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COMPLEMENT OF SPECIAL CHORDAL GRAPHS AND VERTEX DECOMPOSABILITY

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Abstract. In this paper, we introduce a subclass of chordal graphs which contains d-trees and show that their complement are vertex decomposable and so is shellable and sequentially Cohen-Macaulay. This result improves the main result of Ferrarello who used a theorem due to Fröberg and extended a recent result of Dochtermann and Engström.

1. Introduction

Let k be a field. To any finite simple graph G with vertex set $\mathrm{V} = [n] := \{1, \cdots, n\}$ and edge set $\mathrm{E}(G)$ one associates an ideal $I(G) \subset k[x_1, \cdots, x_n]$ generated by all monomials $x_i x_j$ such that $\{i, j\} \in \mathrm{E}(G)$. The ideal $I(G)$ and the quotient ring $k[x_1, \cdots, x_n]/I(G)$ are called the edge ideal of G and the edge ring of G, respectively. The independence complex of G is defined by

$$\mathrm{Ind}\,(G) = \{A \subseteq \mathrm{V} | A \text{ is an independent set in } G\},$$

A is said to be an independent set in G if none of its elements are adjacent. Note that $\mathrm{Ind}\,(G)$ is precisely the simplicial complex with the Stanley-Reisner ideal $I(G)$. We denoted by Δ_G the clique complex of G, which is the simplicial complex with vertex set V and with faces

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the cliques of G. It is easy to see that $\Delta_G = \mathrm{Ind}\,(\overline{G})$, where $\overline{G}$ is the complement of G.

A simplicial complex Δ is recursively defined to be vertex decomposable if it is either a simplex, or has some vertex v so that:

- both $\Delta \setminus v$ and $link_\Delta v$ are vertex decomposable, and
- no face of $link_\Delta v$ is a facet of $\Delta \setminus v$.

A vertex v which satisfies the second condition is called a shedding vertex.

A simplicial complex Δ is called shellable if the facets (maximal faces) of Δ can be ordered a $F_1, \ldots, F_s$ such that for all $1 \leq i < j \leq s$, there exists some $v \in F_j \setminus F_i$ and some $l \in \{1, \ldots, j-1\}$ with $F_j \setminus F_l = \{v\}$, cf [1]. The notion of shellability was discovered in the context of convex polytopes, cf. [8].

The dimension of a face F is $|F| - 1$. Let $d = max\{|F| : F \in \Delta\}$ and define the dimension of Δ to be $dim\Delta = d - 1$. A simplicial complex is pure if all of its facets are of the same dimension. The k-skeleton of Δ is the complex generated by all the k-dimensional faces of Δ. A complex is sequentially Cohen-Macaulay if its k-skeleton is Cohen-Macaulay for each k, k< dimension of the complex. Any shellable complex is sequentially Cohen-Macaulay.

We have the following chains of strict implications:

• vertex decomposable $\stackrel{(i)}{\Longrightarrow}$ shellable $\stackrel{(ii)}{\Longrightarrow}$ sequentially Cohen-Macaulay

• pure vertex decomposable $\stackrel{(iii)}{\Longrightarrow}$ pure shellable $\stackrel{(iv)}{\Longrightarrow}$ Cohen-Macaulay

Where (i) comes from [12, Lemma 6] , (ii) from [10, p. 87] , (iii) from [9, Theorem 2.8] and (iv) from [7].

In recent years there have been a flurry of work investigating how the combinatorial properties of G appear within the algebraic properties of $R/I(G)$, and vice versa. (Sequentially) Cohen-Macaulay rings are of great interest. As a consequence, one particular stream of research has focused on the question of what graph G has the property that $R/I(G)$ is (Sequentially) Cohen-Macaulay.

We can recursively define a generalized d-tree in the following way:

(1) A complete graph with d+1 vertices is a generalized d-tree;
(2) Let G be a graph on the vertex set V(G). Suppose that there is some vertex $v \in$ V(G) such that the followings hold:
    (i) the restriction G_1 of G to $V_1 = V \setminus \{v\}$ is a generalized d-tree;

(ii) there is a subset V_2 of V_1, where the restriction of G to V_2 is a clique of size j with $0 \leq j \leq d$;
(iii) G is the graph generated by G_1 and the complete graph on $V_2 \cup \{v\}$.

