

IMPROVED INFEASIBLE-INTERIOR-POINT ALGORITHM FOR LINEAR COMPLEMENTARITY PROBLEMS

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ABSTRACT. We present a modified version of the infeasible-interior-point algorithm for monotone linear complementary problems introduced by Mansouri et al. (Nonlinear Anal. Real World Appl. 12(2011) 545–561). Each main step of the algorithm consists of a feasibility step and several centering steps. We use a different feasibility step, which targets at the μ^+ -center. It results a better iteration bound.

1. Introduction

For a comprehensive learning about interior-point methods (IPMs), we refer to Roos et al. [5] and Wright [7]. In [4], a full-Newton step infeasible interior-point method (IIPM) for linear optimization (LO) was presented and later this algorithm extended to linear complementarity problems (LCP) by Mansouri et al. [1]. In this paper we present a slightly different algorithm which uses a more natural feasibility step, which targets at the μ^+ -center.

This paper is organized as follows. First, we review some results which

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are due to [1], and then, apply them to analyze the feasibility and the centering steps of our algorithm. Then we present our algorithm. Each main step of the algorithm consists of a feasibility step and several centering steps. Recall that in [1] the feasibility step targets at the μ -center of the next pair of perturbed problems. Since the aim of each main iteration is to get a good approximation of the μ^+ -center of the next pair of perturbed problems, we take a more natural approach to let the feasibility step target at the μ^+ -center of the next pair of perturbed problems. Finally, we give some concluding remarks.

Notations

The notations used throughout the paper is rather standard: capital letters denote matrices, lower case letters denote vectors, script capital letters denote sets, and Greek letters denote scalars. All vectors are considered to be column vectors. The components of a vector $u \in \mathbf{R}^n$ will be denoted by u_i , $i = 1, \dots, n$. The relation $u > 0$ is equivalent to $u_i > 0$, $i = 1, \dots, n$, while $u \geq 0$ means $u_i \geq 0$, $i = 1, \dots, n$. We denote $\mathbf{R}_+^n = \{u \in \mathbf{R}^n : u \geq 0\}$, $\mathbf{R}_{++}^n = \{u \in \mathbf{R}^n : u > 0\}$. For any vector $x \in \mathbf{R}^n$, $x_{\min} = \min(x_1; x_2; \dots; x_n)$ and $x_{\max} = \max(x_1; x_2; \dots; x_n)$. If $u \in \mathbf{R}^n$ then $U := \text{diag}(u)$ denotes the diagonal matrix having the components of u as diagonal entries. If $x, s \in \mathbf{R}^n$, then xs denotes the componentwise (Hadamard) product of the vectors x and s . Furthermore, e denotes the all-one vector of length n . The 2-norm and the infinity norm for vectors are denoted by $\|\cdot\|$ and $\|\cdot\|_\infty$, respectively. The Frobenius matrix norm is given by

$$\|U\|^2 := \sum_{i=1}^m \sum_{j=1}^n U_{ij}^2 = \text{Tr}(U^T U).$$

2. Preliminaries

The monotone linear complementarity problem (LCP) is to find vector pair $(x, s) \in \mathbf{R}^{2n}$ that satisfies the following conditions

$$s = Mx + q, \quad (x, s) \geq 0, \quad x^T s = 0, \quad (P)$$

where $q \in \mathbf{R}^n$ and M is an $n \times n$ matrix supposed positive semidefinite. We denote the feasible set of the problem (P) by

$$\mathcal{F} := \{(x, s) \in \mathbf{R}_+^{2n} : s = Mx + q\}$$

and its solution set by

$$\mathcal{F}^* := \left\{ (x^*, s^*) \in \mathcal{F} : (x^*)^T s^* = 0 \right\}.$$

Throughout this paper it will be assumed that $\mathcal{F}^*$ is not empty, i.e., (P) has at least one solution. As usual for infeasible interior-point methods (IIPMs), we use the starting point as in [1] that one knows some positive scalars ρ_p and ρ_d such that

$$(2.1) \quad \|x^*\|_\infty \leq \rho_p, \quad \max \{ \|s^*\|_\infty, \rho_p \|Me\|_\infty, \|q\|_\infty \} \leq \rho_d,$$

for some $(x^*, s^*) \in \mathcal{F}^*$, and the initial iterates are

$$(2.2) \quad x^0 = \rho_p e, \quad s^0 = \rho_d e, \quad \mu^0 = \rho_p \rho_d.$$

Using $(x^0)^T s^0 = n\rho_p\rho_d$, the total number of iterations in the algorithm of [1] is bounded above by

$$(2.3) \quad 72n \log \frac{\max \{ n\rho_p\rho_d, \|r^0\| \}}{\varepsilon},$$

where r^0 is the initial value of the residual:

$$(2.4) \quad r^0 = s^0 - Mx^0 - q.$$

Up to a constant factor, the iteration bound (2.3) was first obtained by Potra [2], and it is still the best-known iteration bound for IIPMs.

