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GENERALIZED RINGS OF MEASURABLE AND CONTINUOUS FUNCTIONS

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Abstract. We generalize, simultaneously, the ring of real-valued continuous functions and the ring of real-valued measurable functions.

1. **Introduction**

Here, by a ring we always mean a commutative ring with identity. A ring R is called *regular* (in the sense of von Neumann) if for every $a \in R$, there is $b \in R$ such that $a = aba$. A ring R is called an $\aleph_0$-*self-injective* ring if every module homomorphism $g : I \longrightarrow R$ can be extended to a homomorphism $\hat{g} : R \longrightarrow R$, where I is any countably generated ideal of R (see [10] for more details).

Let X be a non-empty set and $\mathbb{R}^X$ be the collection of all real-valued functions on X. Then, $\mathbb{R}^X$ with the (pointwise) addition and multiplication is a reduced commutative ring. Now, suppose that $\mathcal{A}$ is a non-empty family of subsets of X, and $\mathcal{M}(X, \mathcal{A})$ is the collection of all real-valued functions f on X such that for any open subset U of $\mathbb{R}$, $\{x \in X \mid f(x) \in U\} \in \mathcal{A}$. If $\mathcal{A}$ is a σ-algebra of subsets on X, then $\mathcal{M}(X, \mathcal{A})$ is known as the set of all $\mathcal{A}$-*measurable* real-valued functions on X (see, for example, [7] and for more recent articles, [1, 11]). If f and

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g are $\mathcal{A}$-measurable functions, then so are $f+g$ and fg, i.e., $\mathcal{M}(X,\mathcal{A})$ is a subring of $\mathbb{R}^X$ whenever $\mathcal{A}$ is a σ-algebra. Now, if $\mathcal{A}$ is a topology on X, then each member of $\mathcal{M}(X,\mathcal{A})$ is called a continuous function, and $\mathcal{M}(X,\mathcal{A})$ is the ring of (real-valued) continuous functions on X and is denoted by $C(X)$, being under consideration for a long time (see [6] as a classic book in this field, [2, 5] and [9] as some instances of a much larger group of papers on this topic).

For a topological space X, by the family $\mathcal{B}$ of *Borel subsets* of X we mean the smallest σ-algebra containing all open subsets of X. Any element of the ring $\mathcal{M}(X,\mathcal{B})$ is said to be a *Borel measurable function.* The σ-algebra $\mathcal{L}$ of all Lebesgue measurable subsets of $\mathbb{R}$ is an extension of the collection of all Borel subsets of $\mathbb{R}$ (see, for example, [12]). Proceeding as above, $\mathcal{M}(\mathbb{R},\mathcal{L})$ is called the set of all Lebesgue measurable functions. We have the following strict hierarchy of subrings of $\mathbb{R}^X$ when $X=\mathbb{R}$:

$$C(X)\subset\mathcal{M}(X,\mathcal{B})\subset\mathcal{M}(X,\mathcal{L})\subset\mathbb{R}^X.$$

It is well-known that, if $\mathcal{A}$ is a σ-algebra, then $\mathcal{M}(X,\mathcal{A})$, the ring of all $\mathcal{A}$-measurable functions, is a regular ring (see, for example, [7]). In [1], Azadi et al. have also observed that, in this case, $\mathcal{M}(X,\mathcal{A})$ is an $\aleph_0$-self-injective ring and moreover, if $\mathcal{A}$ contains all singletons, then $\mathcal{M}(X,\mathcal{A})$ has an essential socle (see [11] for a necessary and sufficient condition under which socle is essential).

As already mentioned, if $\mathcal{A}$ is either a σ-algebra or a topology on X, then $\mathcal{M}(X,\mathcal{A})$ is a subring of $\mathbb{R}^X$. This brings some general questions to one's attention: which subsets or subrings of $\mathbb{R}^X$ can be written as $\mathcal{M}(X,\mathcal{A})$, for some subset $\mathcal{A}$ of $P(X)$ and on the other hand characterizing subrings (or subsets) of the form $\mathcal{M}(X,\mathcal{A})$? To be more specific, we may ask the following questions: Let $S\subset\mathbb{R}^X$. When does $S=\mathcal{M}(X,\mathcal{A})$, for some $\mathcal{A}\subseteq P(X)$? And if $\mathcal{M}(X,\mathcal{A})$ is a subring of $\mathbb{R}^X$, what can we say about $\mathcal{A}$? Here, we intend to give some light to these questions and related topics.

2. Main results

Definition 2.1. *Let X be a non-empty set and $\mathcal{A}\subseteq P(X)$. By $\mathcal{M}(X,\mathcal{A})$, we mean the set*

$$\{f:X\longrightarrow\mathbb{R}\mid f^{-1}(U)\in\mathcal{A}\text{ for every open subset }U\text{ of }\mathbb{R}\}.$$

Note that for any $\mathcal{A}\subseteq P(X)$, if $\mathcal{M}(X,\mathcal{A})\neq\emptyset$, then $\emptyset,X\in\mathcal{A}$, since if $f\in\mathcal{M}(X,\mathcal{A})$, then $\emptyset=f^{-1}(\emptyset)\in\mathcal{A}$ and $X=f^{-1}(\mathbb{R})\in\mathcal{A}$.

Moreover, if $\mathcal{B} \subseteq \mathcal{A} \subseteq P(X)$, then $\mathcal{M}(X,\mathcal{B}) \subseteq \mathcal{M}(X,\mathcal{A})$. It is also worth mentioning that $\mathbb{R}$ can be seen as the subring of $\mathbb{R}^X$ when we consider it as the ring of all constant functions. The next two results provide an answer to our first question, that is, when a subset or a subring of $\mathbb{R}^X$ is in the form of $\mathcal{M}(X,\mathcal{A})$. But, before stating them, we need some more notations. Let $S \subseteq \mathbb{R}^X$, and define $\mathrm{Coz}(S) = \{\mathrm{Coz}(f) : f \in S\}$, where $\mathrm{Coz}(f) = X \setminus f^{-1}(\{0\})$ and $A_S = \{f^{-1}(U) : \ f \in S,\ U \text{ is an open subset of } \mathbb{R}\}$. Moreover, let $\mathrm{pos}(S) = \{f^{-1}((0,+\infty)) \mid f \in S\}$ and $\mathrm{neg}(S) = \{f^{-1}((-\infty,0)) \mid f \in S\}$.

