ P(t,s,\alpha)-t^{\alpha-1}(1-s)^{\alpha-1}, & \xi<s\le t,\\ -t^{\alpha-1}(1-s)^{\alpha-1}, & t<s\le 1.\end{cases}$$

where

$$P(t,s,\alpha)=(1-\lambda\xi^{\alpha-1})(t-s)^{\alpha-1},$$
$$Q(t,s,\alpha)=t^{\alpha-1}\left[(1-s)^{\alpha-1}-\lambda(\xi-s)^{\alpha-1}\right].$$

By the same approach, we can get the unique solution of

$$D^{\beta}_{0+}y(t)=h(t),\quad t\in(0,1),\tag{3.6}$$

$$\begin{gathered}y(0)=y'(0)=\cdots=y^{(m-2)}(0)=0,\quad y(1)=\lambda y(\xi),\\ 0<\xi<1,\quad 1-\lambda\xi^{\beta-1}>0\end{gathered}\tag{3.7}$$

is

$$y(t)=\int_0^1 G_2(t,s)h(s)ds,$$

where $G_2(t,s)$ is the Green's function which can be obtained by replacing α with β in (3.5). Define the Green's function of the system (1.1) as (G_1,G_2).

Consider the coupled system of the integral equations as follows:

$$\begin{cases}x(t)=\displaystyle\int_0^1 G_1(t,s)f\left(t,y(t),D^{p}_{0+}y(t)\right)ds,\\ y(t)=\displaystyle\int_0^1 G_2(t,s)g\left(t,x(t),D^{q}_{0+}x(t)\right)ds.\end{cases}\tag{3.8}$$

Lemma 3.2. *Assume that $f,g:[0,1]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ are continuous. Then $(x,y)\in X\times Y$ is a solution of* (1.1) *if and only if $(x,y)\in X\times Y$ is a solution of the system* (3.8).

Proof. The proof is immediate from the discussion above, so we omit it. $\square$

Let $T : X \times Y \to X \times Y$ be an operator defined as

$$T(x, y)(t) = (T_1 y(t), T_2 x(t)),$$

where

$$T_1 y(t) = \int_0^1 G_1(t, s) f\left(s, y(s), D_{0+}^p y(s)\right) ds,$$
$$T_2 x(t) = \int_0^1 G_2(t, s) g\left(s, x(s), D_{0+}^q x(s)\right) ds.$$

It is obvious that the fixed-point of the operator T is the solution of the problem (1.1).

For convenience, we will set some notations as follows:

$$\begin{aligned}
d_{11} &= \int_0^1 (1 - s)^{\alpha-1} a(s) ds, \\
d_{12} &= \lambda \int_0^\xi (\xi - s)^{\alpha-1} a(s) ds, \\
l_{11}(k) &= (1 - \lambda\xi^{\alpha-1}) \int_0^1 (1 - s)^{\alpha-k-1} a(s) ds, \\
l_{12}(q) &= (1 - \lambda\xi^{\alpha-1}) \int_0^1 (1 - s)^{\alpha-q-1} a(s) ds, \\
\mu_{11} &= \max_{k \le m-2} \frac{d_{11} + d_{12} + l_{11}(k)}{\Gamma(\alpha - k)(1 - \lambda\xi^{\alpha-1})}, \\
\mu_{12} &= \frac{d_{11} + d_{12} + l_{12}(q)}{\Gamma(\alpha - q)(1 - \lambda\xi^{\alpha-1})}, \\
\mu_1 &= \mu_{11} + \mu_{12}, \\
\omega_{11} &= \max_{k \le m-2} \frac{\alpha(1 - \lambda\xi^{\alpha-1}) + (\alpha - k)(1 + \lambda\xi^\alpha)}{\alpha\Gamma(\alpha - k + 1)(1 - \lambda\xi^{\alpha-1})}, \\
\omega_{12} &= \frac{\alpha(1 - \lambda\xi^{\alpha-1}) + (\alpha - q)(1 + \lambda\xi^\alpha)}{\alpha\Gamma(\alpha - q + 1)(1 - \lambda\xi^{\alpha-1})}, \\
\omega_1 &= \omega_{11} + \omega_{12},
\end{aligned}$$

where $a(t)$, $b(t)$ are nonnegative functions in $L[0,1]$. By replacing α with β and $a(t)$ with $b(t)$ respectively, we can define μ_2 and ω_2.