To describe the aim of this article, we need to recall the main ideas underlying the algorithm in [1]. For any ν with $0 < \nu \leq 1$, we consider the perturbed problem (P_ν) , defined by

$$s - Mx - q = \nu r^0, \quad (x, s) \geq 0. \quad (P_\nu)$$

Note that if $\nu = 1$ then $(x, s) = (x^0, s^0)$ yields a strictly feasible solution of (P_ν) . Owing to the choice of initial iterates, we may conclude that if $\nu = 1$ then (P_ν) has a strictly feasible solution, which means that the perturbed problem then satisfies the well-known interior-point condition (IPC). More generally one has the following lemma.

Lemma 2.1. *If the original problem (P) is feasible then the perturbed problem (P_ν) satisfies the IPC.*

We assume that (P) is feasible, it follows from Lemma 2.1 that the problem (P_ν) satisfies the IPC for each $\nu \in (0, 1]$. But then its central

path exists. This means that the following system has a unique solution for every $\mu > 0$

$$(2.5) \quad \begin{aligned} s - Mx - q &= \nu r^0, & x \geq 0, & s \geq 0, \\ xs &= \mu e. \end{aligned}$$

If $\nu \in (0, 1]$ and $\mu = \nu \rho_p \rho_d$, we denote this unique solution in the sequel as $(x(\nu), s(\nu))$. Using this notation, we have, by taking $\nu = 1$, $(x(1), s(1)) = (x^0, s^0) = (\rho_p e, \rho_d e)$.

We measure proximity of iterates (x, s) to the μ -center of the perturbed problem (P_ν) by quantity $\delta(x, s; \mu)$, which is defined as follows:

$$(2.6) \quad \delta(x, s; \mu) = \frac{1}{\sqrt{2}} \|v - v^{-1}\|, \quad \text{where } v := \sqrt{\frac{xs}{\mu}}.$$

Initially, we have $x = \rho_p e$ and $s = \rho_d e$ and $\mu^0 = \rho_p \rho_d$, where $v = e$ and $\delta(x, s; \mu) = 0$. In the sequel, we assume that at the start of each iteration, $\delta(x, s; \mu)$ is smaller than or equal to a small threshold value $\tau > 0$. So certainly this is true at the start of the first iteration.

3. An iteration of the algorithm

In this section we describe one iteration of our algorithm. As we established above, if $\nu = 1$ and $\mu = \mu^0$, then $(x, s) = (x^0, s^0)$ is the μ -center of the perturbed problem (P_ν) . This is our initial iterate.

We measure proximity to the μ -center of the perturbed problem by the quantity $\delta(x, s; \mu)$ as defined in (2.6). Initially we thus have

$$\delta(x, s; \mu) = 0.$$

In what follows we assume that at the start of each iteration, just before the feasibility step, $\delta(x, s; \mu)$ is smaller than or equal to a small threshold value $\tau > 0$. So this is certainly true at the start of the first iteration.

Suppose that for some $\nu \in (0, 1]$, we have (x, s) satisfying the feasibility condition (2.5) and for $\nu = \frac{\mu}{\mu^0}$, and $x^T s \leq (n + \delta^2) \mu$ and $\delta(x, s; \mu) \leq \tau$. We reduce μ to $\mu^+ = (1 - \theta) \mu$, with $\theta \in (0, 1)$, and find new iterates (x^+, s^+) that satisfy (2.5), with μ replaced by μ^+ and ν by $\nu^+ = \frac{\mu^+}{\mu^0}$, and such that $x^T s \leq (n + \delta^2) \mu^+$ and $\delta(x^+, s^+; \mu^+) \leq \tau$. Note that $\nu^+ = (1 - \theta) \nu$.

To be more precise, this is achieved as follows. Each main iteration consists of feasibility step and a few centering steps. The feasibility step

serves to get iterates (x^f, s^f) that are strictly feasible for (P_{ν^+}) , and close to its μ -center $(x(\mu^+, \nu^+), s(\mu^+, \nu^+))$.