Lemma 2.2. *Suppose $S \subseteq \mathbb{R}^X$, and $\mathcal{A} \subseteq P(X)$. Then*

(a) *$S \subseteq \mathcal{M}(X,\mathcal{A}_S)$, and $S \subseteq \mathcal{M}(X,\mathcal{A})$ if and only if $\mathcal{A}_S \subseteq \mathcal{A}$,and hence $\mathcal{A}_S$ is the smallest $\mathcal{A}$ such that $S \subseteq \mathcal{M}(X,\mathcal{A})$.*

(b) *$\mathcal{A}_S = \mathrm{Coz}(S)$ if and only if for every $f \in S$ and $h \in C(\mathbb{R})$, $\mathrm{Coz}(h \circ f) \in \mathrm{Coz}(S)$ if and only if $\mathcal{M}(X,\mathcal{A}_S) = \mathcal{M}(X,\mathrm{Coz}(S))$.*

(c) *For every $f \in \mathcal{M}(X,\mathcal{A})$ and $h \in C(\mathbb{R})$, $h \circ f \in \mathcal{M}(X,\mathcal{A})$.*

Proof. The verification of part (a) is immediate. For part (b), let $\mathcal{A}_S = \mathrm{Coz}(S)$, $f \in S$ and $h \in C(\mathbb{R})$. Since $(h \circ f)^{-1}(U) \in \mathcal{A}_S$, for every open subset U of $\mathbb{R}$ (and hence in $\mathrm{Coz}(S)$) we observe that $\mathrm{Coz}(h \circ f) = (h \circ f)^{-1}(\mathbb{R} \setminus \{0\}) \in \mathrm{Coz}(S)$. The proof of part (c) is almost straightforward. □

Proposition 2.3. *Suppose $S \subseteq \mathbb{R}^X$. Then, the followings are equivalent:*

(a) *$S = \mathcal{M}(X,\mathcal{A})$, for some $\mathcal{A} \subseteq P(X)$.*

(b) *$S = \mathcal{M}(X,\mathcal{A}_S)$.*

(c) *$S = \mathcal{M}(X,\mathrm{Coz}(S))$.*

(d) *$f \in S$ if and only if for every $h \in C(\mathbb{R})$, $\mathrm{Coz}(h \circ f) \in \mathrm{Coz}(S)$.*

Proof. (a)⇒(b). By Lemma 2.2 (a), $\mathcal{A}_S \subseteq \mathcal{A}$, and hence $\mathcal{M}(X,\mathcal{A}_S) \subseteq \mathcal{M}(X,\mathcal{A}) = S$. On the other hand, again by Lemma 2.2 (a), $S \subseteq \mathcal{M}(X,\mathcal{A}_S)$. This implies that $S = \mathcal{M}(X,\mathcal{A}_S)$.

(b)⇒(c). With regards to Lemma 2.2 (b), it is enough to show that $\mathcal{A}_S = \mathrm{Coz}(S)$. Let $f^{-1}(U) \in \mathcal{A}_S$, where $f \in S$ and U is an open subset in $\mathbb{R}$. Define $d_U : \mathbb{R} \longrightarrow \mathbb{R}$ by $d_U(x) = d(x, \mathbb{R} \setminus U)$, where d is the distance function. It is clear that d_U is a continuous function. Now, we observe that $U = d_U^{-1}(\mathbb{R} \setminus \{0\})$, and hence

$$f^{-1}(U) = f^{-1}d_U^{-1}(\mathbb{R} \setminus \{0\}) = (d_U \circ f)^{-1}(\mathbb{R} \setminus \{0\}).$$

Since d_U is continuous and $f \in S$, we have $d_U \circ f \in S$ and this implies that $\mathcal{A}_S = \mathrm{Coz}(S)$.

(c)⇒(d). By Lemma 2.2 (c), the verification is immediate.

(d)⇒(a). By (d) and Lemma 2.2 (b), $\mathcal{A}_S = \mathrm{Coz}(S)$. Suppose that $f \in \mathcal{M}(X, \mathcal{A}_S)$ and $h \in C(\mathbb{R})$. Hence, $h \circ f \in \mathcal{M}(X, \mathcal{A}_S)$, and so $\mathrm{Coz}(h \circ f) \in \mathcal{A}_S = \mathrm{Coz}(S)$. Now, by (d), $f \in S$. □

Corollary 2.4. *Let $S = \mathcal{M}(X, \mathcal{A})$ be a subset of $\mathbb{R}^X$. Then*

$$\mathcal{A}_S = \mathrm{Coz}(S) = \mathrm{pos}(S) = \mathrm{neg}(S).$$

Proof. In the proof of Proposition 2.3 part (b)⇒(c), we may consider that $U = d_U^{-1}(\mathbb{R} \setminus \{0\}) = d_U^{-1}((0, +\infty)) = (-d_U)^{-1}((-\infty, 0))$. □

What comes in the sequel provides an answer to our second question, characterizing subrings (or subsets) of $\mathbb{R}^X$ in the form $\mathcal{M}(X, \mathcal{A})$. Hence here we begin with a subset of $P(X)$ and investigate the structure of $\mathcal{M}(X, \mathcal{A})$. The words trace and functional trace in the next definitions will make no confusion in the text.

Definition 2.5. *Let $f \in \mathbb{R}^X$. The trace of f is defined by*

$$\mathcal{T}_f = \{f^{-1}(\mathbb{R} \setminus \{r\}) : r \in f(X)\}.$$

It is easily seen that for $f \in \mathbb{R}^X$, f is a constant function if and only if $\mathcal{T}_f = \{\emptyset\}$.

Definition 2.6. *Suppose that $\emptyset \neq \mathcal{B} \subseteq \mathcal{A} \subseteq P(X)$ and $X \notin \mathcal{B}$. Then, $\mathcal{B}$ is said to be a functional trace in $\mathcal{A}$ whenever there exist a subset $T \subseteq \mathbb{R}$ and a bijection $\phi : T \longrightarrow \mathcal{B}$ such that*

(i) *for all $r, s \in T$ with $r \neq s$, $\phi(r) \cup \phi(s) = X$;*
(ii) $\bigcap_{r \in T} \phi(r) = \emptyset$;
(iii) *for every open set $U \subseteq \mathbb{R}$, $(\bigcap_{r \in T \cap U} \phi(r))^c \in \mathcal{A}$.*

Theorem 2.7. *Let $\mathbb{D}_{\mathcal{A}}$ be the set of all functional traces in $\mathcal{A}$. Then, the following is a surjection:*

$$\theta : \mathcal{M}(X, \mathcal{A}) \longrightarrow \mathbb{D}_{\mathcal{A}}, \qquad \theta(f) = \mathcal{T}_f.$$

Proof. First we show that every $\mathcal{T}_f$ belongs to $\mathbb{D}_{\mathcal{A}}$. Let $T = \mathrm{Im} f$ and define $\phi : T \longrightarrow \mathcal{T}_f$ by $\phi(r) = f^{-1}(\mathbb{R} \setminus \{r\})$. We need to show that each $\mathcal{T}_f$ satisfies the three conditions of Definition 2.6. The verification of the first two conditions being evident, we only verify the third one. Suppose that U is an open subset of $\mathbb{R}$. Since $f \in \mathcal{M}(X, \mathcal{A})$, we have

$$(\bigcap_{r \in T \cap U} \phi(r))^c = \bigcup_{r \in T \cap U} \phi(r)^c = \bigcup_{r \in T \cap U} f^{-1}(\{r\}) = f^{-1}(T \cap U) = f^{-1}(U),$$

which belongs to $\mathcal{A}$. Now, we show that θ is surjective. Let $\mathcal{B} \in \mathbb{D}_{\mathcal{A}}$. Then there exist a subset $T \subseteq \mathbb{R}$ and a bijection $\phi : T \longrightarrow \mathcal{B}$ satisfying the conditions of Definition 2.6. For every $x \in X$, there exists a unique $r \in T$ such that $x \notin \phi(r)$, by (i) and (ii). Now, define $f : X \longrightarrow \mathbb{R}$ by $f(x) = r$. One can verify that $f \in \mathcal{M}(X, \mathcal{A})$ and $\mathcal{B} = \mathcal{T}_f$ by (iii). □

Corollary 2.8. *For $\mathcal{A} \subseteq P(X)$, $\mathcal{M}(X, \mathcal{A}) = \emptyset$ if and only if $\mathbb{D}_{\mathcal{A}} = \emptyset$.*

The following example provides a different situation in which $\mathcal{M}(X, \mathcal{A}) = \mathbb{R}$.