Now we present the main results of the paper.

Theorem 3.3. *Let $f, g : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be continuous. Assume that one of the following conditions is satisfied.*

- (H_1) *There exist two nonnegative functions a, $b \in L[0,1]$ such that $|f(t,u,v)| \leq a(t) + \kappa_1 |u|^{\rho_1} + \kappa_2 |v|^{\rho_2}$ and $|g(t,u,v)| \leq b(t) + \chi_1 |u|^{\delta_1} + \chi_2 |v|^{\delta_2}$, where $\kappa_1,\ \kappa_2,\ \chi_1,\ \chi_2 \geq 0,\ 0 < \rho_1,\ \rho_2,\ \delta_1,\ \delta_2 < 1, t \in [0,1]$.*
- (H_2) *$|f(t,u,v)| \leq \kappa_1 |u|^{\rho_1} + \kappa_2 |v|^{\rho_2}$ and $|g(t,u,v)| \leq \chi_1 |u|^{\delta_1} + \chi_2 |v|^{\delta_2}$, where $\kappa_1,\ \kappa_2,\ \chi_1,\ \chi_2 \geq 0,\ \rho_1,\ \rho_2,\ \delta_1,\ \delta_2 > 1$.*

Then the problem (1.1) *has a solution.*

Proof. The proof will be given in two parts, and we'll prove each part in three steps by using the Schauder fixed-point theorem.

Part 1: Let (H_1) be valid. Define

$$B = \left\{(x,y) : (x,y) \in X \times Y,\ \|(x,y)\|_{X\times Y} \leq R\right\},$$

where

$$R \geq \max\left\{3\mu_1,\ b_{12},\ b_{13},\ 3\mu_2,\ b_{11},\ b_{14}\right\},$$

$$b_{11} = (3\omega_1\kappa_1)^{\frac{1}{1-\rho_1}},$$

$$b_{12} = (3\omega_1\kappa_2)^{\frac{1}{1-\rho_2}},$$

$$b_{13} = (3\omega_2\chi_1)^{\frac{1}{1-\delta_1}},$$

and

$$b_{14} = (3\omega_2\chi_2)^{\frac{1}{1-\delta_2}}.$$

Step 1: $T : B \to B$. Let $F(t) = f\left(t, y(t), D_{0+}^{p} y(t)\right)$. For any $(x,y) \in B$, applying Lemma 3.1 and the property $D_{0+}^{q} t^{\alpha-1} = \frac{\Gamma(\alpha)}{\Gamma(\alpha-q)} t^{\alpha-q-1}$ (see [10]) we have