In fact, the feasibility step is designed in a such a way that

$$\delta(x^f, s^f; \mu^+) \leq \frac{1}{\sqrt{2}}.$$

Since (x^f, s^f) is strictly feasible for (P_{ν^+}) , we can easily get iterates (x^+, s^+) that are strictly feasible for (P_{ν^+}) , and such that

$$\delta(x^+, s^+; \mu^+) \leq \tau,$$

just by performing a few centering steps starting at (x^f, s^f) and targeting at the μ^+ -center of (P_{ν^+}) .

Before describing the search directions used in the feasibility step and the centering step, we give a more formal description of the algorithm in figure 1.

Infeasible full-Newton-step algorithm

Input:

Accuracy parameter $\varepsilon > 0$;
 barrier update parameter θ , $0 < \theta < 1$;
 threshold parameter $\tau > 0$;
 $x^0, s^0 > 0$ and $\mu^0 > 0$ such that $x^0 s^0 = \mu^0 e$.

begin

$x := x^0 > 0, s := s^0 > 0; \mu := \mu^0$;

while $\max(n\mu, \|r\|) \geq \varepsilon$ **do**

begin

feasibility step:

$$(x, s) := (x, s) + (\Delta^f x, \Delta^f s);$$

μ -update:

$$\mu := (1 - \theta)\mu;$$

centering steps:

while $\delta(x, s; \mu) \geq \tau$ **do**

begin

$$(x, s) := (x, s) + (\Delta x, \Delta s);$$

end

end

end

FIGURE 1. Infeasible full-Newton-step algorithm

According to the definition of (P_ν) the feasibility equation for (P_ν) is given by

$$s - Mx - q = \nu r^0, \quad (x, s) \geq 0,$$

and this of $(P_{\nu+})$ by

$$s - Mx - q = \nu^+ r^0, \quad (x, s) \geq 0.$$

To get iterates that are feasible for $(P_{\nu+})$ we need search directions $\Delta^f x$ and $\Delta^f s$ such that

$$(s + \Delta^f s) - M(x + \Delta^f x) - q = \nu^+ r^0, \quad (x + \Delta^f x, s + \Delta^f s) > 0.$$

Since (x, s) is feasible for (P_ν) , it follows that $\Delta^f x$ and $\Delta^f s$ should satisfy

$$M\Delta^f x - \Delta^f s = \theta \nu r^0.$$

Therefore, the following system is used to define $\Delta^f x$ and $\Delta^f s$:

$$(3.1) \quad M\Delta^f x - \Delta^f s = \theta \nu r^0,$$

$$(3.2) \quad s\Delta^f x + x\Delta^f s = (1 - \theta)\mu e - xs.$$

It is easy to see that if (x, s) is feasible for the perturbed problem (P_ν) , then after the feasibility step the iterates satisfy the feasibility condition for $(P_{\nu+})$, provided that they are nonnegative. Assuming that before the step $\delta(x, s; \mu) \leq \tau$ holds, and by taking θ small enough, it can be guaranteed that after the step, the iterates

$$(3.3) \quad x^f = x + \Delta^f x,$$

$$(3.4) \quad s^f = s + \Delta^f s.$$

are nonnegative and moreover $\delta(x^f, s^f; \mu^+) \leq \frac{1}{\sqrt{2}}$, where

$$\mu^+ = (1 - \theta)\mu.$$

So, after the μ -update, the iterates are feasible for $(P_{\nu+})$, and μ is such that $\delta(x^f, s^f; \mu^+) \leq \frac{1}{\sqrt{2}}$.

Remark 3.1. For (3.2), we use the linearization of $x^f s^f = (1 - \theta)\mu e$, which means that we target at the μ^+ -center. While in [1], the linearization of $x^f s^f = \mu e$ is used (targeting at the μ -center). As our aim is to calculate a feasible solution to $(P_{\nu+})$, which should also lie in the quadratic convergence neighborhood to its μ^+ -center, the direction used here is more natural and intuitively better.

In the centering steps, starting at iterates $(x, s) = (x^f, s^f)$ and targeting at the μ -center, the search directions Δx , and Δs are the unique directions defined by

$$(3.5) \quad \begin{aligned} \Delta s - M\Delta x &= 0, \\ s\Delta x + x\Delta s &= \mu e - xs. \end{aligned}$$

Denoting the iterates after a centering step as x^+ and s^+ , we recall the following from [1].