Example 2.9. *(i) If $\mathcal{A} = \{X, \emptyset\}$, then $\mathcal{M}(X, \mathcal{A}) = \mathbb{R}$. Note that in this case, $\mathcal{A}$ is both a (trivial) σ-algebra and a topology.*

(ii) Let $X = \mathbb{R}$ and $\mathcal{A} = \{(a, b) \mid a, b \in \mathbb{R} \cup \{\pm\infty\}\}$. We claim that $\mathcal{M}(\mathbb{R}, \mathcal{A}) = \mathbb{R}$. Let $f \in \mathcal{M}(\mathbb{R}, \mathcal{A})$ and $f(x) \neq f(y)$, for some $x, y \in \mathbb{R}$. Suppose that $f(x) \in U$ and $f(y) \in V$, where U and V are two disjoint open intervals in $\mathbb{R}$. We have $f^{-1}(U \cup V) = f^{-1}(U) \cup f^{-1}(V)$. The right hand side is a disjoint union of two open intervals which is never an interval again. Hence, f must be a constant function, i.e., $\mathcal{M}(\mathbb{R}, \mathcal{A}) = \mathbb{R}$.

(iii) Let $|X| > 2$. Put $\mathcal{A} = \{X, \emptyset\} \cup \{\{x\} \mid x \in X\}$. Then, $\mathcal{M}(X, \mathcal{A}) = \mathbb{R}$. Here, $\mathcal{A}$ is neither a σ-algebra nor a topology. To see that $\mathcal{M}(X, \mathcal{A}) = \mathbb{R}$, it is enough to show that all members of $\mathcal{M}(X, \mathcal{A})$ are constant functions. Suppose, on the contrary, there is a non-constant function $f \in \mathcal{M}(X, \mathcal{A})$. So, there are $x, y \in X$ such that $f(x) < f(y)$. Now,

$$f^{-1}((-\infty, f(y))) \cup f^{-1}((f(x), +\infty)) = X,$$

but $x \in f^{-1}((-\infty, f(y)))$, $y \notin f^{-1}((-\infty, f(y)))$, and $f^{-1}((-\infty, f(y)))$ belongs to $\mathcal{A}$. Hence, we must have $f^{-1}((-\infty, f(y))) = \{x\}$. The same argument shows that $f^{-1}((f(x), +\infty)) = \{y\}$. Now, since $|X| > 2$, we have a contradiction. It is worth to mention that if X is infinite and $\mathcal{A} = \{$all finite subsets of $X\} \cup \{X\}$, then $\mathcal{M}(X, \mathcal{A}) = \mathbb{R}$. The proof is the same. Again, in this case, $\mathcal{A}$ is neither a topology nor a σ-algebra.

(iv) In general, let $|X| = \aleph$, where $\aleph$ is an infinite cardinal. Put

$$\mathcal{A} = \{B \subseteq X \mid B = X \text{ or } |B| < \aleph\}.$$

Then, $\mathcal{M}(X, \mathcal{A}) = \mathbb{R}$. When $\aleph > \aleph_0$, in spite of being closed under countable unions, $\mathcal{A}$ is neither a σ-algebra nor a topology.

Corollary 2.10. *For $\mathcal{A} \subseteq P(X)$, the followings are equivalent:*

(a) $\mathcal{M}(X, \mathcal{A}) \neq \emptyset$.

(b) $X, \emptyset \in \mathcal{A}$.
(c) $\mathbb{R} \subseteq \mathcal{M}(X, \mathcal{A})$.
(d) $\{\emptyset\} \in \mathbb{D}_{\mathcal{A}}$.

The next corollary gives a necessary and sufficient condition on $\mathcal{A}$ under which $\mathcal{M}(X, \mathcal{A}) = \mathbb{R}$.

Corollary 2.11. *For $\mathcal{A} \subseteq P(X)$, $\mathcal{M}(X, \mathcal{A}) = \mathbb{R}$ if and only if $\mathbb{D}_{\mathcal{A}} = \{\emptyset\}$.*

The following proposition helps us to make some more useful examples.

Theorem 2.12. *Let X be an infinite set and $\aleph$ be a cardinal number. Then, we have the following assertions.*

(a) *If $S = \{f \in \mathbb{R}^X \ : \ \exists F \subsetneq X, \ |F| \leq \aleph, \ f|_{X \setminus F}$ is constant$\}$, then $S = \mathcal{M}(X, \mathcal{A}_S)$ and $\mathcal{A}_S = \{A \subseteq X : \ |A| \leq \aleph$ or $|X \setminus A| \leq \aleph\}$.*

(b) *If $S = \{f \in \mathbb{R}^X \ : \ \exists F \subsetneq X, \ |F| < \aleph, \ f|_{X \setminus F}$ is constant$\}$, then $S = \mathcal{M}(X, \mathcal{A}_S)$ and $\mathcal{A}_S = \{A \subseteq X : \ |A| < \aleph$ or $|X \setminus A| < \aleph\}$.*

(c) *If $\aleph$ is an infinite cardinal number, then in the above two cases the set S is a subring of $\mathbb{R}^X$.*

Proof. (a): First we show that $\mathcal{A}_S = \{A \subseteq X : \ |A| \leq \aleph \text{ or } |X \setminus A| \leq \aleph\}$. Let $A \subseteq X$ and either $|A| \leq \aleph$ or $|X \setminus A| \leq \aleph$. Suppose that $f = \chi_A$, the characteristic function of A. Since $f^{-1}((\frac{1}{2}, \frac{3}{2})) = A$ and $f \in S$, we have $A \in \mathcal{A}_S$. Now, let $f \in S$ and U be an open subset of $\mathbb{R}$. There exists a set $F \subsetneq X$ such that $|F| \leq \aleph$ and $f|_{X \setminus F} = r$ (the constant). If $r \in U$, then $X \setminus F \subseteq f^{-1}(U)$. Hence, $|X \setminus f^{-1}(U)| \leq |F| \leq \aleph$. If $r \notin U$, then $f^{-1}(U) \subseteq F$. Therefore, $|f^{-1}(U)| \leq \aleph$.

Now, we show that $S = \mathcal{M}(X, \mathcal{A}_S)$. That $S \subseteq \mathcal{M}(X, \mathcal{A}_S)$ is always true. Suppose that for a moment, $\aleph$ is an infinite cardinal. Now, let $f \in \mathcal{M}(X, \mathcal{A}_S)$. We show that $f \in S$. For every rational $t \in \mathbb{Q}$, either $|f^{-1}((-\infty, t))| \leq \aleph$ or $|f^{-1}([t, +\infty))| \leq \aleph$. If for all $t \in \mathbb{Q}$, $|f^{-1}((-\infty, t))| \leq \aleph$, then $X = \bigcup_{t \in \mathbb{Q}} f^{-1}((-\infty, t))$ has cardinality less than or equal to $\aleph$. Similarly, if for every $t \in \mathbb{Q}$, $|f^{-1}([t, +\infty))| \leq \aleph$, then $|X| \leq \aleph$. And in the two cases, $f \in S$. So, let $|f^{-1}((-\infty, t_1))| > \aleph$ and $|f^{-1}([t_2, +\infty))| > \aleph$, for some $t_1, t_2 \in \mathbb{Q}$. Since $|f^{-1}((-\infty, t_2))| \leq \aleph$, $t_2 < t_1$. Put $r = \sup\{t \in \mathbb{Q} \ : \ |f^{-1}((-\infty, t))| \leq \aleph\}$. Since $f^{-1}((-\infty, r)) = \bigcup_{t \in \mathbb{Q}, \, t < r} f^{-1}(-\infty, t)$, we have $|f^{-1}((-\infty, r))| \leq \aleph$. Now, if $t > r$, then $|f^{-1}((-\infty, t))| > \aleph$ and so $|f^{-1}([t, +\infty))| \leq \aleph$. Since