In particular, we say that G is a (d, j)-tree if in the above recursive definition j is fixed. A (d, d)-tree is called a d-tree.

A graph G is called chordal if every cycle of length> 3 has a chord. Recall that a chord of a cycle is an edge which joins two vertices of the cycle but is not itself an edge of the cycle. In [2] Dirac proved that the generalized d-trees are exactly the chordal graphs and so (d, j)-trees are chordal.

Many authors are interested in the case when G or its complement is chordal (in particular d-tree) for example see [4], [5], and sections 3 and 4 of [3].

Ferrarello in [4] showed that the complement of a d-tree is Cohen-Macaulay and Dochtermann and Engström in [3] extended this result by showing that the complement of a d-tree is pure shellable.

It is not hard to show that the complement of a d-tree is pure of dimension d [see Proposition 2.4], but in general the complement of a chordal graph is not pure so it is natural to ask whether the complement of a chordal graph is sequentially Cohen-Macaulay. By giving an example we show that the answer is negative. We show that if G is the complement of a (d, j)-tree, then $Ind(G)$ is vertex decomposable and so shellable and sequentially Cohen-Macaulay. This result is a generalization of [4, Theorem 3.3] and [3, Proposition 3.6].

2. **Main Results**

The definition of vertex decomposable complexes translates nicely to independence complexes as follows:

Lemma 2.1. [13, Lemma 2.2]. *An independence complex $Ind(G)$ is vertex decomposable if G is a totally disconnected graph (with no edges), or if*

(i) *$G \setminus v$ and $G \setminus N[v]$ are both vertex decomposable, and*
(ii) *An independent set in $G \setminus N[v]$ is not a maximal independent set in $G \setminus v$.*

We say that a graph G is vertex decomposable if its independence complex $\mathrm{Ind}\,(G)$ is vertex decomposable. Let $N(v)$ denotes the open

neighborhood of v, that is, all vertices adjacent to v. Let $N[v]$ denotes the close neighborhood of v, which is $N(v)$ together with v itself, so that $N[v] = N(v) \cup \{v\}$.

Theorem 2.2. *Let G be the complement of a (d, j)-tree, then $Ind(G)$ is vertex decomposable and so it is shellable and sequentially Cohen-Macaulay.*

Proof. We use induction on $|V(G)|$. If $|V(G)| = d + 1$, then G is totally disconnected and there is nothing to prove.

Let the statement be true for complement of (d, j)-trees of size $< n$ and let G be a (d, j)-tree with $|V(G)| = n$.

It is easy to see that there is a vertex v in G such that $\overline{G}[N_{\overline{G}}[v]]$ is a clique and $\overline{G}[N_{\overline{G}}(v)]$ is not a maximal clique in $\overline{G} \setminus v$. Now $G \setminus N[v]$ is a totally disconnected graph, so it is a vertex decomposable graph. On the other hand $\overline{G} \setminus v$ is a chordal graph so by the induction hypothesis $G \setminus v$ is vertex decomposable.

Now it suffices to show that no independent set in $G \setminus N[v]$ is a maximal independent set in $G \setminus v$, but it holds obviously, because $\overline{G}[N_{\overline{G}}(v)]$ is not a maximal clique in $\overline{G} \setminus v$.

□

Note that in general it is not the case that the complement of a chordal graph is vertex decomposable. See the following example:

Example 2.3. *Let G be the graph in figure 1:*

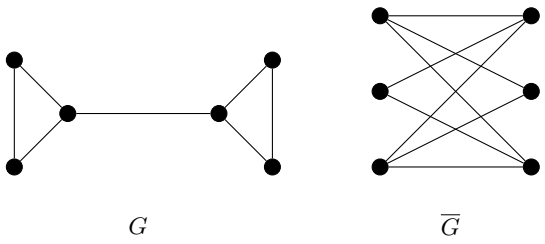

FIGURE 1

Here G is a chordal graph but $\overline{G}$ is bipartite and does not have any end vertex (a vertex of degree 1). Thus by [11, Lemma 3.9] *$\overline{G}$ is not sequentially Cohen-Macaulay and so it is not vertex decomposable.*

By using a result of Fröberg, Ferrarello in [4] showed that the complement of a d-tree is Cohen-Macaulay and Dochtermann and Engström in [3] extended this result by showing that the complement of a d-tree is pure shellable. We state the following result as a generalization:

Corollary 2.4. *Let G be the complement of a d-tree, then $Ind(G)$ is pure vertex decomposable* (*pure shellable and Cohen-Macaulay*).