Lemma 3.2. *If $\delta := \delta(x, s; \mu) \leq 1$, then the new iterates are feasible, i.e. x^+ and s^+ are nonnegative, and $(x^+)^T s^+ \leq (n + \delta^2) \mu$. Moreover, if $\delta = \delta(x, s; \mu) \leq \frac{1}{\sqrt{2}}$, then $\delta = \delta(d, s; \mu) \leq \delta^2$.*

The centering steps serve to get iterates that satisfy

$$x^T s \leq (n + \delta^2) \mu^+ \quad \text{and} \quad \delta(x, s; \mu^+) \leq \tau,$$

where τ is much smaller than $\frac{1}{\sqrt{2}}$. By using Lemma 3.2, the required number of centering steps can easily be obtained. This goes as follows. After μ -update, we have $\delta(x^f, s^f; \mu^+) \leq \frac{1}{\sqrt{2}}$, and hence after k centering steps, the iterates (x, s) satisfy

$$\delta(x, s; \mu^+) \leq \left(\frac{1}{\sqrt{2}} \right)^{2^k}.$$

Just as in [1] this implies that no more than

$$(3.6) \quad \log_2 \left(\log_2 \frac{1}{\tau^2} \right)$$

centering steps are needed.

4. Analysis of the feasibility step

Let (x, s) denote the iterates at the start of an iteration and assume $\delta(x, s; \mu) \leq \tau$. Recall that at the start of the first iteration, this is certainly true because then $\delta(x, s; \mu) = 0$.

4.1. The effect of the feasibility step and the choice of θ .

As we established in Section 3, the feasibility step generates new iterates (x^f, s^f) that satisfy the feasibility condition for (P_{ν^+}) . A crucial element in the analysis is to show that after the feasibility step $\delta(x^f, s^f; \mu^+) \leq \frac{1}{\sqrt{2}}$, i.e., that the new iterates are within the region where the Newton process targeting at μ^+ -center of (P_{ν^+}) is quadratically convergent.

Define

$$(4.1) \quad v = \sqrt{\frac{xs}{\mu}}, \quad d_x^f = \frac{v\Delta^f x}{x}, \quad d_s^f = \frac{v\Delta^f s}{s},$$

by the use of (3.2) and (4.1), we get

$$(4.2) \quad \begin{aligned} x^f s^f &= xs + (x\Delta^f x + s\Delta^f s) + \Delta^f x \Delta^f s \\ &= (1 - \theta)\mu e + \Delta^f x \Delta^f s \\ &= (1 - \theta)\mu e + \frac{xs}{v^2} d_x^f d_s^f = \mu \left((1 - \theta)e + d_x^f d_s^f \right). \end{aligned}$$

Lemma 4.1. *The iterates (x^f, s^f) are strictly feasible if and only if $(1 - \theta)e + d_x^f d_s^f > 0$*

Proof. The proof is similar to the proof of Lemma 3.1 in [1], and is omitted. $\square$

Corollary 4.2. *The iterates (x^f, s^f) are strictly feasible if $\|d_x^f d_s^f\|_\infty < 1 - \theta$.*

Proof. By Lemma 4.1 x^f and s^f are strictly feasible if and only if $(1 - \theta)e + d_x^f d_s^f > 0$. Since the last inequality holds if $\|d_x^f d_s^f\|_\infty < 1 - \theta$, the corollary follows. $\square$

We proceed by deriving an upper bound for $\delta(x^f, s^f; \mu^+)$. According to definition (2.6) one has

$$\delta(x^f, s^f; \mu^+) = \frac{1}{2} \left\| v^f - (v^f)^{-1} \right\|, \quad \text{where } v^f = \sqrt{\frac{x^f s^f}{\mu^+}}.$$

In the sequel we denote $\delta(x^f, s^f; \mu^+)$ also shortly by $\delta(v^f)$, and we have the following result.

Lemma 4.3. *If $\left\|d_x^f d_s^f\right\|_\infty < 1 - \theta$, then*

$$2\delta \left(v^f\right)^2 \leq \frac{\left\|\frac{d_x^f d_s^f}{1-\theta}\right\|^2}{1 - \left\|\frac{d_x^f d_s^f}{1-\theta}\right\|_\infty}.$$

Proof. To simplify the notations in this proof, let $z := \frac{d_x^f d_s^f}{1-\theta}$. After dividing both sides in (4.2) by μ^+ we get

$$\left(v^f\right)^2 = \frac{\mu \left[(1-\theta)e + d_x^f d_s^f\right]}{\mu^+} = \frac{\mu \left[(1-\theta)e + (1-\theta)z\right]}{(1-\theta)\mu} = e + z.$$