$f^{-1}((r,+\infty)) = \bigcup_{t\in\mathbb{Q},\, t>r} f^{-1}([t,+\infty))$, we have $|f^{-1}((r,+\infty))| \leq \aleph$. Hence, $|f^{-1}(\mathbb{R}\setminus\{r\})| \leq \aleph$. Now, by setting $F = f^{-1}(\mathbb{R}\setminus\{r\})$, we see that $f \in S$. For finite cardinals, the above proof works, with the exception that the first two cases in the proof do not happen.

(b): If $\aleph$ is a finite or an uncountable cardinal, then the above proof works as well. For the case $\aleph = \aleph_0$, we divide the proof in the following two parts.

Part $(*)$: Let X be countable. Without loss of generality, we may suppose that $X = \mathbb{N}$, which in this case S is the subset of $\mathbb{R}^{\mathbb{N}}$ consisting of all eventually constant sequences. As above, $\mathcal{A}_S$ is equal to the set of all subsets Y of $\mathbb{N}$, where Y or $\mathbb{N}\setminus Y$ is finite. Now, we will show that $\mathcal{M}(\mathbb{N},\mathcal{A}_S) = S$. To see that $\mathcal{M}(\mathbb{N},\mathcal{A}_S) \subseteq S$, let $f \in \mathcal{M}(\mathbb{N},\mathcal{A}_S)$. We claim that $\mathrm{im}(f)$ is a finite subset of $\mathbb{R}$; otherwise, f has a strictly monotonic subsequence. Therefore, there are integers $n_1 < n_2 < \cdots$ in $\mathbb{N}$ such that $\{f(n_1), f(n_2), \ldots\}$ is a strictly monotonic sequence. Now, we may choose an open subset U of $\mathbb{R}$, which contains all $f(n_1), f(n_3), f(n_5), \ldots$ but does not contain $f(n_2), f(n_4), f(n_6), \cdots$. Now, $f^{-1}(U)$ is not in $\mathcal{A}_S$. Hence, $\mathrm{im}(f)$ must be finite, say, $\{x_1, x_2, \ldots, x_n\}$. But in this case, it is easy to see that, exactly one of the $f^{-1}(\{x_i\})$ is infinite, i.e., f is eventually constant and $f \in S$. Hence, $S = \mathcal{M}(\mathbb{N},\mathcal{A}_S)$.

Part $(*,*)$: Let X be an uncountable set. Then,

$$S = \{f : X \longrightarrow \mathbb{R} \mid f \text{ is constant except on a finite subset of } X\}.$$

We show that $S = \mathcal{M}(X,\mathcal{A}_S)$ where $\mathcal{A}_S = \{Y \subseteq X \mid Y \text{ or } X\setminus Y \text{ is finite}\}$. Let $f \in \mathcal{M}(X,\mathcal{A}_S)$. We claim that the image of f is countable. On the contrary, suppose that the image of f is uncountable. In this case, there is $t \in \mathbb{R}$ such that both $(-\infty,t)\cap \mathrm{im}(f)$ and $(t,+\infty)\cap\mathrm{im}(f)$, are infinite. Hence, $f^{-1}((-\infty,t))$ and $f^{-1}((t,+\infty))$ are not in $\mathcal{A}_S$, giving a contradiction. This implies that the image must be countable. Now, there exists $a \in \mathbb{R}$ such that $f^{-1}(\{a\})$ is uncountable. Since $f \in \mathcal{M}(X,\mathcal{A}_S)$, we must have both $f^{-1}((-\infty,a))$ and $f^{-1}((a,+\infty))$ being finite. So, f is everywhere equal to the constant a, except on a finite set, i.e., $f \in S$.

(c): Let $f, g \in S$. There are subsets F and G of X such that $|F|, |G| \leq \aleph$ and $f|_{X\setminus F}$ and $g|_{X\setminus G}$ are constant. Since $f+g$ and fg are both constant on $X\setminus(F\cup G)$ and $|F\cup G| \leq \aleph$, $f+g$ and fg belong to S. The proof of the second part is the same. □

The examples of this article show that, apparently, no rules govern on $\mathcal{A}$ for $\mathcal{M}(X,\mathcal{A})$ being a subring of $\mathbb{R}^X$. In Example 2.9, we observe that different kinds of subsets of $P(X)$ generate the same subring of $\mathbb{R}^X$, that is, $\mathbb{R}$. These raise a natural question: if S is a subring of $\mathbb{R}^X$ (containing $\mathbb{R}$), then does there exist $\mathcal{A} \subseteq P(X)$ such that $S = \mathcal{M}(X, \mathcal{A})$? The next example answers the question negatively.

Example 2.13. *Let $X = \mathbb{N}$. Then, we may consider $\mathbb{R}^{\mathbb{N}}$ as the collection of all sequences in $\mathbb{R}$. We show that $\mathbb{R}^{\mathbb{N}}$ has a subring which is not of the form $\mathcal{M}(\mathbb{N}, \mathcal{A})$. Let S be the set of all convergent sequences in $\mathbb{R}$. It is clear that S is a subring of $\mathbb{R}^{\mathbb{N}}$. On the contrary, suppose that $S = \mathcal{M}(\mathbb{N}, \mathcal{A})$, for some $\mathcal{A} \subseteq P(\mathbb{N})$. Define $f : \mathbb{N} \longrightarrow \mathbb{R}$ by $f(n) = \frac{1}{n}$, for every $n \in \mathbb{N}$. Then, $f \in S$. Let $T \subseteq \mathbb{N}$. For each $i \in T$, suppose that W_i is an open interval containing only $\frac{1}{i}$ (it does not contain $\frac{1}{j}$, for $j \neq i$). Let $V = \bigcup_{i \in T} W_i$. Hence, $f^{-1}(V) = T$. This implies that $T \in \mathcal{A}$. But this says that $\mathcal{A} = P(\mathbb{N})$, i.e., $\mathcal{M}(\mathbb{N}, \mathcal{A}) = \mathbb{R}^{\mathbb{N}}$, giving a contradiction.*

Although for a subring S of $\mathbb{R}^X$, $\mathcal{M}(X, \mathcal{A}_S)$ need not be a subring of $\mathbb{R}^X$, but we may associate a subring of the form $\mathcal{M}(X, \mathcal{A})$ to S.