$$
\begin{aligned}
&\left|(T_1y)^{(k)}(t)\right| \\
&= \left|\frac{1}{\Gamma(\alpha-k)}\int_0^t (t-s)^{\alpha-k-1}F(s)ds - \frac{t^{\alpha-k-1}}{\Gamma(\alpha-k)(1-\lambda\xi^{\alpha-1})}\right. \\
&\quad \left.\times\left(\int_0^1 (1-s)^{\alpha-1}F(s)ds - \lambda\int_0^\xi (\xi-s)^{\alpha-1}F(s)ds\right)\right| \\
&\le \frac{1}{\Gamma(\alpha-k)(1-\lambda\xi^{\alpha-1})}\left[(1-\lambda\xi^{\alpha-1})\left(\int_0^t (t-s)^{\alpha-k-1}a(s)ds\right.\right. \\
&\quad \left.+\frac{t^{\alpha-k}}{\alpha-k}\left(\kappa_1 R^{\rho_1}+\kappa_2 R^{\rho_2}\right)\right) + t^{\alpha-k-1}\left(\int_0^1 (1-s)^{\alpha-1}a(s)ds\right. \\
&\quad \left.\left.+\lambda\int_0^\xi (\xi-s)^{\alpha-1}a(s)ds + \frac{1+\lambda\xi^\alpha}{\alpha}\left(\kappa_1 R^{\rho_1}+\kappa_2 R^{\rho_2}\right)\right)\right] \\
&\le \frac{1}{\Gamma(\alpha-k)}\left[t^{\alpha-k-1}(1-\lambda\xi^{\alpha-1})\left(\int_0^1 (1-s)^{\alpha-1}a(s)ds\right.\right. \\
&\quad \left.\left.+\lambda\int_0^\xi (\xi-s)^{\alpha-1}a(s)ds\right) + \int_0^t (t-s)^{\alpha-k-1}a(s)ds\right] \\
&\quad +\left[\frac{t^{\alpha-k}}{\Gamma(\alpha-k+1)} + \frac{(1+\lambda\xi^\alpha)t^{\alpha-k-1}}{\alpha\Gamma(\alpha-k)(1-\lambda\xi^{\alpha-1})}\right]\left(\kappa_1 R^{\rho_1}+\kappa_2 R^{\rho_2}\right) \\
&\le \frac{1}{\Gamma(\alpha-k)(1-\lambda\xi^{\alpha-1})}\left[(1-\lambda\xi^{\alpha-1})\int_0^1 (1-s)^{\alpha-k-1}a(s)ds\right. \\
&\quad \left.+\int_0^1 (1-s)^{\alpha-1}a(s)ds + \lambda\int_0^\xi (\xi-s)^{\alpha-1}a(s)ds\right] \\
&\quad +\frac{\alpha(1-\lambda\xi^{\alpha-1})+(\alpha-k)(1+\lambda\xi^\alpha)}{\alpha\Gamma(\alpha-k+1)(1-\lambda\xi^{\alpha-1})}\left(\kappa_1 R^{\rho_1}+\kappa_2 R^{\rho_2}\right),
\end{aligned}
$$

where $k=0,1,2,\cdots,m-2$, and

$$
\begin{aligned}
&\left|D_{0+}^q(T_1y)(t)\right| \\
&= \left|D_{0+}^q I_{0+}^\alpha F(t) - \frac{D_{0+}^q t^{\alpha-1}}{(1-\lambda\xi^{\alpha-1})}\left[I_{0+}^\alpha F(1) - \lambda I_{0+}^\alpha F(\xi)\right]\right| \\
&= \left|I_{0+}^{\alpha-q}F(t) - \frac{\Gamma(\alpha)t^{\alpha-q-1}}{\Gamma(\alpha-q)(1-\lambda\xi^{\alpha-1})}\times\left[I_{0+}^\alpha F(1) - \lambda I_{0+}^\alpha F(\xi)\right]\right|
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{1}{\Gamma(\alpha-q)}\Big[t^{\alpha-q-1}(1-\lambda\xi^{\alpha-1})\Big(\int_0^1(1-s)^{\alpha-1}a(s)ds \\
&\quad +\lambda\int_0^{\xi}(\xi-s)^{\alpha-1}a(s)ds\Big)+\int_0^t(t-s)^{\alpha-q-1}a(s)ds\Big] \\
&\quad +\left[\frac{t^{\alpha-q}}{\Gamma(\alpha-q+1)}+\frac{(1+\lambda\xi^{\alpha})t^{\alpha-q-1}}{\alpha\Gamma(\alpha-q)(1-\lambda\xi^{\alpha-1})}\right](\kappa_1R^{\rho_1}+\kappa_2R^{\rho_2}) \\
&\leq \frac{1}{\Gamma(\alpha-q)(1-\lambda\xi^{\alpha-1})}\Big[(1-\lambda\xi^{\alpha-1})\int_0^1(1-s)^{\alpha-q-1}a(s)ds \\
&\quad +\int_0^1(1-s)^{\alpha-1}a(s)ds+\lambda\int_0^{\xi}(\xi-s)^{\alpha-1}a(s)ds\Big] \\
&\quad +\frac{\alpha(1-\lambda\xi^{\alpha-1})+(\alpha-q)(1+\lambda\xi^{\alpha})}{\alpha\Gamma(\alpha-q+1)(1-\lambda\xi^{\alpha-1})}(\kappa_1R^{\rho_1}+\kappa_2R^{\rho_2}).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\|(T_1y)(t)\| &= \max_{k\leq m-2}\max_{t\in[0,1]}|(T_1y)^{(k)}(t)|+\max_{t\in[0,1]}\left|D^q_{0+}(T_1y)(t)\right| \\
&\leq \mu_1+\omega_1(\kappa_1R^{\rho_1}+\kappa_2R^{\rho_2})\leq\frac{R}{3}+\frac{R}{3}+\frac{R}{3}\leq R.
\end{aligned}
$$