Proof. Using Theorem 2.2 , it remains to prove the purity. The facets of $Ind(G)$ are the maximal independent sets of G, which are in fact the maximal cliques of $\overline{G}$. So it suffices to show that every maximal clique of a d-tree, H, is of size d+1. We use induction on $|V(H)|$, if $|V(H)| = d + 1$ then $H = K_{d+1}$, and there is nothing to prove. Let the statement be true for every d-tree of size $< n$ and let $|V(H)| = n$. It is easy to check that there is a vertex in $V(H)$, say v, such that $\deg v = d$ and $H[N[v]]$ is a clique. Let C be a maximal clique in H if $v \notin C$ then C is a maximal clique in $H \setminus v$, so by the induction hypothesis, $|C| = d+1$. And if $v \in C$ the desired statement holds obviously. □

In the following we give some examples of (pure and non-pure) vertex decomposable graphs.

By using the mentioned theorem of Dirac, one can show that chordal graphs are vertex decomposable, see [3] and [13]. The following example shows that the converse is not true in general.

Example 2.5. *Our first example is $T = P_n$ (a path with n vertices). When $n \geq 5$, $\overline{T}$ is pure vertex decomposable but not chordal. (Note that the complement of a tree T is chordal if and only if $diam\,(T) \leq 3$, where $diam\,(G) = max\{d(u, v)|u, v \in V(G)\}$.*

The complement of a tree T is chordal if and only if $\operatorname{diam}(T) \leq 3$. In the following example we show that this is not true for d-treeS when $d > 1$.

Example 2.6. *The graph G in Figure 2 is a 2-tree with diameter* 2. *$\overline{G}$ is pure vertex decomposable and has an induced cycle of length* 4.

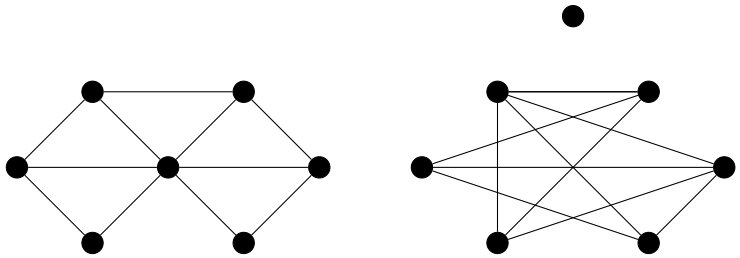

FIGURE 2

Example 2.7. *The graph G in Figure 3 is a $(3,2)$-tree which is not a d-tree. Therefore $\overline{G}$ is vertex decomposable but not pure.*

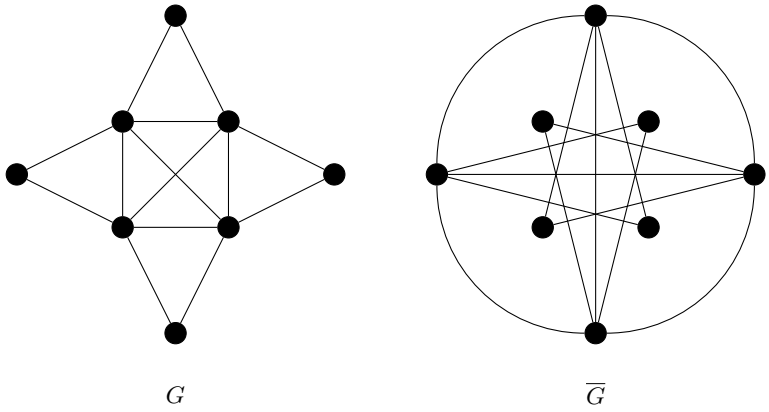

FIGURE 3

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