Definition 2.14. *Let S be a subring of $\mathbb{R}^X$. Then, we put*

$$\overline{S} = \bigcap \{\mathcal{M}(X, \mathcal{A}) \mid \mathcal{M}(X, \mathcal{A}) \text{ is a subring of } \mathbb{R}^X \text{ containing } S\}.$$

We call $\overline{S}$, the ring closure *of S.*

It is a useful fact that for a family $\{\mathcal{A}_i\}_{i \in I}$ of subsets of $P(X)$, we always have $\mathcal{M}(X, \bigcap_{i \in I} \mathcal{A}_i) = \bigcap_{i \in I} \mathcal{M}(X, \mathcal{A}_i)$. Based on this simple fact, we see that if $\mathcal{M}(X, \mathcal{A})$ is a subring of $\mathbb{R}^X$, then there always exists $\mathcal{B} \subseteq P(X)$ which is *minimal* among those subsets $\mathcal{C}$ such that $\mathcal{M}(X, \mathcal{C}) = \mathcal{M}(X, \mathcal{A})$. To observe this, let $\Gamma = \{\mathcal{C} \subseteq P(X) \mid \mathcal{M}(X, \mathcal{C}) = \mathcal{M}(X, \mathcal{A})\}$. Then, put $\mathcal{B} = \bigcap_{\mathcal{C} \in \Gamma} \mathcal{C}$. Since $\bigcap_{\mathcal{C} \in \Gamma} \mathcal{M}(X, \mathcal{C}) = \mathcal{M}(X, \bigcap_{\mathcal{C} \in \Gamma} \mathcal{C}) = \mathcal{M}(X, \mathcal{B})$, we are done. By the aforementioned fact and Lemma 2.2-(a), $\overline{S}$ is a subring of $\mathbb{R}^X$ of the form $\mathcal{M}(X, \mathcal{A}_{\overline{S}})$, where

$$\mathcal{A}_{\overline{S}} = \bigcap \{\mathcal{A} \mid \mathcal{M}(X, \mathcal{A}) \text{ is a subring of } \mathbb{R}^X \text{ containing } S\}.$$

Since $S \subseteq \mathcal{M}(X, P(X)) = \mathbb{R}^X$, the collections in the above definition are never empty. The ring closure of subrings of $\mathbb{R}^X$ has the following properties.

Proposition 2.15. *Let S and T be subrings of $\mathbb{R}^X$. Then*

(a) $\overline{\overline{S}} = \overline{S}$.

(b) *If $S \subseteq T$, then $\overline{S} \subseteq \overline{T}$ and $\mathcal{A}_{\overline{S}} \subseteq \mathcal{A}_{\overline{T}}$.*

(c) *If* $S = \mathcal{M}(X, \mathcal{A})$*, for some* $\mathcal{A} \subseteq P(X)$*, then* $\overline{S} = S$ *and* $\mathcal{A}_{\overline{S}} \subseteq \mathcal{A}$*.*

A question arises immediately: let S be a subring of $\mathbb{R}^X$ and $\overline{S} = \mathcal{M}(X, \mathcal{A}_{\overline{S}})$, the ring closure of S. If S or $\overline{S}$ is a *regular* ring, then is $\mathcal{A}_{\overline{S}}$ necessarily a σ-algebra or even a topology?

The part $(*)$ of the proof of Theorem 2.12 and the next example show that this is not the case. In the part $(*)$, we also observe that $S = \overline{S}$ is a regular ring, but there is no topology or σ-algebra $\mathcal{A}$ such that $S = \mathcal{M}(\mathbb{N}, \mathcal{A})$; since, Otherwise, $\mathcal{A}$ must contain all singletons of $\mathbb{N}$, and hence $\mathcal{A} = P(\mathbb{N})$.

Example 2.16. Let X be a completely regular space which is not a P-space (for example, let $X = [0, 1]$) and put $S = C(X)$. Then, $S = C(X) = \mathcal{M}(X, \tau)$, where τ is the topology on the set X. By Proposition 2.15, $\overline{S} = S$. If $S = \mathcal{M}(X, \mathcal{A})$, for some σ-algebra $\mathcal{A} \subseteq P(X)$, then S has to be a regular ring. But, this is not possible, due to X not being a P-space. For details, see [6].

Remark 2.17. *In rings of continuous functions and rings of measurable functions, regularity (in the sense of von Neumann) and* $\aleph_0$*-self-injectivity always come together (see* [5] *and* [1]*, respectively). However, this is not the case in* $\mathcal{M}(X, \mathcal{A})$*, in general. In the part* $(*,*)$ *of the proof of Theorem 2.12, we observe that* $S = \mathcal{M}(X, \mathcal{A}_S)$ *is a regular ring, and* $\mathcal{A}_S$ *is neither a* σ*-algebra nor a topology. However,* S *is not an* $\aleph_0$*-self-injective ring. For this, by* [8, Theorem 2.2]*, it is enough to show that there are two orthogonal disjoint countable subsets of* S *such that they cannot be separated. Let* $Y_1 = \{x_1, x_2, \dots\} \subseteq X$ *and* $Y_2 = \{y_1, y_2, \dots\} \subseteq X$ *such that* $Y_1 \cap Y_2 = \emptyset$*. Define* $f_i = \chi_{\{x_i\}}$ *and* $g_i = \chi_{\{y_i\}}$*, for* $i = 1, 2, \dots$*. Now,* $\{f_1, f_2, \dots\}$ *and* $\{g_1, g_2, \dots\}$ *are two orthogonal subsets of* S*. But, there is no element* h *in* S*, which can separate them from each other, for if* $hf_i^2 = f_i$ *and* $hg_i = 0$*, then* $h(x_i) = 1$ *and* $h(y_i) = 0$*, for any* $i \in \mathbb{N}$*, but such an* h *does not belong to* S*. It is also worth to mention that if* $S = \mathcal{M}(X, \mathcal{B})$*, for some other subset* $\mathcal{B}$ *of* $P(X)$*, then* $\mathcal{B}$ *is never a* σ*-algebra, while otherwise* S *must be an* $\aleph_0$*-self-injective ring (see* [1]*).*

Example 2.18. Let X be an uncountable set and

$$S = \{f : X \longrightarrow \mathbb{R} \mid f \text{ is constant except on a countable subset of } X\}.$$

We have $\mathcal{A}_S = \{Y \subseteq X \mid Y \text{ or } X \setminus Y \text{ is countable}\}$. By Theorem 2.12, $S = \mathcal{M}(X, \mathcal{A}_S)$. It is notable that, here, $\mathcal{A}_S$ is a σ-algebra and contains all singletons. Hence, $\mathcal{M}(X, \mathcal{A}_S)$ is a regular, $\aleph_0$-self-injective ring with an essential socle.

Proposition 2.19. *Let $\mathcal{A} \subseteq P(X)$. If $f \in \mathcal{M}(X, \mathcal{A})$, then $|f|$ and f^n are in $\mathcal{M}(X, \mathcal{A})$, for every $n \in \mathbb{N}$. Moreover, for every positive real number r, $f^r \in \mathcal{M}(X, \mathcal{A})$, provided that $f(X) \subseteq [0, +\infty)$.*

Proof. The verification is immediate if we remind that for every $h \in C(\mathbb{R})$ and $f \in \mathcal{M}(X, \mathcal{A})$, we have $h \circ f \in \mathcal{M}(X, \mathcal{A})$. □

Corollary 2.20. *Let $\mathcal{A} \subseteq P(X)$ such that $\mathcal{M}(X, \mathcal{A}) \neq \emptyset$. If $\mathcal{M}(X, \mathcal{A})$ is closed under addition, then it is a subring of $\mathbb{R}^X$.*

Proof. Note that $fg = \frac{1}{2}((f+g)^2 - f^2 - g^2)$. □

Since topologies and σ-algebras are closed under finite unions and finite intersections, the first question which comes to mind is that if $\mathcal{A}$ is closed under finite unions and finite intersections, then is $\mathcal{M}(X, \mathcal{A})$ a ring? Or even if $\mathcal{A}$ is a *ring* of subsets of X, that is, when $\mathcal{A}$ is closed under finite unions and complements (and hence closed under finite intersections), then is $\mathcal{M}(X, \mathcal{A})$ necessarily a subring of $\mathbb{R}^X$? Although under this condition, the constant functions $0, 1 \in \mathcal{M}(X, \mathcal{A})$, we do not know whether $\mathcal{M}(X, \mathcal{A})$ is a ring. In general, if $S = \mathcal{M}(X, \mathcal{A})$ is a ring, then $\mathcal{A}$ itself is not necessarily closed under finite unions and finite intersections (see Example 2.9). However, in Theorem 2.21, we see a positive statement about $\mathcal{A}_S$.