Hence we have

$$\begin{aligned} 2\delta \left(v^f\right)^2 &= \sum_{i=1}^n \left(\left(v_i^f\right)^2 + \left(v_i^f\right) - 2 - 2 \right) = \sum_{i=1}^n \left(1 + z_i + \frac{1}{1+z_i} - 2 \right) \\ &= \sum_{i=1}^n \frac{z_i^2}{1+z_i} \leq \sum_{i=1}^n \frac{z_i^2}{1-|z_i|} \leq \sum_{i=1}^n \frac{z_i^2}{1-\|z\|_\infty} = \frac{\|z\|^2}{1-\|z\|_\infty}, \end{aligned}$$

where the inequalities are due to $\|z\|_\infty < 1$. This proves the lemma. $\square$

4.2. First upper bound for θ .

Since we need to have $\delta(v^f) \leq \frac{1}{\sqrt{2}}$, it follows from Lemma 4.3 that it suffices to have

$$(4.3) \quad \frac{\left\|\frac{d_x^f d_s^f}{1-\theta}\right\|^2}{1 - \left\|\frac{d_x^f d_s^f}{1-\theta}\right\|_\infty} \leq 1.$$

As we may easily verify that

$$(4.4) \quad \left\|d_x^f d_s^f\right\|^2 \leq \left(\left\|d_x^f\right\| \left\|d_s^f\right\|\right)^2 \leq \frac{1}{4} \left(\left\|d_x^f\right\|^2 + \left\|d_s^f\right\|^2\right)^2,$$

$$(4.5) \quad \left\|d_x^f d_s^f\right\|_\infty \leq \frac{1}{2} \left(\left\|d_x^f\right\|_\infty^2 + \left\|d_s^f\right\|_\infty^2\right) \leq \frac{1}{2} \left(\left\|d_x^f\right\|^2 + \left\|d_s^f\right\|^2\right).$$

For the moment we assume that $\frac{\|d_x^f\|^2 + \|d_s^f\|^2}{1-\theta} < 2$. Then $\left\| \frac{d_x^f d_s^f}{1-\theta} \right\|_\infty < 1$, whence inequality (4.3) holds if

$$\frac{\frac{1}{4} \left(\frac{\|d_x^f\|^2 + \|d_s^f\|^2}{1-\theta} \right)^2}{1 - \frac{1}{2} \frac{\|d_x^f\|^2 + \|d_s^f\|^2}{1-\theta}} \leq 1.$$

Considering $\frac{\|d_x^f\|^2 + \|d_s^f\|^2}{1-\theta}$ as a single term, and by some elementary calculation, we obtain that (4.3) holds if

$$(4.6) \quad \frac{\|d_x^f\|^2 + \|d_s^f\|^2}{1-\theta} \leq \sqrt{5} - 1 \approx 1.237.$$

Also by Corollary 4.2 and inequality (4.5), the strict feasibility of (x^f, s^f) can be derived from (4.6). In other words, the inequality (4.6) implies that after the feasibility step (x^f, s^f) is strictly feasible and lies in the quadratic convergence neighborhood with respect to μ^+ -center of (P_{ν^+}) .

4.3. The scaled search direction d_x^f and d_s^f .

One may easily check that the system (3.1)-(3.2), which defines the search directions $\Delta^f x$ and $\Delta^f s$, can be expressed in term of the scaled search directions d_x^f and d_s^f as follows.

$$(4.7) \quad MS^{-1}Xd_x^f - d_s^f = \theta\nu\nu s^{-1}r^0,$$

$$(4.8) \quad d_x^f + d_s^f = (1-\theta)v^{-1} - v,$$

where $X = \text{diag}(x)$, $S = \text{diag}(s)$.

Lemma 4.4 (Corollary 2.3 in [2]). *Let $x > 0$ and $s > 0$ be two n -dimensional vectors, and let $M \in \mathbf{R}^{n \times n}$ be a positive semidefinite matrix. Then the solution (u, z) of the linear system*

$$(4.9) \quad MS^{-1}Xu - z = \tilde{a},$$

$$(4.10) \quad u + z = \tilde{b},$$

satisfies the following relations:

$$(4.11) \quad Du = (I + DMD)^{-1} (a + b), \quad Dz = b - Du,$$

$$(4.12) \quad \|Du\| \leq \|a + b\|,$$

$$(4.13) \quad \|Du\|^2 + \|Dz\|^2 \leq \|b\|^2 + 2\|a + b\| \|a\|,$$

where $D = (S^{-1}X)^{\frac{1}{2}}$, $b = D\tilde{b}$ and $a = D\tilde{a}$.