Similarly, we can get

$$
\|(T_2x)(t)\|\leq\mu_2+\omega_2\left(\chi_1R^{\delta_1}+\chi_2R^{\delta_2}\right)\leq\frac{R}{3}+\frac{R}{3}+\frac{R}{3}\leq R.
$$

In consequence, we obtain that $T: B\to B$.

Step 2: T is continuous. By the continuity of G_1, G_2, f and g, we conclude that the operator T is continuous.

Step 3: $T(B)$ is relatively compact. For this we denote

$$
M=\max_{\substack{t\in[0,1]\\|u|\leq R,|v|\leq R}}|f(t,u,v)|,\qquad N=\max_{\substack{t\in[0,1]\\|u|\leq R,|v|\leq R}}|g(t,u,v)|.
$$

Let $(x,y)\in B$, $t_1,t_2\in[0,1]\,(t_1<t_2)$. By Lemma 3.1 we have

$$
\begin{aligned}
&\left|(T_1y)^{(k)}(t_2)-(T_1y)^{(k)}(t_1)\right| \\
=&\left|\frac{1}{\Gamma(\alpha-k)}\Big(\int_0^{t_2}(t_2-s)^{\alpha-k-1}F(s)ds-\int_0^{t_1}(t_1-s)^{\alpha-k-1}F(s)ds\Big)\right.
\end{aligned}
$$

$$
\begin{aligned}
& +\frac{{t_2}^{\alpha-k-1}-{t_1}^{\alpha-k-1}}{\Gamma(\alpha-k)(1-\lambda\xi^{\alpha-1})}\times\bigg[\int_0^1(1-s)^{\alpha-1}F(s)ds \\
& -\lambda\int_0^{\xi}(\xi-s)^{\alpha-1}F(s)ds\bigg]\bigg| \\
= & \bigg|\frac{1}{\Gamma(\alpha-k)}\bigg(\int_0^{t_1}\Big((t_2-s)^{\alpha-k-1}-(t_1-s)^{\alpha-k-1}\Big)F(s)ds \\
& +\int_{t_1}^{t_2}(t_2-s)^{\alpha-k-1}F(s)ds\bigg)+\frac{{t_2}^{\alpha-k-1}-{t_1}^{\alpha-k-1}}{\Gamma(\alpha-k)(1-\lambda\xi^{\alpha-1})} \\
& \times\bigg[\int_0^1(1-s)^{\alpha-1}F(s)ds-\lambda\int_0^{\xi}(\xi-s)^{\alpha-1}F(s)ds\bigg]\bigg| \\
\leq & \frac{M}{\Gamma(\alpha-k)}\bigg[\int_0^{t_1}\Big((t_2-s)^{\alpha-k-1}-(t_1-s)^{\alpha-k-1}\Big)ds \\
& +\int_{t_1}^{t_2}(t_2-s)^{\alpha-k-1}ds\bigg] \\
& +\frac{M(1+\lambda\xi^{\alpha})}{\Gamma(\alpha-k)(1-\lambda\xi^{\alpha-1})}({t_2}^{\alpha-k-1}-{t_1}^{\alpha-k-1}) \\
\leq & \frac{M}{\Gamma(\alpha-k+1)}({t_2}^{\alpha-k}-{t_1}^{\alpha-k}) \\
& +\frac{M(1+\lambda\xi^{\alpha})}{\alpha\Gamma(\alpha-k)(1-\lambda\xi^{\alpha-1})}({t_2}^{\alpha-k-1}-{t_1}^{\alpha-k-1}),
\end{aligned}
$$