Theorem 2.21. *Let $S = \mathcal{M}(X, \mathcal{A})$ be a subring of $\mathbb{R}^X$. Then*

(a) *$\mathcal{A}_S$ is closed under finite unions and finite intersections.*

(b) *S is a regular ring if and only if $\mathcal{A}_S$ is closed under complements.*

Proof. (a) According to Proposition 2.19, if $f \in S$, then $|f| \in S$. Hence, if $f, g \in S$, then $\max(f, g)$ and $\min(f, g)$ are in S, for

$$\max(f,g) = \frac{1}{2}(f + g + |f-g|) \text{ and } \min(f,g) = \frac{1}{2}(f + g - |f-g|).$$

Now, let $U, V \in \mathcal{A}_S$. Then, by Corollary 2.4, $U = f^{-1}((0, +\infty))$ and $V = g^{-1}((0, +\infty))$, for some $f, g \in S$. Hence, $U \cap V = h^{-1}((0, +\infty))$ and $U \cup V = k^{-1}((0, +\infty))$, where $h = \min(f, g)$ and $k = \max(f, g)$, respectively. Therefore, both $U \cap V$ and $U \cup V$ belong to $\mathcal{A}_S$.

(b) First suppose that S is a regular ring and $U \in \mathcal{A}_S$. Then, $U = f^{-1}((0, \infty))$, for some $f \in S$. Since $X \setminus U = f^{-1}((-\infty, 0]) = f^{-1}((-\infty, 0)) \cup f^{-1}(\{0\})$, by part (a) it suffices to show that $f^{-1}(\{0\}) \in \mathcal{A}_S$. There exists $g \in S$ with $f = fgf$. Let $e = fg$. Then, $e^2 = e \in S$

and $f^{-1}(\{0\}) = e^{-1}(\{0\}) = e^{-1}((-1,1)) \in \mathcal{A}_S$. Therefore, $\mathcal{A}_S$ is closed under complements.

Conversely, suppose that $\mathcal{A}_S$ is closed under complements and $f \in S$. Let $U = f^{-1}(\{0\})$. Then, $U = X \setminus f^{-1}(\mathbb{R} \setminus \{0\}) \in \mathcal{A}_S$. Define $g : X \longrightarrow \mathbb{R}$ by $g(x) = 1/f(x)$, if $x \notin U$, and $g(x) = 0$, if $x \in U$. Then, it is easy to show that $g \in S = \mathcal{M}(X, \mathcal{A}_S)$ and $f = fgf$. Hence, S is a regular ring. □

A pm-ring is a commutative ring in which every prime ideal is contained in a unique maximal ideal. In the literature, pm-rings are also called the Gelfand rings. This was first introduced by G. Demarco and A. Orsatti in [4]. Examples of pm-rings are rings of continuous functions, regular rings, local rings, zero-dimensional rings, etc. Also, Contessa [3] showed that a commutative ring is a pm-ring if and only if for every $m \in R$, there exist $a, b \in R$ such that $(1 - am)(1 - bm') = 0$, where $m' = 1 - m$. Now, we show that if $\mathcal{M}(X, \mathcal{A})$ is a subring of $\mathbb{R}^X$, then it is always a pm-ring.

Theorem 2.22. *If $\mathcal{M}(X, \mathcal{A})$ is a ring, then it is a reduced pm-ring.*

Proof. Let $f \in \mathcal{M}(X, \mathcal{A})$. We must find $g, h \in \mathcal{M}(X, \mathcal{A})$ such that $(1 - gf)(1 - hf') = 0$, where $f' = 1 - f$. Let's define $\phi : \mathbb{R} \longrightarrow \mathbb{R}$ as follows: $\phi(x) = 1/x$, if $|x| > 1/3$, and $\phi(x) = 9x$, if $|x| \leq 1/3$. Now, put $g := \phi \circ f$ and $h := \phi \circ f'$. It is easy to verify that $(1 - gf)(1 - hf') = 0$. That $\mathcal{M}(X, \mathcal{A})$ is a reduced ring comes from this fact that $\mathbb{R}^X$ is a reduced ring. Hence, the proof is complete. □

3. Some descriptive examples for $X = \mathbb{R}$

Here, in addition to the previous examples, we present more examples to show that the notion $\mathcal{M}(X, \mathcal{A})$ is not a trivial continuation of rings of continuous or measurable functions. For a measurable subset U of $\mathbb{R}$, let $\mathrm{m}(U)$ be its Lebesgue measure. We refer the reader to the standard text books in measure theory for the definition of the Lebesgue measure and measurable functions.

Example 3.1. *Let $X = \mathbb{R}$ and*

$$\mathcal{A} = \{U \subseteq \mathbb{R} \mid \mathrm{m}(U) = 0\} \cup \{\mathbb{R}\}.$$

Then, $\mathcal{M}(\mathbb{R}, \mathcal{A}) = \mathbb{R}$. Suppose $f \in \mathcal{M}(\mathbb{R}, \mathcal{A})$. We can write $\mathbb{R} = \bigcup_{i=1}^{\infty}(a_i, b_i)$, where $\mathrm{m}((a_i, b_i)) = b_i - a_i = 1$. Since $\mathbb{R} = \bigcup_{i=1}^{\infty} f^{-1}((a_i, b_i))$ and a countable union of zero measure subsets of $\mathbb{R}$ has zero measure,

$f^{-1}((a_i, b_i)) = \mathbb{R}$, for some $i \in \mathbb{N}$. Then, again, we can write the interval $I_1 = (a_i, b_i)$ as a finite union of open intervals with length $\frac{1}{2}$. There must be an interval $I_2 = (c, d) \subset I_1$ such that $\mathrm{m}((c, d)) = d - c = \frac{1}{2}$ and $f^{-1}((c, d)) = \mathbb{R}$. Continuing this process, we get a decreasing sequence $I_1 \supset I_2 \supset \cdots$ of open intervals such that $\mathrm{m}(I_n) = \frac{1}{n}$ and $f^{-1}(I_n) = \mathbb{R}$, for each $n \in \mathbb{N}$. Thus, $f^{-1}(\bigcap_{n=1}^{\infty} I_n) = \bigcap_{n=1}^{\infty} f^{-1}(I_n) = \mathbb{R}$, and hence $\bigcap_{n=1}^{\infty} I_n$ should be a singleton $\{a\}$. Therefore, $f(x) = a$, for all $x \in \mathbb{R}$.