We are now ready to find an upper bound for $\|d_x^f\|^2 + \|d_s^f\|^2$. To this end we first apply Lemma 4.4 with $u = d_x^f$, $z = d_s^f$, $a = \theta\nu Dvs^{-1}r^0$ and $b = D((1 - \theta)v^{-1} - v)$, which implies that

$$(4.14) \quad \begin{aligned} \|Dd_x^f\|^2 + \|Dd_s^f\|^2 &\leq \|D((1 - \theta)v^{-1} - v)\|^2 \\ &+ 2\|\theta\nu Dvs^{-1}r^0 + D((1 - \theta)v^{-1} - v)\| \|\theta\nu Dvs^{-1}r^0\|. \end{aligned}$$

By elementary properties of norms we have

$$\|Dd_x^f\| \leq \|D\| \|d_x^f\|, \quad \|Dd_s^f\| \leq \|D\| \|d_s^f\|,$$

and

$$\begin{aligned} \|\theta\nu Dvs^{-1}r^0\| &\leq \|D\| \|\theta\nu vs^{-1}r^0\|, \\ \|D((1 - \theta)v^{-1} - v)\| &\leq \|D\| \|(1 - \theta)v^{-1} - v\|. \end{aligned}$$

Substituting these bounds in (4.14) we obtain the following weaker condition

$$(4.15) \quad \begin{aligned} \|d_x^f\|^2 + \|d_s^f\|^2 &\leq \|(1 - \theta)v^{-1} - v\|^2 \\ &+ 2(\|\theta\nu vs^{-1}r^0\| + \|(1 - \theta)v^{-1} - v\|) \|\theta\nu vs^{-1}r^0\|. \end{aligned}$$

In order to obtain a bound for $\|\theta\nu vs^{-1}r^0\|$ we write, using $\nu = \frac{\mu}{\mu^0}$ and $v = \sqrt{\frac{xs}{\mu}}$,

$$\begin{aligned}
 \|\theta\nu vs^{-1}r^0\| &= \theta\nu \|vs^{-1}r^0\| \\
 &= \theta \frac{\sqrt{\mu}}{\mu^0} \left\| \sqrt{\frac{x}{s}} r^0 \right\| \\
 &\leq \theta \frac{\sqrt{\mu}}{\mu^0} \left\| \sqrt{\frac{x}{s}} r^0 \right\|_1 \\
 &= \frac{\theta}{\mu^0} \left\| \sqrt{\frac{\mu}{xs}} x r^0 \right\|_1 \\
 &\leq \frac{\theta}{\mu^0 v_{\min}} \|x r^0\|_1 \\
 (4.16) \quad &\leq \frac{\theta}{\mu^0 v_{\min}} \left\| (S^0)^{-1} r^0 \right\|_{\infty} \|s^0\|_{\infty} \|x\|_1.
 \end{aligned}$$

To proceed we have to specify our initial iterates (x^0, s^0) . We assume that ρ_p and ρ_d are such that

$$(4.17) \quad \|x^*\|_{\infty} \leq \rho_p, \quad \max \{\|s^*\|_{\infty}, \rho_p \|Me\|_{\infty}, \|q\|_{\infty}\} \leq \rho_d,$$

for some $(x^*, s^*) \in \mathcal{F}^*$, and as usual we start the algorithm with

$$(4.18) \quad x^0 = \rho_p e, \quad s^0 = \rho_d e, \quad \mu^0 = \rho_p \rho_d.$$

For such starting points we have clearly

$$(4.19) \quad \left\| (S^0)^{-1} r^0 \right\|_{\infty} \leq 1 + \frac{\rho_p}{\rho_d} \|Me\|_{\infty} + \frac{1}{\rho_d} \|q\|_{\infty} \leq 3.$$

By substituting (4.18) and (4.19) into (4.16) we obtain

$$(4.20) \quad \|\theta\nu vs^{-1}r^0\| \leq \frac{3\theta}{\rho_p v_{\min}} \|x\|_1.$$

By using Lemma 3.2 and $\|v\|^2 = \frac{x^T s}{\mu}$ we easily obtain the following inequality

$$(4.21) \quad \|(1-\theta)v^{-1} - v\|^2 \leq 2(1-\theta)\delta^2 + (n+\delta^2)\theta^2.$$