where $k=0,1,\cdots,m-2$, and

$$
\begin{aligned}
& \Big|D_{0+}^{q}(T_1y)(t_2)-D_{0+}^{q}(T_1y)(t_1)\Big| \\
= & \bigg|I_{0+}^{\alpha-q}F(t_2)-I_{0+}^{\alpha-q}F(t_1) \\
& -\frac{\Gamma(\alpha)(t_2^{\alpha-q-1}-t_1^{\alpha-q-1})}{\Gamma(\alpha-q)(1-\lambda\xi^{\alpha-1})}\Big[I_{0+}^{\alpha}F(1)-\lambda I_{0+}^{\alpha}F(\xi)\Big]\bigg| \\
\leq & \frac{M}{\Gamma(\alpha-q)}\bigg[\int_0^{t_1}\Big((t_2-s)^{\alpha-q-1}-(t_1-s)^{\alpha-q-1}\Big)ds \\
& +\int_{t_1}^{t_2}(t_2-s)^{\alpha-q-1}ds\bigg]
\end{aligned}
$$

$$\begin{aligned} & +\frac{M(1+\lambda\xi^{\alpha})}{\Gamma(\alpha-q)(1-\lambda\xi^{\alpha-1})}({t_2}^{\alpha-q-1}-{t_1}^{\alpha-q-1}) \\ \leq\ & \frac{M}{\Gamma(\alpha-q+1)}({t_2}^{\alpha-q}-{t_1}^{\alpha-q}) \\ & +\frac{M(1+\lambda\xi^{\alpha})}{\alpha\Gamma(\alpha-q)(1-\lambda\xi^{\alpha-1})}({t_2}^{\alpha-q-1}-{t_1}^{\alpha-q-1}). \end{aligned}$$

Similarly,

$$\begin{aligned} & \left|D_{0+}^{p}(T_2x)(t_2)-D_{0+}^{p}(T_2x)(t_1)\right| \\ \leq\ & \frac{N}{\Gamma(\beta-p+1)}({t_2}^{\beta-p}-{t_1}^{\beta-p}) \\ & +\frac{N(1+\lambda\xi^{\beta})}{\beta\Gamma(\beta-p)(1-\lambda\xi^{\beta-1})}({t_2}^{\beta-p-1}-{t_1}^{\beta-p-1}), \\ & \left|(T_2x)^{(k)}(t_2)-(T_2x)^{(k)}(t_1)\right| \\ \leq\ & \frac{N}{\beta\Gamma(\beta-k+1)}({t_2}^{\beta-k}-{t_1}^{\beta-k}) \\ & +\frac{N(1+\lambda\xi^{\beta})}{\beta\Gamma(\beta-k)(1-\lambda\xi^{\beta-1})}({t_2}^{\beta-k-1}-{t_1}^{\beta-k-1}), \end{aligned}$$

where $k=0,1,\cdots,m-2$.