Example 3.2. *Let $X = \mathbb{R}$ and*

$$R = \{f \in \mathbb{R}^{\mathbb{R}} \mid f \text{ is almost everywhere constant}\},$$

where by an almost everywhere constant function, we mean a function which is everywhere constant except on a set with zero measure. We claim that $R = \mathcal{M}(\mathbb{R}, \mathcal{A})$, where

$$\mathcal{A} = \{Y \subseteq \mathbb{R} \mid Y \text{ or } \mathbb{R} \setminus Y \text{ has zero measure}\}.$$

Suppose that $f \in R$ and $f(x) = c$ everywhere except on a set with zero measure. Let U be an open subset in $\mathbb{R}$. Either $c \in U$ or $c \notin U$, which implies that $\mathbb{R} \setminus f^{-1}(U)$ or $f^{-1}(U)$ has zero measure. This shows that $R \subseteq \mathcal{M}(\mathbb{R}, \mathcal{A})$. Now, let $f \in \mathcal{M}(\mathbb{R}, \mathcal{A})$. Then, as in Example 3.1, we can find a decreasing sequence $I_1 \supset I_2 \supset \cdots$ of open intervals such that $\mathrm{m}(I_n) = \frac{1}{n}$ and $\mathrm{m}(\mathbb{R} \setminus f^{-1}(I_n)) = 0$, for each $n \in \mathbb{N}$. Since

$$\mathbb{R} \setminus f^{-1}(\bigcap_{n=1}^{\infty} I_n) = \mathbb{R} \setminus \bigcap_{n=1}^{\infty} f^{-1}(I_n) = \bigcup_{n=1}^{\infty} (\mathbb{R} \setminus f^{-1}(I_n))$$

has measure zero, $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$ and so should be a singleton $\{a\}$. Therefore, f is almost everywhere constant a, i.e., $f \in R$. Note that $\mathcal{A}$ is a σ-algebra which is not a topology. Therefore, $\mathcal{M}(\mathbb{R}, \mathcal{A})$ is an $\aleph_0$-self-injective regular ring with essential socle.

Example 3.3. *Let $X = \mathbb{R}$ and R be the set of all measurable functions $f : \mathbb{R} \longrightarrow \mathbb{R}$ such that f is constant everywhere except on a set with finite measure. We show that $R = \mathcal{M}(\mathbb{R}, \mathcal{A})$, where*

$$\mathcal{A} = \{Y \subseteq \mathbb{R} \mid Y \text{ or } \mathbb{R} \setminus Y \text{ has finite measure}\}.$$

It is not difficult to observe that $R \subseteq \mathcal{M}(\mathbb{R}, \mathcal{A})$. Now, let $f \in \mathcal{M}(\mathbb{R}, \mathcal{A})$. It is clear that f is a measurable function. For every $x \in \mathbb{R}$, either $f^{-1}((-\infty, x))$ or $f^{-1}((x, +\infty))$ has finite measure. Without loss of generality, we may suppose that $f^{-1}((-\infty, x))$ has finite measure for some

$x \in \mathbb{R}$. *We put*

$$\alpha = \sup\{x \mid f^{-1}((-\infty, x)) \text{ has finite measure}\}.$$

We claim that $f^{-1}((-\infty, \alpha))$ *has finite measure. Otherwise, there is* $x_1 < \alpha$ *such that* $\mathrm{m}(f^{-1}((-\infty, x_1))) > 1$ *and then we can chose* x_2 *such that* $x_1 < x_2 < \alpha$ *and* $\mathrm{m}(f^{-1}((-\infty, x_2))) > 1 + \mathrm{m}(f^{-1}((-\infty, x_1]))$. *Now, by induction we will have an increasing sequence* $x_1 < x_2 < x_3 < \cdots < \alpha$ *such that*

$$\mathrm{m}(f^{-1}((-\infty, x_{n+1}))) > 1 + \mathrm{m}(f^{-1}((-\infty, x_n])).$$

In particular, $\mathrm{m}(f^{-1}((x_n, x_{n+1}))) > 1$. *Now, consider the open subsets*

$$W_1 = \bigcup_{n=1}^{\infty} (x_{2n-1}, x_{2n}) \text{ and } W_2 = \bigcup_{n=1}^{\infty} (x_{2n}, x_{2n+1}).$$

Then, $W_1 \cap W_2 = \emptyset$ *and the inverse images of both of them have infinite measure, which is a contradiction. Therefore,* $f^{-1}((-\infty, \alpha))$ *has finite measure. In particular,* $\alpha \neq +\infty$. *By the definition of* α, *for each* $x > \alpha$, $f^{-1}((-\infty, x))$ *has infinite measure, and hence* $f^{-1}((x, +\infty))$ *has finite measure. Now, the same line of proof shows that* $f^{-1}((\alpha, +\infty))$ *has finite measure as well. So,* $f(x) = \alpha$, *except on a set of finite measure.*

Notice that $\mathcal{A}$ *is neither a* σ*-algebra nor a topology; however,* $\mathcal{M}(\mathbb{R}, \mathcal{A})$ *is a regular ring which is not* $\aleph_0$*-self-injective. For this, it is enough to show that there are two orthogonal disjoint countable subsets of* R *such that they cannot be separated. Define* $f_i = \chi_{[i-1,i)}$ *and* $g_i = \chi_{[-i,-i+1)}$, *for* $i = 1, 2, \ldots$. *Now,* $\{f_1, f_2, \ldots\}$ *and* $\{g_1, g_2, \ldots\}$ *are two orthogonal subsets of* R. *But, there is no element* h *in* R, *which can separate them from each other, for if* $hf_i^2 = f_i$ *and* $hg_i = 0$, *then* $h(x) = 1$, *for* $x \geq 0$, *and* $h(x) = 0$, *for* $x < 0$, *but such an* h *does not belong to* R.

Example 3.4. *Let* R *be the set of all functions* $f : \mathbb{R} \longrightarrow \mathbb{R}$ *such that for some* $c \in \mathbb{R}$, $f^{-1}(\{c\})$ *contains an open dense subset of* $\mathbb{R}$. *It is easy to see that* R *is a subring of* $\mathbb{R}^{\mathbb{R}}$. *We show that* $R = \mathcal{M}(\mathbb{R}, \mathcal{B})$, *where*

$$\mathcal{B} = \{Y \subseteq \mathbb{R} \mid Y \text{ or } \mathbb{R} \setminus Y \text{ contains an open dense subset of } \mathbb{R}\}.$$

The implication $R \subseteq \mathcal{M}(\mathbb{R}, \mathcal{B})$ *is straightforward. Now, let* $f \in \mathcal{M}(\mathbb{R}, \mathcal{B})$. *For every* $x \in \mathbb{R}$, *either* $f^{-1}((-\infty, x))$ *or* $f^{-1}((x, +\infty))$ *is contained in a closed nowhere dense subset of* $\mathbb{R}$. *Without loss of generality, we may suppose that, for some* $x \in \mathbb{R}$, $f^{-1}((-\infty, x))$ *is contained in a closed nowhere dense subset of* $\mathbb{R}$. *Let* α *be the supremum of the following set*

$$\{x \mid f^{-1}((-\infty, x)) \text{ is contained in a closed nowhere dense subset of } \mathbb{R}\}.$$