Using (4.20) and (4.21) in (4.15) we get

$$(4.22) \quad \left\| d_x^f \right\|^2 + \left\| d_s^f \right\|^2 \leq 2(1 - \theta) \delta^2 + (n + \delta^2) \theta^2 \\ + 2 \left(\frac{3\theta}{\rho_p v_{\min}} \|x\|_1 + \sqrt{2(1 - \theta) \delta^2 + (n + \delta^2) \theta^2} \right) \frac{3\theta}{\rho_p v_{\min}} \|x\|_1.$$

Recall that (x, s) is feasible for (P_ν) and $\delta(x, s; \mu) \leq \tau$; i.e., this iterate is close to the μ -center of (P_ν) . Based on this information, we present the following three lemmas to estimate an upper bound for $\|x\|_1$ and a lower bound for $v_{\min}$.

Lemma 4.5. *Let $\delta = \delta(v)$ be given by (2.6). Then*

$$(4.23) \quad \frac{1}{q(\delta)} \leq v_i \leq q(\delta),$$

where

$$(4.24) \quad q(\delta) := \frac{\sqrt{2}}{2} \delta + \sqrt{\frac{1}{2} \delta^2 + 1}.$$

Proof. The proof of this lemma is exactly the same as that of Lemma II.60 in [5]. $\square$

Lemma 4.6 (Lemma 5.7 in [1]). *Let (x, s) be feasible for the perturbed problem (P_ν) and (x^0, s^0) as defined in (4.18). Then for any $(x^*, s^*) \in \mathcal{F}^*$, we have*

$$\nu \left((s^0)^T x + (x^0)^T s \right) \leq \nu^2 (x^0)^T s^0 + x^T s \\ + \nu(1 - \nu) \left((s^0)^T x^* + (x^0)^T s^* \right) - (1 - \nu) (s^T x^* + x^T s^*).$$

Lemma 4.7 (Lemma 5.8 in [1]). *Let (x, s) be feasible for the perturbed problem (P_ν) and $\delta(v)$ is defined as in (2.6) and (x^0, s^0) as defined in (4.18). Then we have*

$$(4.25) \quad \|x\|_1 \leq \left(2 + q(\delta)^2 \right) n \rho_p,$$

$$(4.26) \quad \|s\|_1 \leq \left(2 + q(\delta)^2 \right) n \rho_p,$$

where $q(\delta)$ as defined in (4.24).

By substituting (4.23) and (4.25) into (4.22) we obtain

$$\begin{aligned}
& \left\| d_x^f \right\|^2 + \left\| d_s^f \right\|^2 \leq 2(1-\theta)\delta^2 + (n+\delta^2)\theta^2 \\
(4.27) \quad & + 2 \left(3n\theta q(\delta) \left(q(\delta)^2 + 2 \right) \right)^2 \\
& + 6 \left(\sqrt{2(1-\theta)\delta^2 + (n+\delta^2)\theta^2} \right) n\theta q(\delta) \left(q(\delta)^2 + 2 \right).
\end{aligned}$$

4.4. Value for θ . We have found that $\delta(v^f) \leq \frac{1}{\sqrt{2}}$ holds if the inequality (4.6) is satisfied. Then by (4.27), inequality (4.6) holds if

$$\begin{aligned}
& 2(1-\theta)\delta^2 + (n+\delta^2)\theta^2 + 2 \left(3n\theta q(\delta) \left(q(\delta)^2 + 2 \right) \right)^2 \\
& + 6 \left(\sqrt{2(1-\theta)\delta^2 + (n+\delta^2)\theta^2} \right) n\theta q(\delta) \left(q(\delta)^2 + 2 \right) \leq 1.237(1-\theta).
\end{aligned}$$

Obviously, the left-hand side of the above inequality is increasing in δ , due to the definition $q(\delta) := \frac{\sqrt{2}}{2}\delta + \sqrt{\frac{1}{2}\delta^2 + 1}$. Using this one may easily verify that the above inequality is satisfied if

$$(4.28) \quad \tau = \frac{1}{8}, \quad \theta = \frac{1}{14n}.$$

Then, according to (3.6), with τ as given, after the feasibility step at most 3 centering steps suffices to get iterates (x^+, s^+) that satisfy $\delta(x^+, s^+; \mu^+) \leq \tau$.

4.5. Complexity analysis. In the previous sections we have found that if at the start of an iteration the iterates satisfy $\delta(x, s; \mu) \leq \tau$, with τ as defined in (4.28), then after the feasibility step, with θ as defined in (4.28), the iterates satisfy $\delta(x^f, s^f; \mu) \leq \frac{1}{\sqrt{2}}$.