Then we can obtain that $T(B)$ is an equicontinuous set, for the fact that $t^{\alpha-q},t^{\beta-p},t^{\alpha-q-1},\ t^{\beta-p-1},t^{\alpha-k},t^{\alpha-k-1},t^{\beta-k},t^{\beta-k-1}\,(k=0,1,\cdots,m-2)$ are uniformly continuous on $[0,1]$. Also it is uniformly bounded for $T(B)\subset B$. By the Arzelà-Ascoli theorem, we conclude that $T(B)$ is relatively compact. Thus, the problem (1.1) has a solution by the Schauder fixed-point theorem.

Part 2: Let (H_2) be valid. In this part, let

$$0<R\leq\min\left\{b_{21},b_{22},b_{23},b_{24}\right\},$$

where

$$b_{21}=\left(\frac{1}{2\omega_1\kappa_1}\right)^{\frac{1}{\rho_1-1}},$$

$$b_{22} = \left(\frac{1}{2\omega_1\kappa_2}\right)^{\frac{1}{\rho_2-1}},$$

$$b_{23} = \left(\frac{1}{2\omega_2\chi_1}\right)^{\frac{1}{\delta_1-1}},$$

and

$$b_{24} = \left(\frac{1}{2\omega_2\chi_2}\right)^{\frac{1}{\delta_2-1}}.$$

We can also get the result by repeating arguments similar to part 1. Here we omit it. This completes the proof . □

Theorem 3.4. *Let $f, g : [0,1]\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$. Assume that the following conditions are satisfied.*

(H_3) *There exist two constants $L_1 > 0$ and $L_2 > 0$ such that*

$$\begin{gathered}\big|f(t,x_1,y_1) - f(t,x_2,y_2)\big| \le L_1\Big(\big|x_1-x_2\big| + \big|y_1-y_2\big|\Big),\\ \big|g(t,x_1,y_1) - g(t,x_2,y_2)\big| \le L_2\Big(\big|x_1-x_2\big| + \big|y_1-y_2\big|\Big),\\ t\in[0,1],\ x_1,\ y_1,\ x_2,\ y_2 \in \mathbb{R}.\end{gathered}$$

(H_4)

$$\begin{gathered}\varepsilon_1 = \max\Big\{\omega_{11},\omega_{12}\Big\},\\ \varepsilon_2 = \max\Big\{\omega_{21},\omega_{22}\Big\},\\ \varepsilon = \max\Big\{L_1\varepsilon_1,\ L_2\varepsilon_2\Big\} < 1.\end{gathered}$$

Then Problem (1.1) *has a unique solution.*

Proof. Let (x_1, y_1), $(x_2, y_2) \in X \times Y$. By Lemma 3.1 and the discussion in theorem 3.4, we have

$$\begin{aligned}\Big|D^q_{0+}\big(T_1y_1 - T_1y_2\big)(t)\Big| &\le L_1\omega_{12}\bigg(\Big|\big(y_1-y_2\big)(t)\Big| + \Big|D^p_{0+}\big(y_1-y_2\big)(t)\Big|\bigg),\\ \Big|\big(T_1y_1 - T_1y_2\big)^{(k)}(t)\Big| &\le L_1\omega_{11}\bigg(\Big|\big(y_1-y_2\big)(t)\Big| + \Big|D^p_{0+}\big(y_1-y_2\big)(t)\Big|\bigg),\end{aligned}$$

where $k = 0, 1, \cdots, m-2$. So,

$$\Big\|T_1y_1 - T_1y_2\Big\|_X \le L_1\varepsilon_1\Big\|y_1 - y_2\Big\|_Y \le \varepsilon\Big\|y_1 - y_2\Big\|_Y.$$

Similarly,

$$\Big\|T_1x_1 - T_1x_2\Big\|_Y \le L_2\varepsilon_2\Big\|x_1 - x_2\Big\|_X \le \varepsilon\Big\|x_1 - x_2\Big\|_X.$$