We have $f^{-1}((-\infty,\alpha)) = \bigcup_{n=1}^{\infty} f^{-1}((-\infty,x_n))$*, where* $\{x_n\}$ *is an increasing sequence converging to* α*. Thus,* $f^{-1}((-\infty,\alpha))$ *is a nowhere dense subset of* $\mathbb{R}$ *and hence* $f^{-1}((-\infty,\alpha))$ *is contained in a closed nowhere dense subset of* $\mathbb{R}$*. In particular,* $\alpha \neq +\infty$*. By the definition of* α*, for each* $x > \alpha$*,* $f^{-1}((-\infty,x))$ *contains an open dense subset of* $\mathbb{R}$*. Hence,* $f^{-1}((x,+\infty))$ *is contained in a closed nowhere dense subset of* $\mathbb{R}$*. Now, the same proof shows that* $f^{-1}((\alpha,+\infty))$ *is contained in a closed nowhere dense subset of* $\mathbb{R}$ *as well. So,* $f^{-1}(\{\alpha\})$ *contains an open dense subset of* $\mathbb{R}$*, and hence* $f \in R$*. Note that* $\mathcal{B}$ *is neither a* σ*-algebra nor a topology, but* $\mathcal{M}(\mathbb{R},\mathcal{B})$ *is a regular ring.*

Observe that if $\mathcal{A} = \{Y \subseteq \mathbb{R} \mid Y \text{ or } \mathbb{R}\backslash Y \text{ is an open dense subset of } \mathbb{R}\}$*, then* $\mathcal{M}(\mathbb{R},\mathcal{A})$ *is a subset of* $\mathbb{R}^X$ *which is not a ring. Let* $K \subseteq \mathbb{R}$ *be the Cantor set and* $f = \chi_K$ *and* $g = \chi_{\mathbb{R}\setminus\{0\}}$*. Since* $K, \mathbb{R}\setminus\{0\} \in \mathcal{A}$*, we have* $f, g \in \mathcal{M}(\mathbb{R},\mathcal{A})$*. But,* $fg = \chi_{K\setminus\{0\}} \notin \mathcal{M}(\mathbb{R},\mathcal{A})$*, and hence* $\mathcal{M}(\mathbb{R},\mathcal{A})$ *is not a subring of* $\mathbb{R}^X$*.*

4. **A remark**

If in the definition of $\mathcal{M}(X,\mathcal{A})$, we replace open subsets of $\mathbb{R}$ by open intervals, then we will have the following subset of $\mathbb{R}^X$:

$$\mathcal{M}'(X,\mathcal{A}) = \{f : X \to \mathbb{R} \mid f^{-1}(U) \in \mathcal{A} \text{ for every open interval } U \text{ in } \mathbb{R}\}.$$

Perhaps the reader asks himself/herself: What is the relation between $\mathcal{M}(X,\mathcal{A})$ and $\mathcal{M}'(X,\mathcal{A})$? In spite of the fact that $\mathcal{M}(X,\mathcal{A}) \subseteq \mathcal{M}'(X,\mathcal{A})$, as the following example shows, they behave differently. But, when $\mathcal{A}$ is a σ-algebra or a topology, then they are equal. Recall that in Theorem 2.12-($*$), if $\mathcal{A}_F = \{Y \subseteq \mathbb{N} \mid Y \text{ or } \mathbb{N}\setminus Y \text{ is finite}\}$, then $\mathcal{M}(\mathbb{N},\mathcal{A}_F)$ is equal to the ring of all eventually constant sequences, while this is not the case for $\mathcal{M}'(\mathbb{N},\mathcal{A}_F)$.

Example 4.1. Let $X = \mathbb{N}$. We show that $\mathbb{R}^{\mathbb{N}}$ has a subring which is not of the form $\mathcal{M}'(\mathbb{N},\mathcal{A})$. Let R be the set of all eventually constant sequences. It is clear that R is a subring of $\mathbb{R}^{\mathbb{N}}$. Suppose that $R \subseteq \mathcal{M}'(\mathbb{N},\mathcal{A})$, for some $\mathcal{A} \subseteq P(\mathbb{N})$. As in Theorem 2.12($*$), we see that the minimal (possible) choice for a subset $\mathcal{A}$ for which $R \subseteq \mathcal{M}'(\mathbb{N},\mathcal{A})$ is $\mathcal{A}_F = \{Y \subseteq \mathbb{N} \mid Y \text{ or } \mathbb{N}\setminus Y \text{ is finite}\}$. But, we observe that $\mathcal{M}'(\mathbb{N},\mathcal{A}_F)$ contains all convergent sequences. Let $f = (a_n)$ be a convergent sequence and $\lim a_n = a$. Now, let U be an open interval in $\mathbb{R}$. Then, either $a \in U$ or $a \notin U$. Hence, $\mathbb{N}\setminus f^{-1}(U)$ or $f^{-1}(U)$ is finite. Therefore, $f \in \mathcal{M}'(\mathbb{N},\mathcal{A}_F)$. This implies that R is a proper subset of $\mathcal{M}'(\mathbb{N},\mathcal{A}_F)$. It is also worth to mention that $\mathcal{M}'(\mathbb{N},\mathcal{A}_F)$ itself is not a ring. For, if

$b_n = n$ and $c_n = \frac{(-1)^n}{n}$, then $g = (b_n)$ and $h = (c_n)$ are in $\mathcal{M}'(\mathbb{N}, \mathcal{A}_F)$, but $gh = (b_n c_n) = ((-1)^n)$ does not belong to $\mathcal{M}'(\mathbb{N}, \mathcal{A}_F)$. Therefore, we have the following hierarchy:

$$R = \mathcal{M}(\mathbb{N}, \mathcal{A}_F) \subset \text{ the set of all convergent sequences } \subset \mathcal{M}'(\mathbb{N}, \mathcal{A}_F).$$

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References

[1] H. Azadi, M. Henriksen and E. Momtahan, Some properties of algebras of real-valued measurable functions, *Acta Math. Hungar.* **124** (2009), no. 1-2, 15–23.

[2] F. Azarpanah and O. A. S. Karamzadeh, Algebraic characterization of some disconnected spaces, *Ital. J. Pure Appl. Math.* **12** (2002) 158–168.

[3] M. Contessa, On pm rings, *Comm. Algebra* **10** (1982), no. 1, 93–108.

[4] G. De Marco and A. Orsatti, Commutative rings in which every prime ideal is contained in a unique maximal ideal, *Proc. Amer. Math. Soc.* **30** (1971) 459–466.

[5] A. Estaji and O. A. S. Karamzadeh, On $C(X)$ modulo its socle, *Comm. Algebra* **31** (2003), no. 4, 1561–1571.

[6] L. Gillman and M. Jerison, Rings of Continuous Functions, Springer-Verlag, New York-Heidelberg, 1976.

[7] A. W. Hager, Algebras of measurable functions, *Duke Math. J.* **38** (1971) 21–27.

[8] O. A. S. Karamzadeh, On a question of Matlis, *Comm. Algebra* **25** (1997), no. 9, 2717–2726.

[9] O. A. S. Karamzadeh and M. Rostami, On the intrinsic topology and some related ideals of $C(X)$, *Proc. Amer. Math. Soc.* **93** (1985), no. 1, 179–184.

[10] E. Momtahan, On $\aleph_0$-injectivity, *Comm. Algebra* **32** (2004), no. 10, 3883–3896.

[11] E. Momtahan, Essential ideals in rings of measurable functions, *Comm. Algebra* **38** (2010), no. 12, 4739–4746.

[12] R. M. dudley, Real Analysis and Probability, Cambridge Studies in Applied Mathematics, 74, Cambridge University Press, Cambridge, 2002.

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