According to (3.6), at most 3 centering steps then suffice to get iterates (x^+, y^+, s^+) that satisfy $\delta(x^+, s^+; \mu^+) \leq \tau$ again. So each main iteration consists of at most 4 so-called inner iterations, in each iteration we need to compute a search direction (for either a feasibility step or a centering step).

usually one measures the complexity of an IPM by inner iterations as many times as needed. In each main iteration both the values of $n\mu$ and the norm of the residual are reduced by the factor $1-\theta$. Hence, the total number of the main iterations is bounded above by

$$\frac{1}{\theta} \log \frac{\max \left\{ (x^0)^T s^0, \|r^0\| \right\}}{\varepsilon}.$$

Due to (4.28) we may take

$$\theta = \frac{1}{14n}.$$

Hence the total number of inner iterations is bounded above by

$$56n \log \frac{\max \left\{ (x^0)^T s^0, \|r^0\| \right\}}{\varepsilon}.$$

Thus we may state without any more proof the main result of the paper.

Theorem 4.8. *If (P) has optimal solution $(x^*, s^*) \in \mathcal{F}^*$ such that $\|x^*\|_\infty \leq \rho_p$ and $\|s^*\|_\infty \leq \rho_d$, then after at most*

$$56n \log \frac{\max \left\{ (x^0)^T s^0, \|r^0\| \right\}}{\varepsilon},$$

iterations the algorithm finds an ε -solution of LCP.

Remark 4.9. *The above iteration bound is derived under the assumption that there exists an optimal solution with $(x^*, s^*) \in \mathcal{F}^*$ such that $\|x^*\|_\infty \leq \rho_p$ and $\|s^*\|_\infty \leq \rho_d$. One might ask what happens if this is not satisfied. In that case, during the course of the algorithm it may happen that after some main steps the proximity measure δ (after the feasibility step) exceeds $\frac{1}{\sqrt{2}}$, because otherwise there is no reason why the algorithm would not generate an ε -solution. So if this happens, either (P) does not have optimal solution in $\mathcal{F}^*$ or the values of ρ_p and ρ_d have been too small. In the latter case one might run the algorithm once more with some larger ρ_p and ρ_d .*

5. Numerical results

In this section we present some numerical results. We consider the following examples:

Example 5.1. [6]

$$M = \begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 0 & 1 \\ 1 & 0 & 1 & 2 \\ -1 & -1 & -2 & 0 \end{bmatrix}, q = \begin{bmatrix} -8 \\ -6 \\ -4 \\ 3 \end{bmatrix}.$$

Example 5.2. [3]

$$M = \begin{bmatrix} 1 & 0 & -0.5 & 0 & 1 & 3 & 0 \\ 0 & 0.5 & 0 & 0 & 2 & 1 & -1 \\ -0.5 & 0 & 1 & 0.5 & 1 & 2 & -4 \\ 0 & 0 & 0.5 & 0.5 & 1 & -1 & 0 \\ -1 & -2 & -1 & -1 & 0 & 0 & 0 \\ -3 & -1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 1 & 4 & 0 & 0 & 0 & 0 \end{bmatrix}, q = \begin{bmatrix} -1 \\ -3 \\ 1 \\ -1 \\ 5 \\ 4 \\ -1.5 \end{bmatrix}.$$

We solve examples 5.1 and 5.2 by using both the short updating algorithm [1] and the algorithm in Figure 1. For both algorithms, the initialization parameters ρ_p and ρ_d are assumed as described in Section 2, and the accuracy parameter ε is set to 10^{-4} . Table 1 shows the number of iterations to obtain ε -solutions for the above two examples with the short updating algorithm and the algorithm in Figure 1. From the table we see that the algorithm in Figure 1 reduced the number of iterations. Since for both algorithms the work in every iteration is almost the same, this is really a huge reduction.

Examples	Algorithm in [1]	Algorithm in Figure 1
Ex.5.1	138	51
Ex.5.2	180	86

TABLE 1. The number of iterations for examples 5.1 and 5.2

6. Concluding remarks

We presented a new IIPM for LCP; each main iteration consists of a feasibility step and three centering steps. Our new feasibility step is more natural, as it targets at the μ^+ -center, which results a better iteration bound in compare with [1]. The ideas underlying this article can be used to extend the algorithm to second-order cone optimization and also to the symmetric cone optimization.

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