Thus,

$$\begin{aligned} &\Big\|T(x_1, y_1) - T(x_2, y_2)\Big\|_{X\times Y} \\ = \ &\Big\| (T_1y_1 - T_1y_2, T_2x_1 - T_2x_2) \Big\|_{X\times Y} \\ = \ &\max\left\{\Big\|T_1y_1 - T_1y_2\Big\|_X, \ \Big\|T_2x_1 - T_2x_2\Big\|_Y\right\} \\ \le \ &\varepsilon\Big\|(x_1, y_1) - (x_2, y_2)\Big\|_{X\times Y}. \end{aligned}$$

Hence, we conclude that the Problem (1.1) has a unique solution by the contraction mapping principle. This ends the proof. □

Example 3.5. *Consider the system*

$$\begin{cases} D_{0+}^{7/2}x(t) = t^2 + \big(y(t)\big)^{1/3} + \big(D_{0+}^{7/6}y(t)\big)^{1/5}, & t \in (0,1), \\ D_{0+}^{10/3}y(t) = t^4 + \big(x(t)\big)^{1/7} + \big(D_{0+}^{4/3}x(t)\big)^{1/9}, & t \in (0,1), \\ x(0) = x'(0) = x''(0) = 0, \quad x(1) = 2x(1/3), \\ y(0) = y'(0) = y''(0) = 0, \quad y(1) = 2y(1/3). \end{cases} \tag{3.9}$$

By Theorem 3.4, the existence of the solution of the System (3.9) *is obvious.*

Example 3.6. *Consider the system*

$$\begin{cases} D_{0+}^{5/2}x(t) = L_1 \arctan y(t) + L_1 D_{0+}^{7/6}y(t), \\ D_{0+}^{10/3}y(t) = L_2\big(x(t) - \ln(1 + e^{x(t)})\big) + L_2 D_{0+}^{4/3}x(t), \\ x(0) = x'(0) = x''(0) = 0, \quad x(1) = 2x(1/3), \\ y(0) = y'(0) = y''(0) = 0, \quad y(1) = 2y(1/3), \end{cases} \tag{3.10}$$

where $t \in (0,1),\ f(t,x,y) = L_1 \arctan x + L_1 y,\ g(t,x,y) = L_2(x - \ln(1+e^x)) + L_2 y$ *and* $L_1,\ L_2 > 0$.

Note that

$$\Big|\big(\arctan x\big)'\Big| = \frac{1}{1+x^2} < 1,\ \ \Big|\big(x - \ln(1+e^x)\big)'\Big| = \frac{1}{1+e^x} < 1,$$

we have

$$\begin{aligned} & \Big|f(t,x_1,y_1) - f(t,x_2,y_2)\Big| \\ \leq\ & L_1\Big(\big|\arctan x_1 - \arctan x_2\big| + \big|y_1 - y_2\big|\Big) \\ \leq\ & L_1\Big(\big|x_1 - x_2\big| + \big|y_1 - y_2\big|\Big), \\ & \Big|g(t,x_1,y_1) - f(t,x_2,y_2)\Big| \\ \leq\ & L_2\Big(\big|(x_1 - \ln(1+e^{x_1})) - (x_2 - \ln(1+e^{x_2}))\big| + \big|y_1 - y_2\big|\Big) \\ \leq\ & L_2\Big(\big|x_1 - x_2\big| + \big|y_1 - y_2\big|\Big),\ t \in [0,1],\ x \in \mathbb{R},\ y \in \mathbb{R}. \end{aligned}$$

A simple computation shows that $\varepsilon_1 \approx 0.3111,\ \varepsilon_2 \approx 0.4426$. *Hence, if we let* $L_1\varepsilon_1,\ L_2\varepsilon_2 < 1$, *we can obtain that the System* (3.10) *has a unique solution.*

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